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THE RIEMANNIAN MEAN AND MATRIX INEQUALITIES RELATED TO THE ANDO–HIAI INEQUALITY AND CHAOTIC ORDER

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Abstract. The Riemannian mean on the convex cone of positive definite matrices is a kind of geometric mean of n-matrices which is an extension of the geometric mean of two-matrices. In this paper, we derive the Ando-Hiai inequality for the Riemannian mean which is an extension of the well-known Ando-Hiai inequality of two-matrices. Moreover, we shall show an extension of a characterization of chaotic order. Lastly, we will give a negative answer for the problem whether the same results are satisfied or not for other geometric means of n-matrices.

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