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# SOME RESULTS RELATED TO THE HEINZ INEQUALITY IN C*–ALGEBRA

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*Abstract.* Let $f$, $g$ be two continuous non-negative real-valued functions defined on the non-negative half-line $[0,\infty)$ that satisfy the condition $f(t)g(t)=t$, for all $t\geqslant 0$, and let $P$ and $Q$ denote two positive elements in an unital C*-algebra $\mathscr{A}$. We shall show that the following model of inequality holds:

$$\forall X\in\mathscr{A},\ \ \|f(P)Xg(Q)+g(P)Xf(Q)\|\geqslant 2\left\|P^{\frac{1}{2}}XQ^{\frac{1}{2}}\right\|.$$

Through this model, we shall establish the universality of the Heinz operator norm inequality and related inequalities within the broad spectrum of any abstract unital C*-algebra.

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