Operators
and
Matrices
Volume 17, Number 2 (2023), 525–538
doi:10.7153/oam-2023-17-34

ON THE MATRIX CAUCHY–SCHWARZ INEQUALITY

MOHAMMAD SABABHEH*, CRISTIAN CONDE AND HAMID REZA MORADI

Abstract. The main goal of this work is to present new matrix inequalities of Cauchy-Schwarz type. In particular, we investigate the so-called Lieb functions, whose definition came as an umbrella of Cauchy-Schwarz-like inequalities, then we consider the mixed Cauchy-Schwarz inequality. This latter inequality has been influential in obtaining several other matrix inequalities, including numerical radius and norm results. Among many other results, we show that

$$\|T\| \leqslant \frac{1}{4}\left(\||T|+|T^*|+2\Re T\|+\||T|+|T^*|-2\Re T\|\right),$$

where $\Re T$ is the real part of the matrix T.

Mathematics subject classification (2020): Primary 47A63; Secondary 15A60, 46L05.
Keywords and phrases: Lieb functions, operator inequality, Cauchy-Schwarz inequality.

REFERENCES

[1] M. ALAKHRASS AND M. SABABHEH, *Lieb functions and sectorial matrices*, Linear Algebra Appl. 58 (2020), 308–324.
[2] T. ANDO, *Geometric mean and norm Schwarz inequality*, Ann. Funct. Anal. 7(1) (2016), 1–8.
[3] T. ANDO AND F. HIAI, *Operator log-convex functions and operator means*, Math. Ann. 350 (2011), 611–630.
[4] T. ANDO AND X. ZHAN, *Norm inequalities related to operator monotone functions*, Math. Ann. 315 (1999), 771–780.
[5] R. BHATIA, *Matrix analysis*, Springer, New York, 1997
[6] R. BHATIA, *Positive definite matrices*, Princeton Univ. Press, Princeton, 2007.
[7] J.-C BOURIN AND E.-Y LEE, *On the Russo-Dye theorem for positive linear maps*, Linear Algebra Appl. 571 (2019), 92–102.
[8] J.-C. BOURIN AND E.-Y. LEE, *Unitary orbits of Hermitian operators with convex or concave functions*, Bull. Lond. Math. Soc. 44 (2012), 1085–1102.
[9] J.-C. BOURIN, E.-Y. LEE AND M. LIN, *On a decomposition lemma for positive semi-definite block-matrices*, Linear Algebra Appl. 437 (2012), 1906–1912.
[10] J. C. BOURIN AND M. UCHIYAMA, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl. 423 (2007), 512–518.
[11] S. DRAGOMIR, *Inequalities for weighted geometric mean in Hermitian unital Banach $*$-algebras via a result of Cartwright and Field*, Oper. Matrices 14 (2) (2020), 417–435.
[12] S. DRURY, *Principal powers of matrices with positive definite real part*, Linear Multilinear Algebra 63 (2015), 296–301.
[13] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.
[14] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ, AND Y. SEO, *Mond-Pečarić method in operator inequalities*, Element, Zagreb, 2005.
[15] I. H. GÜMÜŞ, H. R. MORADI, AND M. SABABHEH, *More accurate operator means inequalities*, J. Math. Anal. Appl. 465(1) (2018), 267–280.
[16] F. HIAI, *A Matrix analysis: matrix monotone functions, matrix means, and majorization* (GSIS selected lectures), Interdiscip. Inform. Sci. 16 (2010) 139–248.
[17] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. 24 (1988), 283–293.

Paper OaM-17-34

[18] E. H. LIEB, *Inequalities for some operator matrix functions*, Adv. Math. 20 (1976), 174–178.
[19] S. LIN AND X. FU, *Inequalities for the $\lambda-$weighted mixed arithmetic-geometric-harmonic means of sector matrices*, Oper. Matrices 14 (2) (2020), 447–454.
[20] J. LIU, J.-J. MEI AND D. ZHANG, *Inequalities related to the geometric mean of accretive matrices*, Oper. Matrices 15 (2) (2021), 581–587.
[21] E. SEILER AND B. SIMON, *An inequality among determinants*, Proc. Nat. Acad. Sci. 72 (1975), 3277–3278.
[22] B. SIMON, *Trace ideals and their applications*, 2nd ed., Amer. Math. Soc. Providence, RI, 2005.
[23] A. TAGHAVI, V. DARVISH, H. M. NAZARI, AND S. S. DRAGOMIR, *Hermite-Hadamard type inequalities for operator geometrically convex functions*, Monatsh. Math. 181 (2016), 187–203.