

RESTRICTED INVERTIBILITY OF CONTINUOUS MATRIX FUNCTIONS

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Abstract. Motivated by an influential result of Bourgain and Tzafriri, we consider continuous matrix functions $A : \mathbb{R} \rightarrow M_{n \times n}$ and lower ℓ_2 -norm bounds associated with their restriction to certain subspaces. We prove that for any such A with unit-length columns, there exists a continuous choice of subspaces $t \mapsto U(t) \subset \mathbb{R}^n$ such that for $v \in U(t)$, $\|A(t)v\| \geq c\|v\|$ where c is some universal constant. We provide two methods. The first relies on an orthogonality argument and it yields an optimal asymptotic dependence for $\dim(U(t))$ on n and $\sup_{t \in \mathbb{R}} \|A(t)\|$ but it does not preserve any structure for $U(t)$. The second is probabilistic and combinatorial in nature and it does not yield the optimal bound for $\dim(U(t))$ but the $U(t)$ obtained in this way are guaranteed to have a canonical representation as joined-together spaces spanned by subsets of the unit vector basis.

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