

## ON INVERSE PROBLEMS FOR LEFT-DEFINITE DISCRETE STURM-LIOUVILLE EQUATIONS

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*Abstract.* We establish an expansion theorem and investigate inverse spectral and inverse scattering problems for the discrete Sturm-Liouville problem

$$-u''(n-1) + q(n)u(n) = \lambda w(n)u(n)$$

where  $q$  is nonnegative and  $w$  may change sign. If  $w$  is positive, the  $\ell^2$ -space weighted by  $w$  is a Hilbert space and it is customary to use that space for setting the problem. In the present situation the right-hand-side of the equation does not give rise to a positive-definite quadratic form and we use instead the left-hand side to define such a form and hence a Hilbert space (such problems are called left-definite). The difference equation gives rise to a linear relation which, upon proper restrictions, generates a self-adjoint operator. For this operator we define a Fourier transform and investigate the relationship between two operators with the same transform (the inverse spectral problem). If  $q - q_0$  and  $w - 1$  are summable one may define the scattering process and we solve the inverse scattering problem. For coefficients decaying sufficiently fast to  $q_0$  and 1, respectively, the concept of a resonance is introduced as a generalization of the notion of an eigenvalue and the set of iso-resonant operators, *i.e.*, operators having the same eigenvalues and resonances, is described.

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### REFERENCES

- [1] C. BENNEWITZ, B. M. BROWN, AND R. WEIKARD, *Inverse spectral and scattering theory for the half-line left-definite Sturm-Liouville problem*, SIAM J. Math. Anal. **40**, 5 (2009), 2105–2131.
- [2] CHRISTER BENNEWITZ, *Spectral theory for pairs of differential operators*, Ark. Mat. **15**, 1 (1977), 33–61.
- [3] CHRISTER BENNEWITZ AND W. NORRIE EVERITT, *The Titchmarsh-Weyl eigenfunction expansion theorem for Sturm-Liouville differential equations*, Sturm-Liouville theory, pages 137–171, Birkhäuser, Basel, 2005.
- [4] PAUL A. BINDING, PATRICK J. BROWNE, AND BRUCE A. WATSON, *Inverse spectral problems for left-definite Sturm-Liouville equations with indefinite weight*, J. Math. Anal. Appl. **271**, 2 (2002), 383–408.
- [5] G. FREILING AND V. YURKO, *Inverse problems for differential equations with turning points*, Inverse Problems **13**, 5 (1997), 1247–1263.
- [6] G. FREILING AND V. YURKO, *Inverse spectral problems for differential equations on the half-line with turning points*, J. Differential Equations **154**, 2 (1999), 419–453.
- [7] G. Š. GUSEĬNOV, *Determination of an infinite nonselfadjoint Jacobi matrix from its generalized spectral function*, Mat. Zametki **23**, 2 (1978), 237–248.
- [8] Q. KONG, H. WU, AND A. ZETTL, *Singular left-definite Sturm-Liouville problems*, J. Differential Equations **206**, 1 (2004), 1–29.
- [9] ALLAN M. KRALL, *Regular left definite boundary value problems of even order*, Quaestiones Math. **15**, 1 (1992), 105–118.

- [10] ALLAN M. KRALL, *Singular left-definite boundary value problems*, Indian J. Pure Appl. Math. **29**, 1 (1998), 29–36.
- [11] L. L. LITTLEJOHN AND R. WELLMAN, *A general left-definite theory for certain self-adjoint operators with applications to differential equations*, J. Differential Equations **181**, 2 (2002), 280–339.
- [12] MARCO MARLETTA AND ANTON ZETTL, *Counting and computing eigenvalues of left-definite Sturm-Liouville problems*, J. Comput. Appl. Math. **148**, 1 (2002), 65–75. On the occasion of the 65th birthday of Professor Michael Eastham.
- [13] H. D. NIESSEN AND A. SCHNEIDER, *Spectral theory for left-definite singular systems of differential equations*, Spectral theory and asymptotics of differential equations (Proc. Conf., Scheveningen, 1973), pages 29–43. North-Holland Math. Studies, Vol. **13**, North-Holland, Amsterdam, 1974.
- [14] H. D. NIESSEN AND A. SCHNEIDER, *Spectral theory for left-definite singular systems of differential equations II*, Spectral theory and asymptotics of differential equations (Proc. Conf., Scheveningen, 1973), pages 45–56. North-Holland Math. Studies, Vol **13**, North-Holland, Amsterdam, 1974.
- [15] RUDI WEIKARD, *A local Borg-Marchenko theorem for difference equations with complex coefficients*, Partial differential equations and inverse problems, volume 362 of Contemp. Math., pages 403–410, Amer. Math. Soc., Providence, RI, 2004.
- [16] H. WEYL, *Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen (2. note)*, Gött. Nachr. (1910), pages 442–467.
- [17] H. WEYL, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. **68** (1910), 220–269.