Operators and Matrices
Volume 14, Number 2 (2020), 417–435
doi:10.7153/oam-2020-14-30

INEQUALITIES FOR WEIGHTED GEOMETRIC MEAN IN HERMITIAN UNITAL BANACH $*$–ALGEBRAS VIA A RESULT OF CARTWRIGHT AND FIELD

S. S. DRAGOMIR

Abstract. Consider the *quadratic weighted geometric mean*

$$x \circledS_\nu y := \left| \left| yx^{-1} \right|^\nu x \right|^2$$

for invertible elements $x,\ y$ in a *Hermitian unital Banach $*$-algebra* and real number ν. In this paper, by utilizing a result of Cartwright and Field, we obtain various upper and lower bounds for the positive difference

$$(1-\nu)\,|x|^2 + \nu\,|y|^2 - x \circledS_\nu y,$$

where $\nu \in [0,1]$, under various assumptions for the elements involved. Applications for the classical *weighted geometric mean*

$$a \sharp_\nu b := a^{1/2} \left(a^{-1/2} b a^{-1/2} \right)^\upsilon a^{1/2}$$

of positive elements $a,\ b$ that satisfy the condition $0 < ka \leqslant b \leqslant Ka$ for certain numbers $0 < k < K,$ are also given.

Mathematics subject classification (2010): 47A63, 47A30, 15A60, 26D15, 26D10.
Keywords and phrases: Weighted geometric mean, weighted harmonic mean, Young's inequality, operator modulus, arithmetic mean-geometric mean inequality, Hermitian unital Banach $*$-algebra.

REFERENCES

[1] H. ALZER, C. M. DA FONSECA AND A. KOVAČEC, *Young-type inequalities and their matrix analogues*, Linear and Multilinear Algebra, **63**(2015), Issue 3, 622–635.
[2] F. F. BONSALL AND J. DUNCAN, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.
[3] D. I. CARTWRIGHT, M. J. FIELD, *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Amer. Math. Soc., **71** (1978), 36–38.
[4] J. B. CONWAY, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.
[5] S. S. DRAGOMIR, *Bounds for the normalized Jensen functional*, Bull. Austral. Math. Soc. **74**(3)(2006), 417–478.
[6] S. S. DRAGOMIR, *Quadratic weighted geometric mean in Hermitian unital Banach $*$-algebras*, Oper. Matrices **12** (2018), no. 4, 1009–1026.
[7] B. Q. FENG, *The geometric means in Banach $*$-algebra*, J. Operator Theory **57** (2007), No. 2, 243–250.
[8] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.
[9] F. KITTANEH, M. KRNIĆ, N. LOVRIČEVIĆ AND J. PEČARIĆ, *Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators*, Publ. Math. Debrecen **80** (2012), no. 3-4, 465–478.
[10] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl. **361** (2010), 262–269.

Paper OaM-14-30

[11] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra, **59** (2011), 1031–1037.
[12] G. J. MURPHY, *C^* -Algebras and Operator Theory*, Academic Press, 1990.
[13] T. OKAYASU, *The Löwner-Heinz inequality in Banach $*$-algebra*, Glasgow Math. J. **42** (2000), 243–246.
[14] S. SHIRALI AND J. W. M. FORD, *Symmetry in complex involutory Banach algebras, II*, Duke Math. J. **37** (1970), 275–280.
[15] K. TANAHASHI AND A. UCHIYAMA, *The Furuta inequality in Banach $*$-algebras*, Proc. Amer. Math. Soc. 128 (2000), 1691–1695.

Operators and Matrices
www.ele-math.com
oam@ele-math.com