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FACTORIZATION OF THE CHARACTERISTIC FUNCTION OF A JACOBI MATRIX

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Abstract. In a recent paper, a class of infinite Jacobi matrices with discrete character of spectra has been introduced. With each Jacobi matrix from this class an analytic function is associated, called the characteristic function, whose zero set coincides with the point spectrum of the corresponding Jacobi operator. Here it is shown that the characteristic function admits Hadamard's factorization in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant. As an intermediate result, a formula for the logarithm of the characteristic function is obtained which is then used to handle the spectral zeta function of the Jacobi matrix. In a number of examples the characteristic function coincides with a special function, and hence to those special functions these general results can be directly applied.

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