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GREEN'S FUNCTION OF THE PROBLEM OF BOUNDED SOLUTIONS IN THE CASE OF A BLOCK TRIANGULAR COEFFICIENT

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Abstract. It is well known that the equation $x'(t) = Ax(t) + f(t)$, $t \in \mathbb{R}$, where A is a bounded linear operator, has a unique bounded solution x for any bounded continuous free term f, provided the spectrum of the coefficient A does not intersect the imaginary axis. This solution can be represented in the form

$$x(t) = \int_{-\infty}^{\infty} \mathscr{G}(s) f(t-s)\,ds.$$

The kernel $\mathscr{G}$ is called Green's function. In this paper, the case when A admits a representation by a block triangular operator matrix is considered. It is shown that the blocks of $\mathscr{G}$ are sums of special convolutions of Green's functions of the diagonal blocks of A.

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