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ON THE BEHAVIOR AT INFINITY OF SOLUTIONS TO DIFFERENCE EQUATIONS IN SCHRÖDINGER FORM

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Abstract. We study the behavior at infinity of solutions of discrete Schrödinger equations. First we study pairs of discrete Schrödinger equations whose potential functions differ by a quantity that can be considered small in a suitable sense as the index $n \to \infty$. With simple assumptions on the growth rate of the solutions of the original system, we show that the perturbed system has a fundamental set of solutions with the same behavior at infinity, employing a variation-of-constants scheme to produce a convergent iteration for the solutions of the second equation in terms of those of the original one.

Next, we present a sharp discrete analogue of the Liouville-Green (JWKB) transformation, making it possible to derive exponential behavior at infinity of a single difference equation, by explicitly constructing a comparison equation to which the perturbation results apply. After that we use the relations between the solution sets of two discrete Schrödinger equations differing by a perturbation to derive exponential dichotomy of solutions and to elucidate the structure of transfer matrices.

A final section contains illustrative examples, including some with large, oscillatory potentials, and an appendix discusses the connection between the discrete Schrördinger problem and orthogonal polynomials on the real line.

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REFERENCES

[1] R. P. AGARWAL, *Difference equations and inequalities, theory, methods and applications*, Second Edition, Revised and Expanded, Marcel Dekker, New York, 2000, pp. 998.
[2] S. AGMON, *Lectures on exponential decay*, Princeton Univ. Press, Mathematical Notes 29. Princeton, 1982.
[3] Z. BENZAID AND D. A. LUTZ, *Asymptotic representation of solutions of perturbed systems of linear difference equations*, Stud. Appl. Math. **77** (1987) 195–221.
[4] G. D. BIRKHOFF, *General theory of linear difference equations*, Trans. Amer. Math. Soc. **12** (1910) 243–284.
[5] P. A. BRAUN, *WKB method for three-term recursion relations and quasienergies of an anharmonic oscillator*, Teoret. Mat. Fiz. **37** (1978) 355–370, translated as Theoret. and Math. Phys. **37** (1978) 1070–1081.
[6] N. A. CHERNYAVSKAYA, *Stability and absolute stability of a three-point difference scheme*, Computers and Math. Appl. **45** (2003) 1181–1194.
[7] N. A. CHERNYAVSKAYA AND L. SHUSTER, *Necessary and sufficient conditions for solvability of the Hartman–Wintner problem for difference equations*, J. Difference Eqns. Appl. **14** (2008) 1215–1254.
[8] C. V. COFFMAN, *Asymptotic Behavior of Solutions of Ordinary Difference Equations*, Trans. Amer. Math. Soc. **110** (1964) 22–51.
[9] E. B. DAVIES AND E. M. HARRELL II, *Conformally flat Riemannian metrics, Schrödinger operators, and semiclassical approximation*, J. Diff. Eqs. **66** (1987) 165–188.

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[10] M. S. P. EASTHAM, *The asymptotic solution of linear differential systems. Applications of the Levinson Theorem*, Oxford: Clarendon Press, 1989.

[11] J. S. GERONIMO AND D. T. SMITH, *WKB (Liouville-Green) Analysis of Second Order Difference Equations and Applications*, J. Approx. Theory **69** (1992) 269–301.

[12] J. S. GERONIMO AND W. VAN ASSCHE, *Relative asymptotics for orthogonal polynomials with unbounded recurrence coefficients*, J. Approx. Theory **62** (1990) 47–69.

[13] F. GESZTESY AND B. SIMON, *The Xi Function*, Acta Math. **176** (1994) 49–71.

[14] E. M. HARRELL II AND M. L. WONG, *On a transformation of Bohl and its discrete analogue*, to appear in Proc. Symp. Pure Math.

[15] E. M. HARRELL II AND B. SIMON, *The Mathematical Theory of Resonances whose Widths are Exponentially Small*, Duke Math. J. **47** (1980) 845–902.

[16] P. HARTMAN, *Ordinary Differential Equations*, Hartman, Baltimore, 1973.

[17] P. HISLOP AND I. M. SIGAL, *Introduction to spectral theory, with applications to Schrödinger operators*, Springer Applied Mathematical Sciences 113. New York: Springer-Verlag, 1996.

[18] M. G. KREIN, *On certain new studies in the perturbation theory for self-adjoint operators*, in M. G. Krein, Topics in differential and integral equations and operator theory, I. Gohberg, ed. Birkhäuser, Basel, 1983, 107–172.

[19] N. LEVINSON, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. **15** (1948) 111–126.

[20] A. MÁTÉ, P. NEVAI, V. TOTIK, *Asymptotics for orthogonal polynomials defined on a recurrence relation*, Constr. Approx. **1** (1985) 231–248.

[21] F. W. J. OLVER, *Asymptotics and Special Functions*, New York: Academic, 1974.

[22] O. PERRON, *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen*, J. reine angew. Math. (Crelles Journal) **161** (1929) 41–64;

[23] H. POINCARÉ, *Sur les équations linéaires aux differentielles ordinaires et aux differences finies*, Amer. J. Math. **7** (1885) 203–258.

[24] B. SIMON, *Orthogonal polynomials on the unit circle. Part 1: Classical Theory*, AMS Colloquium Series, Amer. Math. Soc., Providence, RI, 2005.

[25] B. SIMON, *The Christoffel-Darboux kernel*, Proc. Sympos. Pure Math., **79**, Amer. Math. Soc., Providence, RI, 2008.

[26] D. T. SMITH, *Exponential Decay of Resolvents of Banded Infinite Matrices and Asymptotics of Solutions of Linear Difference Equations*, Georgia Institute of Technology, March, 1990.

[27] W. F. TRENCH, *Linear perturbations of a nonoscillatory second order difference equation*, J. Math. Analysis Appl. **273** (2002), 548–557.

[28] W. F. TRENCH, *Linear perturbations of a nonoscillatory second order difference equation II*, J. Math. Analysis Appl. **255** (2001), 627–635.

[29] M.-W. L. WONG, *Generalized bounded variation and inserting point masses*, Const. Approx. **30** (2009), 1–15.

[30] M.-W. L. WONG, *Asymptotics of orthogonal polynomials and point perturbation on the unit circle*, J. Approx. Theory **162** (2010), 1294–1321.

[31] M.-W. L. WONG, *Asymptotic analysis of orthogonal polynomials via the transfer matrix approach*, preprint, http://arxiv.org/abs/1009.1639v2

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