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ORTHONORMAL SEQUENCES AND TIME FREQUENCY LOCALIZATION RELATED TO THE RIEMANN–LIOUVILLE OPERATOR

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Abstract. For every real number $p>0$, we define the p-dispersion $\rho_{p,\nu_\alpha}(f)$ of a measurable function f on $[0,+\infty[\times\mathbb{R}$, where ν_α is some positive measure. We prove that for every orthonormal basis $(\varphi_{m,n})_{(m,n)\in\,\mathbb{N}^2}$ of $L^2(d\nu_\alpha)$, the sequences $\Big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\Big)_{(m,n)\in\,\mathbb{N}^2}$, $\Big(\rho_{p,\nu_\alpha}\big(\widetilde{\mathscr{F}}_\alpha(\varphi_{m,n})\big)\Big)_{(m,n)\in\,\mathbb{N}^2}$ can not be simultaneously bounded, where $\widetilde{\mathscr{F}}_\alpha$ is some Fourier transform. The main tool is a time frequency localization inequality for orthonormal sequences in $L^2(d\nu_\alpha)$.

On the other hand, we construct an orthonormal sequence $(\psi_{m,n})_{(m,n)\in\,\mathbb{N}^2}\subset L^2(d\nu_\alpha)$ such that the sequence $\Big(\rho_{p,\nu_\alpha}(\psi_{m,n})\rho_{p,\nu_\alpha}\big(\widetilde{\mathscr{F}}_\alpha(\psi_{m,n})\big)\Big)_{(m,n)\in\,\mathbb{N}^2}$ is bounded.

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