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CONTRACTION IN L^1 FOR A SYSTEM ARISING IN CHEMICAL REACTIONS AND MOLECULAR MOTORS

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Abstract. We prove a contraction in L^1 property for the solutions of a nonlinear reaction–diffusion system whose special cases include a system related to intracellular transport as well as reversible chemical reactions. We then consider the special case of the linear molecular motor problem and prove the existence and uniqueness of the stationary solution up to a multiplicative constant, extending to arbitrary space dimension results which were already known in the one dimensional case; this in turn implies the convergence to stationary solutions of the solutions of the time evolution linear molecular motor problem.

1. Introduction

We start with two specific reaction-diffusion systems. The first one describes a reversible reaction and the other one a molecular motor. We first consider the reversible chemical reaction (see also Bothe [3], Bothe and Hilhorst [4], Desvillettes and Fellner [9] and Érdi and Tóth [10]). It involves a reaction-diffusion system of the form

$$\begin{aligned} u_t &= d_1\Delta u - \alpha k\big(r_A(u) - r_B(v)\big) \quad \text{in} \quad \Omega\times(0,T),\ \ \Omega\subset\mathbb{R}^d, \\ v_t &= d_2\Delta v + \beta k\big(r_A(u) - r_B(v)\big) \quad \text{in} \quad \Omega\times(0,T),\ \ \Omega\subset\mathbb{R}^d, \end{aligned} \tag{1.1}$$

together with homogeneous Neumann boundary conditions, where d_1, d_2, α, β, k and T are positive constants and where Ω is a bounded subset of $\mathbb{R}^d$ with smooth boundary. Such systems describe, with a suitable choice of the functions r_A and r_B, chemical reactions for two mobile species. For example, functions $r_A(u) = u^k$, $r_B(v) = v^m$ correspond to a reversible reaction $kA \rightleftharpoons mB$. Reactions of the type $q_1A_1 + \dots q_kA_k \rightleftharpoons q_1B_1 + \dots q_mB_m$ can also be described by similar systems with more complicated reactions terms.

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Another model problem is a system in $d = 1$ space dimension and n unknown variables $u_1, \dots, u_n$, $n > 1$, for intracellular transport, namely

$$\begin{aligned} \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x}\left(\sigma \frac{\partial u_i}{\partial x} + u_i \psi_i'\right) \\ + \sum_{j=1}^{n} a_{ij} u_j \quad \text{in} \quad Q_T = (0,1) \times (0,T) \\ \sigma \frac{\partial u_i}{\partial x} + u_i \psi_i' = 0 \quad \text{on} \quad \partial Q_T = \{0,1\} \times (0,T), \end{aligned}$$

where

$$\begin{aligned} a_{ii} \leqslant 0,\ a_{ij} \geqslant 0 \ \text{ for all } \ i \in \{1,\dots,n\}, i \neq j, \\ \sum_{i=1}^{n} a_{ij} = 0 \ \text{ for all } \ i,j \in \{1,\dots,n\}. \end{aligned} \tag{1.2}$$

It models transport via motor proteins in the eukaryotic cell where chemical energy is transduced into directed motion. A derivation of the system from a mass transport viewpoint is given in [6]. For an analysis of the steady state solutions and for further references we refer to [5], [11], [12], [15] and [16].

In this paper we study the corresponding system in higher space dimension, namely

$$\frac{\partial u_i}{\partial t} = \operatorname{div}\big(\sigma_i \nabla u_i + u_i \nabla \psi_i\big) + \alpha_i \left(\sum_{j=1}^{n} \lambda_{ij} r_j\big(u_j(x,t), x\big) \right) \quad \text{in } Q_T, \tag{1.3a}$$

where $i \in \{1,\dots,n\}$, and $u_i(x,t) : Q_T \to \mathbb{R}^+$, with $Q_T = \Omega \times (0,T)$, Ω an open bounded subset of $\mathbb{R}^d$ with smooth boundary, and T some positive constant. We supplement this system with the Robin (no-flux) boundary conditions

$$\sigma_i \frac{\partial u_i}{\partial \nu} + u_i \frac{\partial \psi_i}{\partial \nu} = 0, \quad i \in \{1,\dots,n\}, \quad \text{on} \quad \partial\Omega \times (0,T), \tag{1.3b}$$

where ν is the outward normal vector to $\partial\Omega$, and the initial conditions

$$u_1(x,0) = u_{0,1}(x),\ \dots\ , u_n(x,0) = u_{0,n}(x), \quad x \in \Omega. \tag{1.3c}$$

We assume that the following hypotheses hold

1. The constants σ_i and $\alpha_i \in \mathbb{R}$, where $i \in \{1,\dots,n\}$, are strictly positive;
2. For $i,j \in \{1,\dots,n\}$, $\lambda_{ii} \leqslant 0$, $\lambda_{ij} \geqslant 0$ if $i \neq j$, $\sum_{k=1}^{n} \lambda_{kj} = 0$;
3. for all $i \in \{1,\dots,n\}$, the smooth functions r_i are nondecreasing with respect to the first variable; $r_i(0,x) = 0$ and we assume that the functions ψ_i are smooth as well;
4. $u_i(.,0) = u_{0i} \in C(\overline{\Omega})$, $u_{0i} \geqslant 0$.

In the linear case of the molecular motor, this amount to choosing

$$r_i(s,x) = s,\ \lambda_{ij} = a_{ij} \text{ and } \alpha_i = 1 \text{ for all } i,j \in \{1,\dots,n\}. \tag{1.4}$$

We denote by Problem (P) the system (1.3a) together with the boundary and initial conditions (1.3b), (1.3c), and admit without proof that Problem (P) possesses a unique smooth and bounded solution on each time interval $(0,T]$. An essential idea for proving the existence of a solution would be to apply the Comparison principle Theorem 2.2 below to deduce that any solution of Problem (P) has to be nonnegative and bounded from above by a stationary solution.

Finally, we note that because of the boundary conditions (1.3b) the quantity

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} u_i(x,t)\,\mathrm{d}x \tag{1.5}$$

is conserved in time.

The organization of this paper is as follows. In Section 2 we prove a comparison principle for Problem (P). The main idea, which permits to show that Problem (P) is cooperative, is a change of functions which transforms the Robin boundary conditions into homogeneous Neumann boundary conditions. In Section 3 we establish a contraction in L^1 property for the corresponding semigroup solution. Let us point out the similarity with an old result due to Crandall and Tartar [7] where they proved in a scalar case that in the presence of a conservation of the integral property such as (1.5), a comparison principle such as Theorem 2.2 is equivalent to a contraction in L^1 property such as the inequality (3.4) below. As far as we know such an abstract result is not known in the case of systems. Finally we show in Section 4 the existence and uniqueness of the stationary solution of the linear molecular motor problem up to a multiplicative constant. This result holds in arbitrary space dimension whereas the proofs in [11] and [5] only hold in the one dimensional case. The convergence to the stationary solution as $t \to \infty$ of the solution of the time evolution linear molecular motor problem then follows as in [11]; their proof of convergence to the stationary solution is based on a similar Krein-Rutman idea.

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2. Comparison principle

First, we remark that the system of equations (1.3a) is cooperative. However, since nothing is known about the sign of the coefficients $\dfrac{\partial \psi_i}{\partial \nu}$ in the Robin boundary conditions (1.3b), we cannot decide whether Problem (P) is cooperative. This leads us to perform a change of variables which transforms the Robin boundary conditions into homogeneous Neumann boundary conditions.

2.1. The change of unknown functions

Performing the change of variables

$$w_i(x,t) = u_i(x,t)\,\mathrm{e}^{\psi_i(x)/\sigma_i}, \quad i \in \{1,\dots,n\}, \tag{2.1}$$

we deduce from (1.3) that $\vec{w} := (w_1,\dots,w_n)$ satisfies the parabolic system

$$\begin{aligned}\frac{\partial w_i}{\partial t} &= \sigma_i\,\mathrm{e}^{\psi_i(x)/\sigma_i}\mathrm{div}\big(\mathrm{e}^{-\psi_i(x)/\sigma_i}\nabla w_i\big)\\ &\quad + \alpha_i\,\mathrm{e}^{\psi_i(x)/\sigma_i}\Bigg(\sum_{j=1}^{n}\lambda_{ij}r_j\big(w_j(x,t)\,\mathrm{e}^{-\psi_j(x)/\sigma_j},x\big)\Bigg) \quad \text{in } Q_T,\end{aligned} \tag{2.2}$$

together with the homogeneous Neumann boundary conditions

$$\frac{\partial w_i}{\partial \nu} = 0, \quad i \in \{1,\dots,n\}, \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

and the initial conditions

$$w_i(x,0) = u_{0,i}(x)\,\mathrm{e}^{\psi_i(x)/\sigma_i}, \quad i \in \{1,\dots,n\}, \quad x \in \Omega. \tag{2.4}$$

In the following, we denote by Problem (P_N) — the problem (2.2), (2.3), (2.4). To begin with we define the operators

$$\begin{aligned}\mathscr{L}_i(w_i) &= \frac{\partial w_i}{\partial t} - \sigma_i\,\mathrm{e}^{\psi_i(x)/\sigma_i}\mathrm{div}\big(\mathrm{e}^{-\psi_i(x)/\sigma_i}\nabla w_i\big)\\ &\quad - \alpha_i\,\mathrm{e}^{\psi_i(x)/\sigma_i}\Bigg(\sum_{j=1}^{n}\lambda_{ij}r_j\big(w_j(x,t)\,\mathrm{e}^{-\psi_j(x)/\sigma_j},x\big)\Bigg) \quad \text{in } Q_T.\end{aligned} \tag{2.5}$$

We say that $(\underline{w}_1,\dots,\underline{w}_n)$ is a subsolution of Problem (P_N) if

$$\begin{aligned}&\mathscr{L}_i(\underline{w}_i) \leqslant 0 \quad \text{in} \quad Q_T,\\ &\frac{\partial \underline{w}_i}{\partial \nu} \leqslant 0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ &\underline{w}_i(x,0) \leqslant w_i(x,0), \quad x \in \Omega\end{aligned} \tag{2.6}$$

for all $i \in \{1,\dots,n\}$. We define similarly a supersolution $(\overline{u}_1,\dots,\overline{u}_n)$ of Problem (P_N) by the inequalities

$$\begin{aligned}&\mathscr{L}_i(\overline{w}_i) \geqslant 0 \quad \text{in} \quad Q_T,\\ &\frac{\partial \overline{w}_i}{\partial \nu} \geqslant 0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ &\overline{w}_i(x,0) \geqslant w_i(x,0), \quad x \in \Omega.\end{aligned} \tag{2.7}$$

The following comparison theorem holds ([1], [17]).

THEOREM 2.1. *Let* $(\underline{w}_1,\dots,\underline{w}_n)$ *and* $(\overline{w}_1,\dots,\overline{w}_n)$*, be a sub- and a super-solution, respectively, for the operators* $\mathscr{L}_j$ *defined by* (2.5) *with* $j\in\{1,\dots,n\}$*, which means that* (2.6) *and* (2.7) *hold for* $i\in\{1,\dots,n\}$*. Then* $\underline{w}_i\leqslant\overline{w}_i$ *in* Q_T*. Moreover, for all* $i\in\{1,\dots,n\}$ *such that* $\underline{w}_i\leqslant\overline{w}_i$ *and* $\underline{w}_i\not\equiv\overline{w}_i$ *on* $\{t=0\}\times\Omega$*, we have that* $\underline{w}_i<\overline{w}_i$ *in* Q_T*.* □

This comparison theorem immediately translates into a comparison theorem for solutions of the original Problem (P). For all $i\in\{1,\dots,n\}$, we define the operators

$$L_i(u_i)=(u_i)_t-\operatorname{div}\big(\sigma_i\nabla u_i+u_i\nabla\psi_i\big)-\alpha_i\left(\sum_{j=1}^{n}\lambda_{ij}\,r_j\,(u_j,x)\right)\quad\text{in}\quad Q_T. \tag{2.8}$$

The following result holds.

THEOREM 2.2. *Let* $(\underline{u}_1,\dots,\underline{u}_n)$ *and* $(\overline{u}_1,\dots,\overline{u}_n)$*, be a sub- and a super-solution, respectively, for the operators* L_j*, defined by* (2.8) *with* $j\in\{1,\dots,n\}$*. Then* $\underline{u}_i\leqslant\overline{u}_i$ *in* Q_T*. Moreover, for all* $i\in\{1,\dots,n\}$ *such that* $\underline{u}_i\leqslant\overline{u}_i$ *and* $\underline{u}_i\not\equiv\overline{u}_i$ *on* $\{t=0\}\times\Omega$ *then* $\underline{u}_i<\overline{u}_i$ *in* Q_T*.* □

Next we state two immediate corollaries of Theorem 2.2.

COROLLARY 2.3. (uniqueness) *If* $(u_1^1,\dots,u_n^1)$ *and* $(u_1^2,\dots,u_n^2)$ *are solutions of Problem (P) with the same initial condition* $(u_{0,1},\dots,u_{0,n})\in\big(C(\overline{\Omega})\big)^n$*, then for all* $i\in\{1,\dots,n\}$*,* $u_i^1=u_i^2$*.* □

COROLLARY 2.4. (positivity) *If* $(u_1,\dots,u_n)$ *is the solution of Problem (P) with the nonnegative initial condition* $(u_{0,1},\dots,u_{0,n})\in\big(C(\overline{\Omega})\big)^n$*, then for all* $i\in\{1,\dots,n\}$*,* $u_i\geqslant 0$*. Moreover, for all* $i\in\{1,\dots,n\}$*, such that* $u_{0,i}\geqslant 0$ *and* $u_{0,i}\not\equiv 0$*,* $u_i>0$ *in* Q_T*.* □

3. Contraction property

The purpose of this section is to show a contraction in $\big(L^1(\Omega)\big)^n$ property for solutions of Problem (P) with initial conditions belonging to $\big(L^\infty(\Omega)\big)^n$. The main steps of the proof rely upon arguments due to [2] and [14].

We first introduce some notation. We suppose that the functions $(u_1^1,\dots,u_n^1)$ and $(u_1^2,\dots,u_n^2)$ are the solutions of Problem (P) with the initial conditions $(u_{0,1}^1,\dots,u_{0,n}^1)$ and $(u_{0,1}^2,\dots,u_{0,n}^2)$, respectively. Define

$$(U_1,\dots,U_n):=(u_1^1-u_1^2,\dots,u_n^1-u_n^2). \tag{3.1}$$

Then

$$
\begin{aligned}
(U_i)_t &= \operatorname{div}\big(\sigma_i \nabla U_i + U_i \nabla \psi_i\big) \\
&\quad + \alpha_i \sum_{j=1}^{n} \lambda_{ij}\big(r_j(u_j^1(x,t),x) - r_j(u_j^2(x,t),x)\big) \quad \text{in} \quad Q_T, \\
\sigma_i \frac{\partial U_i}{\partial \nu} &+ U_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\
U_i(x,0) &= U_{0,i}(x) \quad \text{for} \quad x \in \Omega,
\end{aligned}
\tag{3.2}
$$

together with

$$
U_{0,i} = u_{0,i}^1 - u_{0,i}^2, \tag{3.3}
$$

for each $i \in \{1,\ldots,n\}$.

Next we prove the following contraction in L^1 property.

THEOREM 3.1. *For all $t > 0$,*

$$
\begin{aligned}
\frac{1}{\alpha_1}\|U_1(\cdot,t)\|_{L^1(\Omega)} + \ldots + \frac{1}{\alpha_n}\|U_n(\cdot,t)\|_{L^1(\Omega)} \\
\leqslant \frac{1}{\alpha_1}\|U_{0,1}(\cdot)\|_{L^1(\Omega)} + \ldots + \frac{1}{\alpha_n}\|U_{0,n}(\cdot)\|_{L^1(\Omega)},
\end{aligned}
\tag{3.4}
$$

where U_i and $U_{0,i}$, $i \in \{1,\ldots,n\}$, are defined by (3.1) *and* (3.3), *respectively.*

Proof. Dividing each partial differential equation of (3.2) by α_i and summing them up, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} \frac{1}{\alpha_i} U_i\right) &= \sum_{i=1}^{n} \frac{1}{\alpha_i} \operatorname{div}(\sigma_i \nabla U_i + U_i \nabla \psi_i) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij}\Big(r_j(u_j^1(x,t),x) - r_j(u_j^2(x,t),x)\Big) \\
&= \sum_{i=1}^{n} \frac{1}{\alpha_i} \operatorname{div}(\sigma_i \nabla U_i + U_i \nabla \psi_i) \\
&\quad + \sum_{j=1}^{n} \left\{ \Big(r_j(u_j^1(x,t),x) - r_j(u_j^2(x,t),x)\Big) \sum_{i=1}^{n} \lambda_{ij} \right\} \\
&= \sum_{i=1}^{n} \frac{1}{\alpha_i} \operatorname{div}(\sigma_i \nabla U_i + U_i \nabla \psi_i),
\end{aligned}
$$

where we have used Hypothesis 2.

This, together with the boundary conditions (1.3b), implies the conservation in time property

$$
\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} \frac{1}{\alpha_i} \int_{\Omega} U_i(x,t)\,\mathrm{d}x = 0. \tag{3.5}
$$

Let us look closer at the nonlinear term in (3.2). We can write, for fixed index i

$$\sum_{j=1}^{n}\lambda_{ij}\big(r_j(u_j^1(x,t),x)-r_j(u_j^2(x,t),x)\big)$$
$$=\sum_{j=1}^{n}\lambda_{ij}U_j\int_0^1\frac{\partial}{\partial u}r_j(\theta u_j^1+(1-\theta)u_j^2,x)\mathrm{d}\theta=\sum_{j=1}^{n}A_{ij}U_j.$$

Freezing the functions u_i^k for $i\in\{1,\dots,n\}$, $k\in\{1,2\}$, we deduce that the functions $U_1,\dots,U_n$ satisfy a system of the form

$$(U_i)_t=\operatorname{div}\Big(\sigma_i\nabla U_i+U_i\nabla\psi_i\Big)+\sum_{j=1}^{n}A_{ij}U_j\quad\text{in}\quad Q_T,\tag{3.6}$$

with the boundary and initial conditions

$$\begin{gathered}\sigma_i\frac{\partial U_i}{\partial\nu}+U_i\frac{\partial\psi_i}{\partial\nu}=0\quad\text{on}\quad\partial\Omega\times(0,T),\\ U_i(x,0)=U_{0,i}(x),\quad x\in\Omega,\end{gathered}\tag{3.7}$$

for $i\in\{1,\dots,n\}$, where A_{ij} are functions of space and time.

In order to make the notation more concise, we write

$$\begin{aligned}\vec{U}_0&=\big(U_{0,1},\dots,U_{0,n}\big),\\ \vec{U}&=\big(U_1,\dots,U_n\big),\\ \vec{U}_0^{\pm}&=\big(U_{0,1}^{\pm},\dots,U_{0,n}^{\pm}\big),\\ \vec{U}^{\pm}&=\big(U_1^{\pm},\dots,U_n^{\pm}\big),\end{aligned}$$

where $s^+=\max\{s,0\}$, $s^-=\max\{-s,0\}$. By (3.6), (3.7) and Corollary 2.3 we can write $\vec{U}$ in the form

$$\vec{U}(x,t)=\mathscr{S}(t)\,\vec{U}_0(x)=\big(\mathscr{S}_1(t)\vec{U}_0,\dots,\mathscr{S}_n(t)\vec{U}_0\big)(x).$$

We set

$$\big(W_1,\dots,W_n\big)=-\big(U_1\,\mathrm{e}^{\psi_1(x)/\sigma_1},\dots,U_n\,\mathrm{e}^{\psi_n(x)/\sigma_n}\big),$$

and $\widetilde{A}_{ij}=A_{ij}\,\mathrm{e}^{\psi_i(x)/\sigma_i}\,\mathrm{e}^{-\psi_j(x)/\sigma_j}$. Then, the system of equations (3.6) can be expressed in the form

$$\big(W_i\big)_t=\sigma_i\,\mathrm{e}^{\psi_i(x)/\sigma_i}\operatorname{div}\Big(\mathrm{e}^{-\psi_i(x)/\sigma_i}\nabla W_i\Big)+\sum_{j=1}^{n}\widetilde{A}_{ij}W_j\quad\text{in}\quad Q_T,\tag{3.8}$$

with the boundary and initial conditions

$$\frac{\partial W_i}{\partial\nu}=0\quad\text{on}\quad\partial\Omega\times(0,T),\tag{3.9}$$

$$W_i(x,0)=-U_{0,i}\,\mathrm{e}^{\psi_i(x)/\sigma_i},\quad x\in\Omega,\tag{3.10}$$

for $i\in\{1,\dots,n\}$.

Next we show that the solutions W_i of the problem (3.8) – (3.10) with nonpositive initial conditions are nonpositive in $\overline{\Omega}$ for all $t \in (0,T)$. To that purpose we consider the auxiliary problem

$$\big(W_i\big)_t - \vartheta_i(x)\mathrm{div}\Big(\zeta_i(x)\nabla W_i\Big) - \sum_{j=1}^{n} \gamma_{ij} W_j \leqslant 0 \quad \text{in} \quad Q_T, \tag{3.11}$$

$$\frac{\partial W_i}{\partial \nu} \leqslant 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{3.12}$$

$$W_i(x,0) = W_{0,i}(x) \leqslant 0 \quad x \in \Omega, \tag{3.13}$$

for $i \in \{1,\ldots,n\}$. We assume that $\vartheta_i(x)$ and $\zeta_i(x)$ are nonnegative in $\overline{\Omega}$ and that the coefficients γ_{ij} satisfy the same assumptions as the coefficients λ_{ij} in Problem (P). The following result holds.

LEMMA 3.2. *Let $(W_1,\ldots,W_n)$ be a smooth and bounded solution of the problem* (3.11) – (3.13) *with nonpositive initial conditions $W_{0,i}$ on a time interval $[0,T]$. Then $W_i(x,t) \leqslant 0$ in $\overline{\Omega} \times (0,T]$. Moreover, for each $i \in \{1,\ldots,n\}$ such that $W_{0,i} \leqslant 0$ and $W_{0,i} \not\equiv 0$, $W_i < 0$ in $\overline{\Omega} \times (0,T]$.*

Proof. The result of Lemma 3.2 follows from the fact that the system (3.11), (3.12), (3.13), with the inequalities $\{\leqslant\}$ replaced by the equalities $\{=\}$, is a cooperative system. However, for the sake of completeness, we present a proof below. We first remark that, in view of [17, Remark (i), p. 191], one can always satisfy the condition

$$\sum_{j=1}^{n} \gamma_{ij} \leqslant 0 \text{ for all } i \in \{1,\ldots,n\}, \tag{3.14}$$

for the matrix of coefficients $\big(\gamma_{ij}\big)_{i,j=1}^{n}$ by performing the change of variables $\overline{W}_i = W_i \mathrm{e}^{-ct}$ for all $i \in \{1,\ldots,n\}$ and $c > 0$ large enough.
Thanks to the regularity of each W_i, we can apply Theorem 15, p. 191 from [17] to conclude that $W_i - M \leqslant 0$ in $\overline{\Omega} \times [0,T]$ for some $M > 0$ and all $i \in \{1,\ldots,n\}$. In fact, we can deduce that $W_i - M < 0$ in $\overline{\Omega} \times (0,T)$.
Indeed, if for some $k \in \{1,\ldots,n\}$, $W_k = M$ in an interior point $(\tilde{x},\tilde{t}) \in \Omega \times (0,T)$, then Theorem 15, p. 191 in [17] implies that $W_k \equiv M$ for all $0 \leqslant t < \tilde{t}$, which is impossible since $W_k(x,0) \leqslant 0$. If the maximum M of W_k is attained at a boundary point $P \in \partial\Omega \times (0,T)$ then either there exists an open ball $K \subset \Omega \times (0,T)$ such that $P \in \partial K$ and $W_k - M < 0$ in K, and the last part of Theorem 15, p. 191 in [17] contradicts the boundary inequality (3.12), or for all open balls $K \subset \Omega \times (0,T)$ such that $P \in \partial K$ there exists a point $(\tilde{x},\tilde{t}) \in K$ such that $W_i(\tilde{x},\tilde{t}) = M$, and we proceed as in the case before.
Hence, there exists $\widetilde{M} > 0$, such that $W_i \leqslant \widetilde{M} < M$ in $\overline{\Omega} \times [0,T]$ for all $i \in \{1,\ldots,n\}$. Then we can repeat the reasoning for all $M > 0$ until $M = 0$. Indeed, if this would not be the case, we find the least real number $\overline{M} > 0$, with $W_i \leqslant \overline{M} \leqslant \widetilde{M}$ in $\overline{\Omega} \times [0,T]$, which leads again to the existence of a real number $0 \leqslant \widehat{M} < \overline{M}$ with the same property. This contradicts the fact that $\overline{M}$ was defined as the least such real number. $\square$

Since the functions u_i^1, u_i^2 are bounded on $\overline{\Omega}\times[0,T]$, it follows that the functions W_i are bounded on $\overline{\Omega}\times[0,T]$ for all $i\in\{1,\dots,n\}$.
Then we are in a position to apply Lemma 3.2 with $\vartheta_i(x)=\mathrm{e}^{\psi_i/\sigma_i}$, $\zeta_i(x)=\sigma_i\,\mathrm{e}^{-\psi_i/\sigma_i}$ and $\gamma_{ij}=\widetilde{A}_{ij}$ for $i,j\in\{1,\dots,n\}$. We deduce that the solutions W_i of the problem (3.8)–(3.10) with nonpositive initial conditions are nonpositive in $\overline{\Omega}$ for all $t\in(0,T)$.

Next we remark that the above reasoning can be applied either with $\vec{U}_0$ replaced by $\vec{U}_0^+$ or with $\vec{U}_0$ replaced by $\vec{U}_0^-$. This permits to show that $\mathscr{S}_i(t)\vec{U}_0^+,\mathscr{S}_i(t)\vec{U}_0^-\geqslant 0$ and that

$$\mathscr{S}_i(t)\vec{U}_0^\pm>0 \quad \text{if} \quad \vec{U}_0^\pm\not\equiv 0. \tag{3.15}$$

We easily compute

$$\begin{aligned}
&\sum_{i=1}^n\frac{1}{\alpha_i}\big\|U_i(\cdot,t)\big\|_{L^1(\Omega)}-\sum_{i=1}^n\frac{1}{\alpha_i}\big\|U_{0,i}(\cdot)\big\|_{L^1(\Omega)}\\
&\quad=\sum_{i=1}^n\frac{1}{\alpha_i}\big\|\mathscr{S}_i(t)\vec{U}_0^+-\mathscr{S}_i(t)\vec{U}_0^-\big\|_{L^1(\Omega)}-\sum_{i=1}^n\frac{1}{\alpha_i}\big\|U_{0,i}(\cdot)\big\|_{L^1(\Omega)}\\
&\quad=\sum_{i=1}^n\int_\Omega\frac{1}{\alpha_i}\Big\{\max\big\{\mathscr{S}_i(t)\vec{U}_0^+,\mathscr{S}_i(t)\vec{U}_0^-\big\}\\
&\qquad\quad-\frac{1}{\alpha_i}\min\big\{\mathscr{S}_i(t)\vec{U}_0^+,\mathscr{S}_i(t)\vec{U}_0^-\big\}\Big\}\,\mathrm{d}x-\sum_{i=1}^n\frac{1}{\alpha_i}\int_\Omega\big\{U_{i,0}^++U_{i,0}^-\big\}\,\mathrm{d}x\\
&\quad=\sum_{i=1}^n\int_\Omega\frac{1}{\alpha_i}\big(\mathscr{S}_i(t)\vec{U}_0^++\mathscr{S}_i(t)\vec{U}_0^-\big)\,\mathrm{d}x-\sum_{i=1}^n\frac{1}{\alpha_i}\int_\Omega\big\{U_{i,0}^++U_{i,0}^-\big\}\,\mathrm{d}x\\
&\qquad\quad-2\sum_{i=1}^n\int_\Omega\frac{1}{\alpha_i}\min\big\{\mathscr{S}_i(t)\vec{U}_0^+,\mathscr{S}_i(t)\vec{U}_0^-\big\}\,\mathrm{d}x\\
&\quad=-2\sum_{i=1}^n\int_\Omega\frac{1}{\alpha_i}\min\big\{\mathscr{S}_i(t)\vec{U}_0^+,\mathscr{S}_i(t)\vec{U}_0^-\big\}\,\mathrm{d}x\leqslant 0,
\end{aligned}\tag{3.16}$$

which completes the proof of (3.4). □

COROLLARY 3.3. *Let* $(u_{0,1}^1,\dots,u_{0,n}^1)$, $(u_{0,1}^2,\dots,u_{0,n}^2)\in\big(C(\overline{\Omega})\big)^n$ *be as in Theorem* 3.1. *Moreover, let us assume that for at least one index* $k\in\{1,\dots,n\}$ *the difference* $u_{0,k}^1-u_{0,k}^2$ *changes the sign. Then, the inequality* (3.4) *is strict for all* $t>0$*, so that solution satisfies a strict contraction property.* □

4. Stationary solutions of the linear molecular motor problem

In this section we show the existence and uniqueness up to a multiplicative constant of the classical stationary solution of the molecular motor problem. We suppose that Ω is an open bounded subset of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$.

We consider the linear system

$$\operatorname{div}\big(\sigma_i \nabla v_i(x) + v_i(x)\nabla \psi_i(x)\big) + \sum_{j=1}^{n} a_{ij} v_j(x) = 0 \quad \text{in} \quad \Omega, \tag{4.1}$$

where $i \in \{1,\dots,n\}$, $n > 1$. The system (4.1) is supplemented with the Robin boundary conditions

$$\sigma_i \frac{\partial v_i}{\partial \nu} + v_i \frac{\partial \psi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \tag{4.2}$$

where $i \in \{1,\dots,n\}$. Thus, the problem can be written as

$$\mathscr{A}\vec{v} = 0,$$

with a linear operator $\mathscr{A}$ in a suitable Banach space $\mathscr{X}$ of functions on Ω, to be made precise later. Moreover, we impose the integral constraint

$$\sum_{i=1}^{n} \int_{\Omega} v_i(x)\,\mathrm{d}x = 1. \tag{4.3}$$

The adjoint problem $\mathscr{A}^*\vec{\varphi} = 0$ to (4.1), in a dual space $\mathscr{X}^*$, is now

$$\sigma_i \Delta\varphi_i - \nabla\psi_i \cdot \nabla\varphi_i + \sum_{j=1}^{n} a_{ji}\varphi_j = 0, \quad \text{in} \quad \Omega, \tag{4.4}$$

with the Neumann boundary conditions for each $i = 1, \dots, n$

$$\frac{\partial \varphi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{4.5}$$

Since $\sum_{j=1}^{n} a_{ji} = 0$, the problem (4.4) has the obvious solution

$$\vec{\varphi} = (\varphi_1, \dots, \varphi_n) = (1, \dots, 1). \tag{4.6}$$

We are going to apply the Krein-Rutman theorem about the first eigenvalue and eigenvector of positive operators, which will permit to conclude that the problem (4.1)–(4.2) has a one-dimensional space of solutions. Therefore, under the additional constraint (4.3), the original problem (4.1)–(4.2) has a unique solution.
Perthame and Souganidis sketched this argument for $n > 1$ and $d = 1$ in [16].

THEOREM 4.1. *Under the assumption $\sum_{j=1}^{n} a_{ji} = 0$, there exists a unique smooth solution $\vec{v}$ of the system* (4.1)*–*(4.3)*.*

Before proving Theorem 4.1 we recall some basic definitions as well as the Krein-Rutman theorem from [8, Ch. VIII, p. 188–191].

DEFINITION 4.2. (Reproducing cone) We say that a closed set K in $\mathscr{X}$ is a cone, if it possesses the following properties:

i) $0 \in K$,

ii) $u,\ v \in K \Longrightarrow \alpha u + \beta v \in K$, for all $\alpha,\ \beta \geqslant 0$,

iii) $v \in K$ and $-v \in K \Longrightarrow v = 0$.

A cone $K \subset \mathscr{X}$ is said to be reproducing if $\mathscr{X} = K - K \equiv \{k_1 - k_2 :\ k_1,\ k_2 \in K\}$.

DEFINITION 4.3. (Dual cone) If K is a cone in $\mathscr{X}$, then the dual cone $K^* \subset \mathscr{X}^*$ is defined by

$$K^* = \{f^* \in \mathscr{X}^* \text{ such that } \langle f^*, v\rangle \geqslant 0 \text{ for all } v \in K\}.$$

DEFINITION 4.4. (Strict positivity) Let $\mathscr{B}$ be a linear operator on $\mathscr{X}$. Then $\mathscr{B}$ is said to be strongly positive if $\mathscr{B}v \in K^o$ for all $v \in K$ such that $v \neq 0$.

THEOREM 4.5. *Let K be a reproducing cone in a Banach space $\mathscr{X}$, with nonempty interior $K^o \neq \emptyset$, and let $\mathscr{B}$ be a strongly positive compact operator on K in the sense of Definition* 4.4. *Then the spectral radius of $\mathscr{B}$, $r(\mathscr{B})$, is a simple eigenvalue of $\mathscr{B}$ and $\mathscr{B}^*$, and their associated eigenvectors belong to K^o and $(K^*)^o$. More precisely, there exists a unique associated eigenvector in K^o (resp. $(K^*)^o$) of norm* 1. *Furthermore, all other eigenvalues are strictly less in absolute value than $r(\mathscr{B})$.*

Proof. We will apply Theorem 4.5 to the space $\mathscr{X} = \left(C(\overline{\Omega})\right)^n \subset \left(L^1(\Omega)\right)^n$ endowed with the usual supremum norm, and to the operators

$$\mathscr{B} = (\lambda I - \mathscr{A})^{-1} :\ \mathscr{X} \to \mathscr{X},$$
$$\mathscr{B}^* = (\lambda I - \mathscr{A}^*)^{-1} :\ \mathscr{X}^* \to \mathscr{X}^*,$$

where $\lambda > 0$ is a strictly positive real number to be fixed later.

Let

$$K = \left\{\vec{u} \in \mathscr{X} :\ u_i(x) \geqslant 0 \text{ for each } x \in \overline{\Omega},\ i = 1, \ldots, n\right\}.$$

We remark that K is a reproducing cone, with nonempty interior

$$K^o = \left\{\vec{u} \in \mathscr{X} :\ \inf_{x \in \overline{\Omega}} u_i(x) > 0,\ i = 1, \ldots, n\right\}.$$

From the standard theory [13, Theorem 2.1 and Theorem 3.1, Ch. 7] for elliptic partial differential linear systems, the boundary value problem

$$\sigma_i \Delta\varphi_i - \nabla\psi_i \cdot \nabla\varphi_i + \sum_{i=1}^{n} a_{ji}\varphi_j - \lambda\varphi_i = f_i \text{ in } \Omega, \tag{4.7}$$

with the homogeneous Neumann conditions (4.5) on $\partial\Omega$, and $\lambda = \widetilde{\lambda} > 0$ sufficiently large, has a solution $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) \in \mathscr{X}$ for each $\vec{f} = (f_1, \ldots, f_n) \in \mathscr{X}$. Moreover, if $f_i(x) \geqslant 0$ for each $i = 1, \ldots, n$, and $x \in \overline{\Omega}$, then $\varphi_i(x) \geqslant 0$ (in fact, $\varphi_i(x) > 0$ in Ω), which is a consequence of the maximum principle (cf. also Example 3 on p. 196–197 in [8]). Thus, the operator $\mathscr{B}^* = \left(\widetilde{\lambda} I - \mathscr{A}^*\right)^{-1}$ is a strongly positive and compact operator, and by Theorem 4.5, the largest eigenvalue μ of $\mathscr{B}$ and $\mathscr{B}^*$ is simple.

Since

$$-\sigma_i\Delta\varphi_i + \nabla\psi_i \cdot \nabla\varphi_i - \sum_{j=1}^{n} a_{ji}\varphi_j + \widetilde{\lambda}\,\varphi_i = \widetilde{\lambda}\,\varphi_i \quad \text{in} \quad \Omega$$

$$\frac{\partial \varphi_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

for all $i \in \{1,\ldots,n\}$, with $\vec{\varphi} = (\varphi_1,\ldots,\varphi_n) = (1,\ldots,1)$, and since $(1,\ldots,1) \in (K^*)^o$, it follows that $\frac{1}{\widetilde{\lambda}} = r\Big(\big(\widetilde{\lambda} I - \mathscr{A}^*\big)^{-1}\Big)$ is a simple eigenvalue of the operator $\big(\widetilde{\lambda} I - \mathscr{A}^*\big)^{-1}$. Applying again Theorem 4.5, we deduce that $\frac{1}{\widetilde{\lambda}}$ is the largest eigenvalue of the operator $\big(\widetilde{\lambda} I - \mathscr{A}\big)^{-1}$, that it is simple, and that there exists $\vec{v} \in K^o \subset \mathscr{X}$ such that

$$\Big(\widetilde{\lambda} I - \mathscr{A}\Big)^{-1}\vec{v} = \frac{1}{\widetilde{\lambda}}\,\vec{v},$$

which is equivalent to

$$\mathscr{A}\vec{v} = 0.$$

This completes the proof of the existence of the solution of the problem (4.1)–(4.3). □

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