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A RIEMANNIAN STRUCTURE FOR CORRELATION MATRICES

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Abstract. In this paper we present a new approach to viewing the set of non-degenerate correlation matrices $Corr(n)$ as a manifold and provide an optimization procedure using its newfound Riemannian structure. First we give a proof that $Corr(n)$ is a quotient submanifold of the symmetric positive-definite matrices $SPD(n)$ obtained via a Lie group action of positive diagonal matrices $Diag_{+}(n)$. With this structure $Corr(n)$ naturally inherits a Riemannian metric from $SPD(n)$ and therefore enables us to develop a Riemannian-based Newton's method on $Corr(n)$. We subsequently compare this Newton method to other optimization methods on $Corr(n)$.

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