

## CYLINDRICAL HARDY TYPE INEQUALITIES WITH BESSEL PAIRS

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**Abstract.** Using factorizations of suitable partial differential operators and the notion of Bessel pairs, we establish several cylindrical Hardy's type identities and inequalities in the sense of Badiale-Tarantello [2].

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