

A REVIEW OF A RIESZ BASIS PROPERTY FOR INDEFINITE STURM-LIOUVILLE PROBLEMS

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Abstract. For an indefinite weight function r on $[-1, 1]$ with $xr(x) > 0$ we consider connections between a Riesz basis property of the indefinite Sturm-Liouville eigenvalue problem

$$-y'' = \lambda ry, \quad y(-1) = y(1) = 0$$

and various different conditions, for example HELP-type inequalities

$$\left(\int_0^1 |h'|^2 \frac{1}{r} dx \right)^2 \leq k \left(\int_0^1 |h|^2 dx \right) \left(\int_0^1 \left| \left(\frac{h'}{r} \right)' \right|^2 dx \right)$$

for certain classes of functions h on $[0, 1]$. We show that for so-called strongly odd dominated functions r (including odd r) these problems are equivalent. This allows us to apply known results from the theory of one problem to the others.

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