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POSITIVE ANSWER TO THE HYPERINVARIANT SUBSPACE PROBLEM FOR M–HYPONORMAL OPERATORS

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Abstract. The question whether every operator on infinite dimensional Hilbert space H has a non-trivial invariant subspace or a non-trivial hyperinvariant subspace is one of the most difficult problem in operator theory. This problem is open for more than half a century. A subnormal operator has a non-trivial invariant subspace, but the existence of non-trivial invariant subspace for a hyponormal operator T still open. In this paper, we present a review of the history of the problem and we give an affirmative answer of the existence of a non-trivial hyperinvariant subspace for a m-hyponormal operator S such that $S = T + N$, where T is m-hyponormal and N is normal such that $TN = NT$.

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REFERENCES

[1] P. AIENA, M. BIONDI, *Weyl's and Browder's theorems through the quasi-nilpotent part of an operator*, in J. Castillo, W. Johnson (Eds.): Methods in Banach Space Theory (London Mathematical Society Lecture Note Series, pp. 239–252), Cambridge: Cambridge University Press, 2006, doi:10.1017/CBO9780511721366.013.
[2] E. ALBRECHT, J. ESCHMEIER AND M. M. NEUMANN, *Some topics in the theory of decomposable operators*, in Advances in invariant subspaces and other results of Operator Theory: Advances and Applications **17**, Birkhauser Verlag, Basel, 1986.
[3] C. AMBROZIE, V. MULLER, *Invariant subspaces for polynomially bounded operators*, J. Funct. Anal., 2004, **213**, 321–345.
[4] R. ANISCA, V. G. TROITSKY, *Minimal vectors of positive operators*, Indiana Univ. Math. J., 2005, **54**, 861–872.
[5] S. ANSARI, P. ENFLO, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc., 1998, **350**, 539–558.
[6] S. A. ARGYROS, R. G. HAYDON, *A hereditarily indecomposable $L1$-space that solves the scalar-plus-compact problem*, Acta Math., 2011, **206**, 1–54.
[7] N. ARONSZAJN, K. T. SMITH, *Invariant subspaces of completely continuous operators*, Ann. of Math. (2), 1954, **60**, 345–350.
[8] A. ATZMON, *An operator without invariant subspace on a nuclear Fréchet space*, Ann. of Math. (2), 1983, **117**, 669–694.
[9] B. BEAUZAMY, *Un opérateur sans sous-espace invariant: simplification de l'exemple de P. Enflo*, Integral Equations Operator Theory, 1985, **8**, 314–384.
[10] H. BERCOVICI, *Notes on invariant subspaces*, Bull. Amer. Math. Soc. (N.S.) **23** (1990) 1–33.
[11] H. BERCOVICI, C. FOIAS, C. PEARCY, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional conference series in mathematics **56**, A.M.S., Providence, 1985.
[12] H. BERCOVICI, C. FOIAS, C. PEARCY, *Two Banach space methods and dual operator algebras*, J. Funct. Anal., 1988, **78**, 306–345.
[13] A. R. BERNSTEIN, A. ROBINSON, *Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos*, Pacific J. Math., 1966, **16**, 421–431.

[14] R. BHATIA AND R. ROSENTHAL, *How and why to solve the operator equation* $AX - XB = Y$, Bull. London Math. Soc. **29** (1997), 1–21.
[15] E. BISHOP, *A duality theorem for an arbitrary operator*, Pacific J. Math. **9**: 379–424, 1959.
[16] S. W. BROWN, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory, 1978, **1**, 310–333.
[17] S. W. BROWN, B. CHEVREAU, C. PEARCY, *On the structure of contraction operators II*, J. Funct. Anal., 1988, **76**, 30–55.
[18] I. CHALENDAR, J. R. PARTINGTON, *An overview of some recent developments on the Invariant Subspace Problem*, Review Article, doi:10.2478/conop-2012-0001, CO. 2013, 1–10.
[19] I. CHALENDAR, J. R. PARTINGTON, *Convergence properties of minimal vectors for normal operators and weighted shifts*, Proc. Amer. Math. Soc., 2005, **133**, 501–510.
[20] I. CHALENDAR, J. R. PARTINGTON, *Modern approaches to the invariant-subspace problem*, Cambridge Tracts in Mathematics **188**, Cambridge University Press, Cambridge, 2011.
[21] I. CHALENDAR, J. R. PARTINGTON, *Variations on Lomonosov's theorem via the technique of minimal vectors*, Acta Sci. Math. (Szeged), 2005, **71**, 603–617.
[22] I. COLOJOARA AND C. FOIAS, *Theory of Generallized spectral Operators*, Mathematics and its Applications, vol. 9, Gordon and Breach, Science Publishers, New York-London-Paris, 1968.
[23] H. R. DOWSOW, *Spectral theory of linear operators*, Academic Press, London 1973.
[24] N. DUNFORD, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc. **64**: 217–274, 1958.
[25] N. DUNFORD, *Spectral operators*, Pacific J. Math. **4** (1954) 321–354.
[26] P. ENFLO, *Extremal vectors for a class of linear operators*, Functional analysis and economic theory (Samos, 1996), 61–64, Springer, Berlin, 1998.
[27] P. ENFLO, *On the invariant subspace problem for Banach spaces*, Acta Math., 1987, **158**, 213–313.
[28] P. ENFLO, *On the invariant subspace problem in Banach spaces*, Seminaire Maurey-Schwartz (1975–1976) Espaces Lp, applications radonifiantes et geometrie des espaces de Banach, Exp. Nos. 14–15, 7 pp., Centre Math., Ecole Polytech., Palaiseau, 1976.
[29] P. ENFLO, T. HOIM, *Some results on extremal vectors and invariant subspaces*, Proc. Amer. Math. Soc., 2003, **131**, 379–387.
[30] J. ESCHMEIER, *Invariant subspaces for subscalar operators*, Arch. Math. (Basel). **52** (1989), 562–570.
[31] J. ESCHMEIER, K. B. LAURSEN AND M. M. NEUMANN, *Multipliers with natural local spectra on commutative Banach algebras*, J. Funct. Anal. **138**: 273–294, 1996.
[32] J. ESCHMEIER AND M. PUTINAR, *Spectral Decompositions and Analytic Sheaves*, London Mathematical Society Monographs, no. 10, Clarendon Press, Oxford, 1996.
[33] S. R. FOGUEL, *Normal operators of finite multiplicity*, Comm. Pure Appl. Math. **11** (1958) 297–313.
[34] C.-K. FONG, *On* M*-hyponormal operators*, Studia Math, LXV (1979), 1–5.
[35] W. T. GOWERS, B. MAUREY, *Banach spaces with small spaces of operators*, Math. Ann., 1997, **307**, 543–568.
[36] J. W. HELTON, *Operators with a representation as multiplication by* x *on a Sobolev space*, Colloquia Math. Soc., Janos Bolyai 5, Hilbert space operators, Thany, Hungary 279–287, 1970.
[37] D. A. HERRERO, *Approximation of Hilbert space operators*, Pitnam Res. Notes Math. Ser., vol. 224, Longman Scientific and Technical, Harlow, 1989.
[38] H. J. KIM, *Hyperinvariant subspaces for operators having a normal part*, Oper. Matrices, 2011, **5**, 487–494.
[39] I. H. KIM AND W. Y. LEE, *The spectrum is continuous on the set of quasi-*n*-hyponormal operators*, J. Math. Anal. Appl. **335** (2007), 260–267.
[40] H. W. KIM, C. H. PEARCY, *Subnormal operators and hyperinvariant spaces*, Illinois J. Math. **23** (1979), 459–463.
[41] E. KO, *On Hypo-Jordan operators*, Glasg. Math. J. **43**: 411–418, 2001.
[42] R. LANGE, *Duality and asymptotic spectral decompositions*, Pacific J. Math. **121** (1986), 93–108.
[43] K. B. LAURSEN AND M. M. NEUMANN, *An introduction to Local Spectral Theory*, London Mathematical society Monographs, Oxford 2000.
[44] K. B. LAURSEN AND M. M. NEUMANN, *Automatic continuity of intertwining linear operators on Banach spaces*, Rend. Circ. Mat. Palermo II, **40**: 325–341, 1991.

[45] K. B. LAURSEN AND M. M. NEUMANN, *Decomposable operators, and automatic continuity*, J. Operator Theory **15** (1986), 33–51.
[46] V. I. LOMONOSOV, *Invariant subspaces for operators commuting with compact operators*, Funct. Anal. Appl., 1973, **7**, 213–214.
[47] S. MECHERI, *Bishop property (β) and Riesz idempotent for k-quasi-paranormal operators*, Banach J. Math. Anal. **6** (2012), 147–154.
[48] S. MECHERI, *Bishops property (β), hypercyclicity and hyperinvariant subspaces*, Operators and Matrices **8** (2014), 555–562.
[49] S. MECHERI, *Bishop's property, SVEP and Dunford property (C)*, Elect. Jour. Lin. Alg. **23** (2012), 523–529.
[50] S. MECHERI, *Isolated points of spectrum of k-quasi-*-class A operators*, Studia Math. **208** (2012), 87–96.
[51] S. MECHERI, *On a new class of operators and Weyl type theorems*, Filomat. **27** (2013), 629–636.
[52] S. MECHERI, *Subscalarity, Invariant, and Hyperinvariant Subspaces for Upper Triangular Operator Matrices*, Bull. Malays. Math. Sci. Soc., doi:10.1007/s40840-016-0377-4.
[53] S. Z. NAGY, C. FOIAS, *Harmonic Analysis of operators in Hilbert spaces*, North Holland, Amsterdam, 1970.
[54] M. OUDGHIRI, *Weyl's and Browder's theorem for operators satisfying the SVEP*, Studia Mathematica **163**, 1, (2004), 85–101.
[55] C. M. PEARCY, *Some recent developments in operator theory*, Conference board of the mathematical sciences regional conference series in mathematics **36** (1975).
[56] G. PISIER, *A polynomially bounded operator on Hilbert space which is not similar to a contraction*, J. Amer. Math. Soc., 1997, **10**, 351–369.
[57] M. PUTINAR, *Hyponormal operators are subscalar*, J. Operator Theory **12** (1984), no. 2, 385–395.
[58] H. RADJAVI AND P. ROSENTHAL, *Hyperinvariant subspaces for spectral and n-normal operators*, Acta Sci. Math. (Szeged) **32** (1971) 121–126.
[59] H. RADJAVI AND P. ROSENTHAL, *Invariant subspaces*, Springer, Heidelberg, 1973.
[60] C. READ, *A short proof concerning the invariant subspace problem*, J. London Math. Soc. (2), 1986, **34**, 335–348.
[61] C. READ, *A solution to the invariant subspace problem*, Bull. London Math. Soc., 1984, **16**, 337–401.
[62] C. READ, *A solution to the invariant subspace problem on the space 1*, Bull. London Math. Soc., 1985, **17**, 305–317.
[63] C. READ, *Quasinilpotent operators and the invariant subspace problem*, J. London Math. Soc. (2), 1997, **56**, 595–606.
[64] C. READ, *Strictly singular operators and the invariant subspace problem*, Studia Math., 1999, **132**, 203–226.
[65] SHANLI SUN, *The sum and product of decomposable operators* (in Chinese), Northeast. Math. J. **5** (1989), 105–117.
[66] J. E. THOMSON, *Invariant subspaces for algebras of subnormal operators*, Proc. Amer. Math. Soc., 1986, **96**, 462–464.
[67] V. G. TROITSKY, *Lomonosov's theorem cannot be extended to chains of four operators*, Proc. Amer. Math. Soc., 2000, **128**, 521–525.
[68] V. G. TROITSKY, *Minimal vectors in arbitrary Banach spaces*, Proc. Amer. Math. Soc., 2004, **132**, 1177–1180.

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