

## EXTENSIONS OF CONVEXITY-BASED INEQUALITIES VIA POSITIVE LINEAR FUNCTIONALS

RABIA BIBI\*, ALVINA YASMIN, FARHANA JABEEN AND MUHAMMAD BASHIR

*Abstract.* This paper establishes refined extensions of fundamental inequalities such as Hardy, Hölder, Minkowski, AM-GM, and Ky Fan types. Our approach employs partition-based weights, positive linear functionals, and time scale calculus to obtain sharper estimates valid across continuous, discrete, and quantum domains. We derive new bounds for Csiszár divergences, Kullback-Leibler divergence, Shannon entropy, and related measures. These results link convexity-based refinements with modern information theory and provide analytic tools for optimization, error estimation, and hybrid dynamical systems.

*Mathematics subject classification (2020):* 26D15, 34N05, 94A17.

*Keywords and phrases:* Jensen's inequality, classical inequalities, divergence measures, positive linear functionals, time scales calculus.

### REFERENCES

- [1] R. BIBI, *Refinements of some classical inequalities on time scales*, Math. Inequal. Appl. **25** (1) (2022), 251–260.
- [2] R. BIBI AND H. RAZA, *Some converse of functional Hölder-type inequalities*, Math. Inequal. Appl. **27** (2024), 613–626.
- [3] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
- [4] T. M. COVER AND J. A. THOMAS, *Elements of Information Theory*, 2nd ed., Wiley-Interscience, Hoboken, NJ, 2006.
- [5] I. CSISZÁR, *I-divergence geometry of probability distributions and minimization problems*, Ann. Probab. **3** (1) (1975), 146–158.
- [6] I. CSISZÁR, *Information-type measures of difference of probability distributions and indirect observations*, Studia Sci. Math. Hungar. **2** (1967), 299–318.
- [7] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.
- [8] K. A. KHAN, T. NIAZ, D. PEČARIĆ AND J. PEČARIĆ, *Refinement of Jensen's inequality and estimation of  $f$ - and Rényi divergence via Montgomery identity*, J. Inequal. Appl. **2018** (318) (2018), 1–22.
- [9] M. A. KHAN, J. PEČARIĆ AND Y.-M. CHU, *Refinements of Jensen's and McShane's inequalities with applications*, AIMS Math. **5** (5) (2020), 4931–4945.
- [10] M. A. KHAN, Z. HUSAIN AND Y. M. CHU, *New estimates for Csiszár divergence and Zipf-Mandelbrot entropy via Jensen-Mercer's inequality*, Complexity **2020** (2020), Article ID 8928691, 1–8.
- [11] S. KULLBACK AND R. A. LEIBLER, *On information and sufficiency*, Ann. Math. Stat. **22** (1) (1951), 79–86.
- [12] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, Dordrecht, 1993.
- [13] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Math. Sci. Eng. vol. 187, Academic Press, Boston, 1992.
- [14] Z. M. SAYED, A. KHAN, M. S. KHAN AND J. PEČARIĆ, *A refinement of the integral Jensen inequality pertaining to certain functions with applications*, J. Funct. Spaces **2022** (1) (2022), 1–11.

- [15] Y. ZHAO, L. CHENG AND Y. ZHANG, *Refinements of Jensen and Hardy inequalities on time scales*, Math. Inequal. Appl. **23** (2020), 735–754.