

ON COUPLINGS OF SYMMETRIC OPERATORS WITH POSSIBLY UNEQUAL AND INFINITE DEFICIENCY INDICES

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Abstract. In the paper the known results on couplings of symmetric operators A_j , $j \in \{1, 2\}$, in the sense of A.V. Shtraus are extended to the case of operators A_j with arbitrary (possibly unequal and infinite) deficiency indices. In particular, we generalize to this case the coupling method based on the theory of boundary triplets for symmetric operators. This enables us to obtain the abstract Titchmarsh formula, which gives the representation of the Weyl function of the coupling in terms of Weyl functions of boundary triplets for A_1^* and A_2^* . In applications to differential operators on \mathbb{R} this formula turns into the classical Titchmarsh formula, which gives a representation of the characteristic matrix $\Omega(\cdot)$ in terms of Titchmarsh-Weyl functions on semiaxes \mathbb{R}_+ and \mathbb{R}_- . Moreover, by using the coupling method we parameterize all Naimark exit space extensions $\tilde{A} = \tilde{A}^*$ of the second kind of a densely defined symmetric operator A with finite possibly unequal deficiency indices.

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