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INDEFINITE HAMILTONIAN SYSTEMS WHOSE TITCHMARSH—WEYL COEFFICIENTS HAVE NO FINITE GENERALIZED POLES OF NON–POSITIVE TYPE

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Abstract. The two-dimensional Hamiltonian system

$$(*) \qquad y'(x) = zJH(x)y(x), \quad x \in (a,b),$$

where the Hamiltonian H takes non-negative 2×2-matrices as values, and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, has attracted a lot of interest over the past decades. Special emphasis has been put on operator models and direct and inverse spectral theorems. Weyl theory plays a prominent role in the spectral theory of the equation, relating the class of all equations $(*)$ to the class $\mathcal{N}_0$ of all Nevanlinna functions via the construction of Titchmarsh–Weyl coefficients.

In connection with the study of singular potentials, an indefinite (Pontryagin space) analogue of equation $(*)$ was proposed, where the 'general Hamiltonian' is allowed to have a finite number of inner singularities. Direct and inverse spectral theorems, relating the class of all general Hamiltonians to the class $\mathcal{N}_{<\infty}$ of all generalized Nevanlinna functions, were established.

In the present paper, we investigate the spectral theory of general Hamiltonians having a particular form, namely, such which have only one singularity and the interval to the left of this singularity is a so-called indivisible interval. Our results can comprehensively be formulated as follows.

– We prove direct and inverse spectral theorems for this class, i.e. we establish an intrinsic characterization of the totality of all Titchmarsh–Weyl coefficients corresponding to general Hamiltonians of the considered form.

– We determine the asymptotic growth of the fundamental solution when approaching the singularity.

– We show that each solution of the equation has 'polynomially regularized' boundary values at the singularity.

Besides the intrinsic interest and depth of the presented results, our motivation is drawn from forthcoming applications: the present theorems form the core for our study of Sturm–Liouville equations with two singular endpoints and our further study of the structure theory of general Hamiltonians (both to be presented elsewhere).

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