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ON NUMBER THEORETIC PROPERTIES OF THE KDV FREQUENCIES

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Abstract. In this paper we investigate some number theoretic properties of the frequencies of the Korteweg–de Vries equation on the torus, relevant for the stability of finite gap solutions.

Mathematics subject classification (2020): 37K10, 35Q53, 37K45, 11D25, 11D41, 11G30, 14G05.
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