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THE MATRIX TODA EQUATIONS FOR COEFFICIENTS OF A MATRIX THREE–TERM RECURRENCE RELATION

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Abstract. For $q \times q$ positive measures of the form $e^{-xt}\sigma(dx)$ on $[0,\infty)$ with respect to x and $t \geqslant 0$, we derive the matrix Toda equations for the three-term recurrence relation coefficients of the corresponding orthogonal matrix polynomials. Additionally, relations for the matrix version of the Volterra lattice and associated orthogonal polynomials are attained.

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