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FINITE SECTION METHOD FOR APERIODIC SCHRÖDINGER OPERATORS

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Abstract. We consider 1D discrete Schrödinger operators with aperiodic potentials given by a Sturmian word, which is a natural generalisation of the Fibonacci Hamiltonian. Via a standard approximation by periodic potentials, we establish Hausdorff convergence of the corresponding spectra for the Schrödinger operators on the axis as well as for their compressions to the half-axis.

Based on the half-axis results, we study the finite section method, which is another operator approximation, now by compressions to finite but growing intervals, that is often used to solve operator equations approximately. We find that, also for this purpose, the aperiodic case can be studied via its periodic approximants. Our results on the finite section method of the aperiodic operator are illustrated by confirming a result on the finite sections of the special case of the Fibonacci Hamiltonian.

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