

BOUNDARY INTEGRAL FORMULATIONS OF EIGENVALUE PROBLEMS FOR ELLIPTIC DIFFERENTIAL OPERATORS WITH SINGULAR INTERACTIONS AND THEIR NUMERICAL APPROXIMATION BY BOUNDARY ELEMENT METHODS

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Abstract. In this paper the discrete eigenvalues of elliptic second order differential operators in $L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, with singular δ - and δ' -interactions supported on hypersurfaces are studied. We show the self-adjointness of these operators and derive equivalent formulations for the eigenvalue problems involving boundary integral operators. These formulations are suitable for the numerical computations of the discrete eigenvalues and the corresponding eigenfunctions by boundary element methods. We provide convergence results and show numerical examples.

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