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ON BOUNDS OF LOGARITHMIC MEAN AND MEAN INEQUALITY CHAIN

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Abstract. An upper bound of the logarithmic mean is given by a convex combination of the arithmetic mean and the geometric mean. In addition, a lower bound of the logarithmic mean is given by a geometric bridge of the arithmetic mean and the geometric mean. In this paper, we study the bounds of the logarithmic mean. We give operator inequalities and norm inequalities for the fundamental inequalities on the logarithmic mean. We give monotonicity of the parameter for the unitarily invariant norm of the Heron mean, and give its optimality as the upper bound of the unitarily invariant norm of the logarithmic mean. We study the ordering of the unitarily invariant norms for the Heron mean, the Heinz mean, the binomial mean and the Lehmer mean. Finally, we give a new mean inequality chain as an application of the point–wise inequality.

Mathematics subject classification (2020): 15A60, 26E60, 26D07, 47A30, 47A63.

Keywords and phrases: Arithmetic mean, geometric mean, harmonic mean, logarithmic mean, Heinz mean, binomial mean, Heron mean, Lehmer mean, matrix inequality, norm inequality, unitarily invariant norm, positive definite function and infinitely divisible function.

REFERENCES

[1] R. BHATIA, *Interpolating the arithmetic–geometric mean inequality and its operator version*, Linear Alg. Appl., **413** (2006), 355–363, 10.1016/j.laa.2005.03.005.

[2] R. BHATIA, *Positive definite matrices*, Princeton University Press, 2007, https://press.princeton.edu/books/paperback/9780691168258/positive-definite-matrices.

[3] R. BHATIA AND H. KOSAKI, *Mean matrices and infinite divisibility*, Linear Alg. Appl., **424** (2007), 36–54, 10.1016/j.laa.2006.03.023.

[4] P. S. BULLEN, *Handbook of means and their inequalities*, Kluwer Academic Publishers, 2003, https://link.springer.com/book/10.1007/978-94-017-0399-4.

[5] T. H. DINH, R. DUMITRU AND J. A. FRANCO, *The matrix power means and interpolations*, Adv. Oper. Theory, **3** (3) (2018), 647–654, 10.15352/aot.1801-1288.

[6] D. DRISSI, *Sharp inequalities for some operator means*, SIAM J. Matrix Anal. Appl., **28** (2006), 822–828, 10.1137/050648444.

[7] S. FURUICHI AND K. YANAGI, *Bounds of the logarithmic mean*, J. Inequal. Appl., **2013** (535) (2013), 1–11, 10.1186/1029-242X-2013-535.

[8] F. HIAI AND H. KOSAKI, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J., **48** (1999), 899–936, 10.1512/IUMJ.1999.48.1665.

[9] F. HIAI AND H. KOSAKI, *Means of Hilbert space operators*, Springer, 2003, https://link.springer.com/book/10.1007/b13213.

[10] F. HIAI, H. KOSAKI, D. PETZ AND M. B. RUSKAI, *Families of completely positive maps associated with monotone metrics*, Linear Alg. Appl., **439** (2013), 1749–1791, 10.1016/j.laa.2013.05.012.

[11] Y. KAPIL AND M. SINGH, *Contractive maps on operator ideals and norm inequalities*, Linear Alg. Appl., **459** (2014), 475–492, 10.1016/j.laa.2014.06.055.

[12] H. KOSAKI, *On infinite divisibility of positive definite functions arising from operator means*, J. Func.Anal., **254** (2008), 84–108.

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[13] H. KOSAKI, *Positive definiteness of functions with applications to operator norm inequalities*, Mem. Amer. Math. Soc., **212** (997), 2011, https://www.ams.org/books/memo/0997/.

[14] H. KOSAKI, *Strong monotonicity for various means*, J. Func. Anal., **267** (2014), 1917–1958, 10.1016/j.jfa.2014.07.021.

[15] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (3) (1980), 205–224, 10.1007/BF01371042.

[16] E. B. LEACH AND M. C. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl., **92** (1983), 207–223, 10.1016/0022-247X(83)90280-9.

[17] T.-P. LIN, *The power mean and the logarithmic mean*, Amer. Math. Monthly, **81** (8) (1974), 879–883, 10.1080/00029890.1974.11993684.

[18] Z.-H. YANG, *New sharp bounds of logarithmic mean and identric mean*, J. Inequal. Appl., **2013** (2013), Art. 116, 10.1186/1029-242X-2013-116.

[19] L. ZHENG, *On the upper bound and lower bound of the logarithmic mean*, J. Math. Research and Exposition, **20** (4) (2000), 623–626. (in Chinese).

[20] L. ZOU, *Matrix versions of the classical Polya inequality*, ScienceAsia, **39** (1) (2013), 204–207, 10.2306/scienceasia1513-1874.2013.39.204.