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THE ARVESON BOUNDARY OF A FREE QUADRILATERAL IS GIVEN BY A NONCOMMUTATIVE VARIETY

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Abstract. Let $SM_n(\mathbb{R})^g$ denote the set of g-tuples of $n\times n$ real symmetric matrices and set $SM(\mathbb{R})^g = \cup_n SM_n(\mathbb{R})^g$. A free quadrilateral is the collection of tuples $X \in SM(\mathbb{R})^2$ which have positive semidefinite evaluation on the linear equations defining a classical quadrilateral. Such a set is closed under a generalized type of convex combination called a matrix convex combination. That is, given elements $X = (X_1,\dots,X_g) \in SM_{n_1}(\mathbb{R})^g$ and $Y = (Y_1,\dots,Y_g) \in SM_{n_2}(\mathbb{R})^g$ of a free quadrilateral $\mathscr{Q}$, one has

$$V_1^T X V_1 + V_2^T Y V_2 \in \mathscr{Q}$$

for any contractions $V_1 : \mathbb{R}^n \to \mathbb{R}^{n_1}$ and $V_2 : \mathbb{R}^n \to \mathbb{R}^{n_2}$ satisfying $V_1^T V_1 + V_2^T V_2 = I_n$. These matrix convex combinations are a natural analogue of convex combinations in the dimension free setting.

Elements of a free quadrilateral which cannot be expressed as a nontrivial matrix convex combination of other elements of the free quadrilateral are called free extreme points. Free extreme points serve as the minimal set which recovers a free quadrilateral through matrix convex combinations. In this way, free extreme points are the natural type of extreme point for a free quadrilateral.

In this article we show that the set of free extreme points of a free quadrilateral is determined by the zero set of a collection of noncommutative polynomials. More precisely, given a free quadrilateral $\mathscr{Q}$, we construct noncommutative polynomials p_1, p_2, p_3, p_4 such that a tuple $X \in SM(\mathbb{R})^2$ is a free extreme point of $\mathscr{Q}$ if and only if $X \in \mathscr{Q}$ and $p_i(X) = 0$ for $i = 1,2,3,4$ and X is irreducible.

In addition, we establish several basic results for projective maps of free spectrahedra and for homogeneous free spectrahedra. In particular, we show that that the image of a free extreme point under an invertible projective map is again a free extreme point. We also extend a kernel condition for a tuple to be a free extreme point to the setting of homogeneous free spectrahedra.

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