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$\mathsf{D}^2 = \mathsf{H} + \frac{1}{4}$ WITH POINT INTERACTIONS

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Abstract. Let D and H be the self-adjoint, one-dimensional Dirac and Schrödinger operators in $L^2(\mathbb{R};\mathbb{C}^2)$ and $L^2(\mathbb{R};\mathbb{C})$ respectively. It is well known that, in absence of an external potential, the two operators are related through the equality $\mathsf{D}^2 = (\mathsf{H} + \frac{1}{4})\mathbb{1}$. We show that such a kind of relation also holds in the case of n-point singular perturbations: given any self-adjoint realization $\widehat{\mathsf{D}}$ of the formal sum $\mathsf{D} + \sum_{k=1}^{n} \gamma_k \delta_{y_k}$, we explicitly determine the self-adjoint realization $\widehat{\mathsf{H}}$ of $\mathsf{H}\mathbb{1} + \sum_{k=1}^{n}(\alpha_k \delta_{y_k} + \beta_k \delta'_{y_k})$ such that $\widehat{\mathsf{D}}^2 = \widehat{\mathsf{H}} + \frac{\mathbb{1}}{4}$. The found correspondence preserves the subclasses of self-adjoint realizations corresponding to both the local and the separating boundary conditions. Some connections with supersymmetry are provided. The case of nonlocal boundary conditions allows the study of the relation $\mathsf{D}^2 = \mathsf{H} + \frac{1}{4}$ for quantum graphs with (at most) two ends; in particular, the square of the extension corresponding to Kirchhoff-type boundary conditions for the Dirac operator on the graph gives the direct sum of two Schrödinger operators on the same graph, one with the usual Kirchhoff boundary conditions and the other with a sort of reversed Kirchhoff ones.

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