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EXISTENCE, NONEXISTENCE AND MULTIPLICITY RESULTS FOR SEMILINEAR ELLIPTIC PROBLEMS WITH MEASURE DATA AND ABSORPTION–REACTION TERM

B. ABDELLAOUI, A. PRIMO, AND T. M. TOUAOULA

To Ireneo Peral,
master and friend,
in the occasion of his 60th birthday.

(*Communicated by M. Kirane*)

Abstract. In the case where $g(u)$ appears as an absorption term, then under some additional hypotheses on g we prove that the main problem has a solution for all $\lambda > 0$ and for all positive $\mu \in L^1(\Omega)$. In the case where g appears as a reaction term, then we prove that the main problem has at least two positive solutions under suitable hypotheses on μ. The asymptotic linear case is also studied.

1. Introduction

This paper is devoted to obtain existence and nonexistence results for nonlinear elliptic equations of the form

$$\begin{cases} -\Delta u \pm g(u) = \lambda \dfrac{u}{|x|^2} + \alpha\mu \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset I\!R^N$ is an open bounded domain which contains the origin, g is a continuous function under suitable hypotheses and $\lambda, \alpha \in I\!R$. In the whole of this work, we suppose that μ is a radon positive measure with some additional hypotheses that we will precise later.

In the case where $g \equiv 0$, we refer to problem (1.1) as the elliptic Baras-Goldstein problem (see [10]). By setting

$$\Lambda_N = \inf_{\{\phi \in \mathscr{C}_0^\infty(\Omega), \phi \not\equiv 0\}} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega \frac{\phi^2}{|x|^2} dx} = \left(\frac{N-2}{2}\right)^2,$$

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we obtain that if $\lambda > \Lambda_N$, then the above problem, with $g \equiv 0$, has no positive solution. In the case where $\lambda \leqslant \Lambda_N$, then the above problem has a solution if and only if $\langle|\mu|, |x|^{-a(\lambda)}\rangle < \infty$ where $a(\lambda) = (N-2)/2 - \sqrt{\Lambda_N - \lambda}$.

The case where $\lambda = 0$ was considered by several authors and it is well known in the literature, see for instance [11], [13], [9] and the references therein. Let begin by quoting the result in the absorption case. From the result of [11], we know that the above problem has a solution for all measure μ if g is an increasing function such that $g(s) \leqslant C|s|^q$ as $|s| \to \infty$ with $q < N/(N-1)$. We refer to [13] and [19] for a complete discussion in this case.

The case of reaction term was considered in [12]. Using capacity estimates and some tools from convex analysis, under some conditions on g, the authors proved that problem (1.1) has a solution for all $\alpha \in (0, \alpha^*)$. Recently in [24], the authors consider the above problem for $\lambda = 0$ with the reaction term. They give an alternative proof of the existence result based on an iteration schema. In the case where $\lambda > 0$, we know apriori that any supersolution to problem $-\Delta u - \lambda \frac{u}{|x|^2} \geqslant 0$ is not bounded at the origin and then to insure the existence of positive solution to problem (1.1) we need some additional hypotheses on the measure.

The paper is organized as follows. In section 2 we give functional tools that we need in the paper, and we define the main spaces where we will work.

In Section 3 we consider the case of absorption term, namely we will prove that under some hypotheses on g, the above problem has a solution for all $\lambda > 0$ and positive $\mu \in L^1(\Omega)$. At the end of the section, we show that the condition on g is optimal to get the existence result.

In Section 4 we study the case of reaction term. We will assume that $\lambda = \Lambda_N$, which is the critical case. In Subsection 4.1 we deal with the superlinear case. Notice that in this case, problem (1.1) has some convex-concave behavior, therefore under some hypotheses on μ and g, we prove the existence of α^* such that (1.1) has two positive solutions for $\alpha < \alpha^*$, at least one solution for $\alpha = \alpha^*$ and no positive solution for $\alpha > \alpha^*$. The proof of the existence result for $\alpha = \alpha^*$ and the proof of the existence of the second positive solution are different from the proofs obtained in [24].

In Subsection 4.2 we consider the asymptotic linear case, namely we will prove that independently of value of α, problem (1.1) has a solution. In subsection 4.3 we prove an antimaximum principle for the linear case. At the end, in the appendix, we continue with asymptotic linear case in variational framework. Following the argument used in [14], under additional hypotheses on μ, we show that problem (1.1) has exactly two solutions, one solution or no solution. This is related to the beautiful result of Ambrosetti-Prodi, see [6] and [7].

2. Functional setting.

Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geqslant 3$ such that $0 \in \Omega$. Given $\alpha \in \mathbb{R}$ we note the weighted Lebesgue space,

$$L^r_\alpha(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable}, \ \int_\Omega |u|^r |x|^{-r\alpha} dx < \infty\Big\}.$$

We consider the Sobolev space $W^{1,2}(\Omega)$, which is defined as the closure of $\mathscr{C}^\infty(\Omega)$, with respect to the norm

$$\|\phi\| = \left(\int_\Omega (|\phi|^2 + |\nabla\phi|^2)dx\right)^{1/2}$$

and we denote by $W_0^{1,2}(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ with respect to the norm $\|.\|$. If $0 \in \Omega$, then for $u \in W_0^{1,2}(\Omega)$ we have the following Hardy-Sobolev inequality

$$\Lambda_N \int\limits_\Omega \frac{u^2}{|x|^2}dx \leqslant \int\limits_\Omega |\nabla u|^2 dx,$$

where $\Lambda_N = (\frac{N-2}{2})^2$ is optimal and is never achieved in $W_0^{1,2}(\Omega)$. Using the improved Caffarelli-Kohn-Nirenberg inequalities in [31] (see also [2] for an alternative proof) we can define the space $H(\Omega)$ as the closure of $\mathscr{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_H^2 = \int_\Omega \left[|\nabla u|^2 - \left(\frac{N-2}{2}\right)^2 \frac{|u|^2}{|x|^2}\right]dx.$$

$H(\Omega)$ is a Hilbert space with the following inner product,

$$\langle u,v\rangle_H = \int_\Omega \left[\nabla u \nabla v - \left(\frac{N-2}{2}\right)^2 \frac{uv}{|x|^2}\right]dx,\ \forall u,v \text{ in } H(\Omega).$$

Furthermore, the following embeddings are followed:

$$W_0^{1,2}(\Omega) \hookrightarrow H(\Omega) \hookrightarrow W_0^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega) \quad \text{for all} \quad 1 \leqslant r < q^* = \frac{qN}{N-q} \quad \text{and } q < 2. \tag{2.1}$$

Let us denote by $H'(\Omega)$ the dual space of $H(\Omega)$ and by $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H',H}$ the duality product. We define:

$$H(\Omega) \to H'(\Omega)$$
$$u \mapsto Lu = -\Delta u - \Lambda_N \frac{u}{|x|^2}.$$

It follows that:

a) $L : H(\Omega) \to H'(\Omega)$ is a continuous linear isomorphism (uniformly continuous on bounded sets).

b) $\langle Lu,v\rangle_{H,H'} = \langle u,v\rangle_H$, so $\|u\|_H = \|Lu\|_{H'}$ and therefore, L is an isometry.

c) L is a self adjoint operator, i.e., $\langle Lu,v\rangle_{H',H} = \langle u,v\rangle_H = \langle u,Lv\rangle_{H,H'}$.

d) The restriction of L^{-1} to $(L^r)'(\Omega)$ with $1 \leqslant r < 2^*$ is a compact map in $L^r(\Omega)$.

Let consider the following eigenvalue problem:

$$(EP^{\beta}) \equiv \begin{cases} Lu = \lambda |x|^{-2\beta} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

with Ω as in the introduction and $\beta < 1$. From classical theory of Hilbert Spaces we get the next classical result.

THEOREM 2.1. *There exists an eigenvalue sequence $\{\lambda_k(|x|^{-2\beta})\} \subset \mathbb{R}^+$, with $\lambda_k(|x|^{-2\beta}) \to \infty$ as $k \to \infty$ for which problem (EP^{β}) has nontrivial solution. Noting E_{λ_i} the eigenvectors associated to λ_i, then $\{E_{\lambda_i}\}_i$ is a decomposition of $H(\Omega)$. Furthermore, the first eigenvalue to (EP^{β}) is simple and isolated and the corresponding eigenfunctions don't change sign in Ω.*

REMARK 2.2. We denote ϕ_k, $k \in \mathbb{N}$, the eigenvector associated to the eigenvalue $\lambda_k(|x|^{-2\beta})$. Without loss of generality, we can consider that $\|\phi_k\|_H = 1$.

Let $a = \frac{N-2}{2}$, and consider $v(x) = |x|^a u(x)$, then

$$\begin{aligned} &H'(\Omega) \to H(\Omega) \to W_0^{1,2}(|x|^{-2a}dx, \Omega) \to W^{-1,2}(|x|^{-2a}dx, \Omega), \\ &f = Lu \mapsto u \mapsto v = |x|^a u \mapsto -\operatorname{div}(|x|^{-2a}\nabla v) = |x|^{-a} f. \end{aligned} \tag{2.3}$$

We obtain the following equivalence of norms

$$\begin{aligned} \|u\|_H^2 &= \langle Lu, u\rangle_{H',H} = \langle -|x|^a \operatorname{div}(|x|^{-2a}\nabla v), |x|^{-a} v\rangle_{H,H} \\ &= \langle -\operatorname{div}(|x|^{-2a}\nabla v), v\rangle_{W_0^{-1,2}(|x|^{-2a}dx), W_0^{1,2}} = \|v\|^2_{W_0^{1,2}(|x|^{-2a}dx)}. \end{aligned}$$

Therefore, we get the equivalent eigenvalue problem

$$(EP^{\beta})' \equiv \begin{cases} -\operatorname{div}(|x|^{-2a}\nabla v) = \mu |x|^{-(2a+2\beta)} v \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

THEOREM 2.3. *There exists an eigenvalue sequence $\{\mu_k(|x|^{-(2a+2\beta)})\} \subset \mathbb{R}^+$, with $\mu_k \to \infty$ as $k \to \infty$ for which problem $(EP^{\beta})'$ has nontrivial solution. Noting E'_{λ_i} the eigenvectors associated to λ_i, then $\{E'_{\lambda_i}\}_i$ is a decomposition of the Hilbert space $W_0^{1,2}(|x|^{-2a}dx)$. Furthermore, the first eigenvalue to $(EP^{\beta})'$ is simple and isolated and the corresponding eigenfunctions don't change sign in Ω.*

REMARK 2.4. We denote φ_k, $k \in \mathbb{N}$, the eigenvector associated to the eigenvalue $\mu_k(|x|^{-(2a+2\beta)})$. Without loss of generality, we can consider that $\|\varphi_k\|_{W_0^{1,2}(|x|^{-2a}dx)} = 1$. Furthermore, $\varphi_k = |x|^a \phi_k$ and $\lambda_k(|x|^{-2\beta}) = \mu_k(|x|^{-(2a+2\beta)})$. For the simplicity of notation, we just write λ_k and μ_k, except in the case where we need to precise the correct weight.

PROPOSITION 2.5. *Let $v \in W_0^{1,2}(|x|^{-2a}dx)$ be a weak solution to problem,*

$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla v) = g \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.5}$$

where $g \in L^r_\eta$, $\eta = -(N-2)(r-1)/r$ with $r > N/2$. Then $v \in L^\infty(\Omega)$.

Proof. It is sufficient to see the proof of Lemma 2.8 in [4] with $\gamma = (N-2)/2$.

COROLLARY 2.6. *Let $u \in H(\Omega)$ be a solution to $Lu = h$, where $h \in L^r_k(\Omega)$, $k = (N-2)(2-r)/(2r)$ and $r > N/2$. Then it follows that $v = |x|^a u \in L^\infty(\Omega)$.*

LEMMA 2.7. *Under the same condition as in Corollary 2.6, if $u \in H(\Omega)$ is a solution to $Lu = h$, with $h \in L^r_k(\Omega)$, $k = (N-2)(2-r)/(2r)$ and $r > N/2$, then $v = |x|^a u \in C^\alpha(\bar{\Omega})$, for some $0 < \alpha < 1/2$.*

Proof. It is sufficient to consider $\gamma = (N-2)/2$ in Theorem 5.1 in [21]. □

Given u a measurable function we will consider the k-truncation of u defined by

$$T_k(u) = \begin{cases} u, & |u| \leqslant k, \\ k\dfrac{u}{|u|}, & |u| > k. \end{cases}$$

3. Existence of weak positive solutions

In this section we prove the existence of a positive solution to the problem

$$\begin{cases} -\Delta u + g(u) = \lambda \dfrac{u}{|x|^2} + \mu \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with $0 \in \Omega$. Moreover under some additional hypotheses on g, we will show that the above problem has a non-negative solution for all $\lambda > 0$ and for a suitable class of positive radon measures.

We will assume that g is an increasing function such that $g(0) = 0$ and

$$\lim_{s \to +\infty} \frac{g(s)}{s^q} = \infty \text{ for some } q > \frac{N}{N-2}. \tag{3.2}$$

Let begin by the following definition.

DEFINITION 3.1. We say that u is a minimal solution to (3.1) if $u \gneqq 0$ and for any non-negative solution v to problem (3.1), we have $u \leqslant v$.

Our first existence result is the following.

THEOREM 3.2. *Assume that the above hypothesis holds and let* $0 \leqslant \mu \in L^1(\Omega)$*, then problem* (3.1) *has a minimal solution for all* $\lambda > 0$.

Proof. Since $\mu \in L^1(\Omega)$, then define $\mu_n = T_n(\mu)$. Consider w_n, the minimal solution to the problem

$$\begin{cases} -\Delta w_n + g(w_n) = \lambda \frac{1}{|x|^2+\frac{1}{n}} T_n w_n + \mu_n(x), \\ w_n \in W_0^{1,2}(\Omega), \quad w_n \geqslant 0. \end{cases} \tag{3.3}$$

It is clear that $w_n \leqslant w_{n+1}$. Using $T_k(w_n)$ as a test function in (3.3) it follows that

$$\int_\Omega |\nabla T_k(w_n)|^2 dx + k \int_{w_n \geqslant k} g(w_n) dx \leqslant k\lambda \int_{w_n \geqslant k} \frac{w_n}{|x|^2} dx + \lambda k^2 C(\Omega) + k \int_\Omega \mu dx.$$

By the main hypotheses on g we know that $g(s) \geqslant cs^q$ as $s \to \infty$, for some $q > \frac{N}{N-2}$. Hence we conclude

$$\begin{aligned} \int_\Omega |\nabla T_k(w_n)|^2 dx + k \int_{w_n \geqslant k} w_n^q dx &\leqslant \lambda k \int_{w_n \geqslant k} \frac{w_n}{|x|^2} dx + C(k,\lambda,\mu,\Omega) \\ &\leqslant \varepsilon k \int_{w_n \geqslant k} w_n^q dx + C(k,\lambda,\varepsilon) \int_\Omega \frac{1}{|x|^{2q'}} dx + C(k,\lambda,\mu,\Omega). \end{aligned}$$

Since $q > \frac{N}{N-2}$, then $2q' < N$, moreover choosing ε small enough it follows that

$$\int_\Omega |\nabla T_k(w_n)|^2 dx + k \int_{w_n \geqslant k} w_n^q dx \leqslant C(k,\lambda,\varepsilon,\mu,\Omega).$$

Hence,

$$i) \int_\Omega g(w_n) dx \leqslant C, \ ii) \int_\Omega \frac{w_n}{|x|^2} dx \leqslant C, \text{ and } iii) \int_\Omega |\nabla T_k(w_n)|^2 \leqslant Ck.$$

Therefore we get the existence of $w \in W_0^{1,p}(\Omega)$, $p < \frac{N}{N-1}$ such that $g(w_n) \to g(w)$, $\dfrac{T_n(w_n)}{|x|^2 + \frac{1}{n}} \to \dfrac{w}{|x|^2}$ strongly in $L^1(\Omega)$. Thus there results that w solves (3.1) and the result follows. □

REMARK 3.3. It is not difficult to see that the existence result holds for all $\mu \in W^{-1,2}(\Omega) + L^1(\Omega)$, namely for any measure that is continuous with respect to the $W_0^{1,2}(\Omega)$ capacity.

To see the optimality condition imposed on g in (3.2), we have the following nonexistence result.

THEOREM 3.4. *Assume that* $g(s) = |s|^q$ *where* $1 < q < \frac{N}{N-2}$, *then for* $\lambda > \Lambda_N$, *the problem*

$$-\Delta u + g(u) = \lambda \frac{u}{|x|^2}, u > 0 \ in\ \Omega,$$

has no positive supersolution with $g(u), \dfrac{u}{|x|^2} \in L^1_{loc}(\Omega)$.

Proof. We argue by contradiction. Assume that the above equation has a supersolution u^* such that $g(u^*), \dfrac{u^*}{|x|^2} \in L^1_{loc}(\Omega)$ for some $\lambda > \Lambda_N$. Choosing a subdomain $\Omega_1 \subset\subset \Omega$ such that $0 \in \Omega_1$, $g(u^*), \dfrac{u^*}{|x|^2} \in L^1_{loc}(\Omega)$ and let $\lambda^* = \lambda - \sigma > \Lambda_N$, then u^* satisfies

$$-\Delta u^* + g(u^*) \geqslant \lambda^* \frac{u^*}{|x|^2} + h^*(x),\ \ u^* > 0 \text{ in } \Omega_1$$

where $h^*(x) = \sigma \dfrac{u^*}{|x|^2}$. Hence an iteration argument allows us to prove that the problem

$$\begin{cases} -\Delta u + g(u) = \lambda^* \dfrac{u}{|x|^2} + h^* \text{ in } \Omega_1, \\ \qquad u > 0 \text{ in } \Omega_1, \\ \qquad u = 0 \text{ on } \partial\Omega_1, \end{cases} \tag{3.4}$$

has a minimal solution $u_* \leqslant u^*$ obtained as a limit of solutions to some approximated problems. Using the strong maximum principle we obtain that $u_* > 0$ in Ω_1. Notice that $u_* \in W_0^{1,p}(\Omega_1)$ for all $p < \frac{N}{N-1}$, then for all $\varepsilon > 0$ and for all $a < \frac{N}{N-2}$, we get the existence of $\eta > 0$ such that $\displaystyle\int_{B_\eta(0)} u_*^a dx \leqslant \varepsilon$ with $B_\eta(0) \subset\subset \Omega_1$. We choose ε such that $\dfrac{\lambda^*}{(1 + S^{-1}\varepsilon^{\frac{2}{N}})} > \Lambda_N + \sigma$, where S is the Sobolev constant, and fixed η as above.

Let $\phi \in \mathscr{C}_0^\infty(B_\eta(0))$, then using Picone type inequality as in [4] it follows that

$$\int_{B_\eta(0)} |\nabla\phi|^2 dx \geqslant \int_{B_\eta(0)} \frac{-\Delta u_*}{u_*} \phi^2 dx.$$

Thus

$$\int_{B_\eta(0)} |\nabla\phi|^2 dx + \int_{B_\eta(0)} u_*^{q-1}\phi^2 dx \geqslant \lambda^* \int_{B_\eta(0)} \frac{\phi^2}{|x|^2} dx.$$

Using Hölder and Sobolev inequalities there results that

$$\begin{aligned} \int_{B_\eta(0)} u_*^{q-1}\phi^2 dx &\leqslant \Big(\int_{B_\eta(0)} \phi^{2^*} dx\Big)^{\frac{2}{2^*}} \Big(\int_{B_\eta(0)} u_*^{\frac{(q-1)N}{2}} dx\Big)^{\frac{2}{N}} \\ &\leqslant S^{-1}\Big(\int_{B_\eta(0)} u_*^{\frac{(q-1)N}{2}} dx\Big)^{\frac{2}{N}} \int_{B_\eta(0)} |\phi|^2 dx. \end{aligned}$$

Since $\frac{(q-1)N}{2} < \frac{N}{N-2}$, we get $\int_{B_\eta(0)} u_*^{\frac{(q-1)N}{2}} dx \leqslant \varepsilon$. Therefore we conclude that

$$\int_{B_\eta(0)} |\nabla\phi|^2 dx \geqslant \frac{\lambda^*}{(1+S^{-1}\varepsilon^{\frac{2}{N}})} \int_{B_\eta(0)} \frac{\phi^2}{|x|^2} dx \geqslant (\Lambda_N+\sigma) \int_{B_\eta(0)} \frac{\phi^2}{|x|^2} dx,$$

a contradiction with the optimality of the Hardy inequality. Hence we conclude. □

4. The case of reaction term

4.1. The superlinear case

In this part we consider the next problem

$$\begin{cases} Lu \equiv -\Delta u - \Lambda_N \dfrac{u}{|x|^2} = g(u) + \alpha\mu \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that g is a regular convex function such that the following hypotheses hold:

i) $g'(s) > 0$ for all $s > 0$,

ii) $|g(s)| \leqslant \lambda|s| + C|s|^p$, $p < \frac{N}{N-2}, \lambda \in (0,\lambda_1)$,

iii) $\lim_{s\to\infty} \frac{g(s)}{s^{1+\sigma}} = \infty$ for some $0 < \sigma < \dfrac{2}{N-2}$.

We begin by the following result (a part of the proof can be found in [5]).

THEOREM 4.1. *Consider the problem*

$$\begin{cases} Lw = \mu \ in\ \Omega, \\ w > 0 \ in\ \Omega, \\ w = 0 \ on\ \partial\Omega, \end{cases} \tag{4.2}$$

where μ is a positive Radon measure. Then problem (4.2) *has a positive solution if and only if $\langle \mu, |x|^{-a}\rangle < \infty$. Moreover w verifies:*

1) $\dfrac{w}{|x|^{2+a-\varepsilon}} \in L^1(\Omega)$ *for all* $\varepsilon > 0$,

2) $|\nabla w||x|^{-a-\gamma+\varepsilon} \in L^1(\Omega)$, *where* $\gamma = \frac{N+2}{2N}$ *and* $\varepsilon > 0$.

Recall that $a = \frac{N-2}{2}$.

Proof. If $\mathcal{M}_1(\Omega)$ is the set of radon measures μ such that $\langle |\mu|, |x|^{-a}\rangle < \infty$, then existence and uniqueness result to problem (4.2) is obtained in [5] where it is also proved that the condition $\mu \in \mathscr{M}_1(\Omega)$ is necessary and sufficient for the existence of a positive solution. It is not difficult to show that $w = \lim_{n\to\infty} w_n$, where w_n solves

$$\begin{cases} Lw_n = f_n \text{ in } \Omega, \\ \quad w > 0 \text{ in } \Omega, \\ \quad w = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.3}$$

with $f_n \in L^\infty(\Omega)$, $||f_n||_{L^1(\Omega)} \leqslant C$ and $f_n \rightharpoonup \mu$ in the sense of measures. Using the theory of renormalized solutions, we obtain that $w_n \to w$ strongly in $W_0^{1,q}(\Omega)$, for all $q < N/(N-1)$.

Let begin by proving 1). Assume that $\lambda < \Lambda_N$ and consider $\psi \in W_0^{1,2}(\Omega)$, the unique solution to the problem

$$-\Delta\psi = \lambda\frac{\psi}{|x|^2} + \frac{1}{|x|^2}.$$

It is clear that $\psi \simeq c|x|^{-a+\sqrt{\Lambda_N-\lambda}}$. Since $\langle\mu,\psi\rangle < \infty$, hence using an approximation argument we can use ψ as a test function in (4.2). Therefore we get

$$(\Lambda_N - \lambda)\int_\Omega \frac{\psi w}{|x|^2}dx + \langle\mu,\psi\rangle = \int_\Omega \frac{w}{|x|^2}dx < \infty.$$

Choosing λ very close to Λ_N it follows that $\frac{w}{|x|^{2+a-\varepsilon}} \in L^1(\Omega)$ where $\varepsilon = \sqrt{\Lambda_N - \lambda}$. Hence the proof of the point 1) follows.

We continue to prove the point 2). We observe that $|\nabla w||x|^{-a-\gamma+\varepsilon} \in L^1(B\backslash B_r(0))$. Hence we have just to prove that $|\nabla w||x|^{-a-\gamma+\varepsilon} \in L^1(B_r(0))$, for r small.

Using $T_k(w_n)\psi$ as a test function in (4.3) and letting $n \to \infty$, we get

$$\int_\Omega |\nabla T_k w|^2 \psi dx \leqslant k\langle f,\psi\rangle \leqslant Ck.$$

Thus from the result of [4] and using the behavior of ψ near the origin we conclude that

$$\int_\Omega |\nabla w|^q |x|^{-a+\varepsilon} dx < \infty \text{ for all } \varepsilon > 0 \text{ and for all } q < \frac{N}{N-1}.$$

Fixed $q < N/(N-1)$ such that q is very close to $N/(N-1)$, then using Hölder inequality it follows that

$$\begin{aligned}\int_\Omega |\nabla w||x|^{-a-\gamma+\varepsilon}dx &= \int_\Omega |\nabla w||x|^{\frac{-a+\varepsilon}{q}}|x|^{\frac{-(q-1)(a-\varepsilon)}{q}-\gamma}dx \\ &\leqslant \Big(\int_\Omega |\nabla w|^q|x|^{-a+\varepsilon}dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |x|^{-a+\varepsilon-\frac{q\gamma}{q-1}}dx\Big)^{\frac{q-1}{q}}.\end{aligned}$$

Using the hypothesis on γ it follows that

$$\int_{\Omega} |x|^{-a+\varepsilon-\frac{q\gamma}{q-1}} dx < \infty.$$

Hence the result follows. $\square$

The next result will play an important role in the proof of the main result of this section.

THEOREM 4.2. *If w is the solution to problem (4.2), then $\int_{\Omega} |x|^{-a} w^q dx < \infty$, for all $q < \frac{N}{N-2}$. As a consequence the next problem*

$$\begin{cases} Lw_1 = w^q \text{ in } \Omega, \\ w_1 > 0 \text{ in } \Omega, \\ w_1 = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

has a unique positive solution.

Proof. Using Theorem (4.1) we have just to show that $\int_{\Omega} |x|^{-a} w^q dx < \infty$. Without loss of generality, we can assume that $B_1(0) \subset \Omega$. Let $\phi(x) = |x|^{-a} - 1$, then $\phi \in H(B_1(0))$ and ϕ solves

$$L\phi = \frac{\Lambda_N}{|x|^2} \text{ in } B_1(0),\ \phi = 0 \text{ on } \partial B_1(0).$$

Since $\langle \mu, \phi \rangle < \infty$, then using an approximation argument, we obtain that

$$\begin{aligned} -\Delta(w\phi^\alpha) = \phi^\alpha(\Lambda \frac{w}{|x|^2} + \mu) + w(\alpha\phi^{\alpha-1}(-\Delta\phi) - \alpha(\alpha-1)\phi^{\alpha-2}|\nabla\phi|^2) \\ + 2\alpha\phi^{\alpha-1}\nabla\phi\nabla w = \Lambda(\alpha+1)\frac{\phi^\alpha w}{|x|^2} + \alpha\frac{\phi^{\alpha-1} w}{|x|^2} \\ - \alpha(\alpha-1)\phi^{\alpha-2} w|\nabla\phi|^2 + 2\alpha\phi^{\alpha-1}\nabla w\nabla\phi + \langle \mu, \phi \rangle. \end{aligned}$$

Notice that $\frac{w}{|x|^{2+a-\varepsilon}} \in L^1(\Omega)$ for all $\varepsilon > 0$. Therefore using the main properties of w and ϕ, it follows that

$$\frac{\phi^\alpha w}{|x|^2},\ \phi^{\alpha-2} w|\nabla\phi|^2 \in L^1_{loc}(\Omega).$$

From Theorem 4.1, we obtain that $|\phi^{\alpha-1}\nabla w\nabla\phi| \equiv |\nabla w||x|^{-\alpha a-1} \in L^1(\Omega)$. Hence $w\phi^\alpha \in L^1_{loc}$ and $-\Delta(w\phi^\alpha) \in L^1_{loc}$. Then, from [13], we obtain that $(w\phi^\alpha)^q \in L^1_{loc}$. Thus $w^q\phi \in L^1(\Omega)$ and then the result follows. $\square$

As a consequence of the previous result we have the main Theorem of this section.

THEOREM 4.3. *Assume that the above hypotheses on g hold. If $\mu \in \mathcal{M}_1(\Omega)$ is a positive radon measure, then there exists $\alpha_0 > 0$ such that for all $\alpha \in [0,\alpha_0]$ problem* (4.1) *has a minimal solution.*

Proof. We divide the proof in several steps.

Step 1. Let $E = \{\alpha > 0$ such that problem (4.1) has a minimal solution $\}$. We claim that if $E \neq \emptyset$, then E is an interval. To prove the claim we suppose that $E \neq \emptyset$. Let $\alpha_0 \in E$ and consider u_0 a solution to (4.1) with $\alpha = \alpha_0$, then u_0 is a supersolution to (4.1) with $\alpha < \alpha_0$. It is clear that 0 is a subsolution. Hence using an iteration argument we get the existence result. Therefore $\alpha \in E$ for all $\alpha \in (0,\alpha_0)$. Thus we conclude that E is an interval and then the claim follows.

Step 2. We will show that $E \neq \emptyset$. We follow closely the argument used in [24].

Let $w_{1,\alpha}$ be the solution to the problem

$$\begin{cases} Lw_{1,\alpha} = \alpha\mu \text{ in } \Omega, \\ w_{1,\alpha} > 0 \text{ in } \Omega, \\ w_{1,\alpha} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

It is clear that $w_{1,\alpha} = \alpha\tilde{w}_1$, the unique positive solution to the problem

$$L\tilde{w}_1 = \mu.$$

By iteration, we define $w_{k+1,\alpha}$ ($\tilde{w}_{k+1}$ respectively) as the unique solution to the problem

$$\begin{cases} Lw_{k+1,\alpha} = g(\sum_{i=1}^{k} w_{i,\alpha}) - g(\sum_{i=1}^{k-1} w_{i,\alpha}) \text{ in } \Omega, \\ w_{k+1,\alpha} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.6}$$

and

$$\begin{cases} L\tilde{w}_{k+1} = g(\sum_{i=1}^{k} \tilde{w}_i) - g(\sum_{i=1}^{k-1} \tilde{w}_i) \text{ in } \Omega, \\ \tilde{w}_{k+1} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.7}$$

respectively. Using the hypotheses on g and Theorem 4.2, we obtain that $g(\sum_{i=1}^{k} w_{i,\alpha})$ is well defined and that $w_{k+1,\alpha} > 0$. If we suppose that problem (4.1) has a solution u, then by iteration we define $u_{k+1,\alpha} = u_{k,\alpha} - w_{k+1,\alpha}$. Hence $u_{k+1,\alpha}$ solves

$$\begin{cases} Lu_{k+1,\alpha} = g(u_{k+1,\alpha} + \sum_{i=1}^{k+1} w_{i,\alpha}) - g(\sum_{i=1}^{k} w_{i,\alpha}) \text{ in } \Omega, \\ u_{k+1,\alpha} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.8}$$

For simplicity of typing we set $v_{k,\alpha} = \sum_{i=1}^{k+1} w_{i,\alpha}$, then we define the odd function $h_{k,\alpha}(x,s)$ by

$$h_{k,\alpha}(x,s) = \begin{cases} g(s+v_{k,\alpha}) - g(v_{k,\alpha}) & \text{if } s \geqslant 0 \\ h_{k,\alpha}(x,s) = -h_{k,\alpha}(x,-s) & \text{if } s \leqslant 0. \end{cases}$$

It is clear that $u_{k+1,\alpha}$ solves

$$Lw = h_{k,\alpha}(x,w) + \beta_{k,\alpha}(x) \;\; w|_{\partial\Omega} = 0, \tag{4.9}$$

where $\beta_{k,\alpha}(x) = g(v_{k,\alpha}(x)) - g(v_{k-1,\alpha}(x))$. Thus to prove the existence of a solution u to problem (4.1) we have just to prove that problem (4.9) has a solution for some $k > 1$.

Notice that by using the main properties of g, we have

$$h_{k,\alpha}(x,s) \leqslant (\lambda_1 + C(v_{k,\alpha}+s)^{p-1})s \text{ for all } x \in \Omega, s \geqslant 0,$$

and

$$h'_{k,\alpha}(x,s) \leqslant \lambda_1 + C(v_{k,\alpha}+s)^{p-1} \text{ for all } x \in \Omega, s \geqslant 0.$$

We claim the existence of $m \in \mathbb{N}$ such that $w_{m,\alpha} \in H(\Omega)$ and $\beta_{m,\alpha} \in L^\theta(\Omega)$ with $\theta > \frac{2N}{N+2}$. Notice that

$$\beta_{k,\alpha} = g(v_{k,\alpha}) - g(v_{k-1,\alpha}) \leqslant g(v_{k,\alpha})w_{k,\alpha} \leqslant (\lambda_1 + Cv_{k,\alpha}^{p-1})w_{k,\alpha}.$$

Let begin by proving the existence of $m \in \mathbb{N}$ such that $w_{m,\alpha} \in H(\Omega)$. From the result of [24] we get the existence of $m_0 \in \mathbb{N}$ such that $w_{m,\alpha} \in L^\infty(\Omega \backslash B_r(0))$ for r small and for all $m \geqslant m_0$. Hence we have just to prove the desired regularity in $B_r(0)$. We will show that $w_{m_1,\alpha} \simeq C|x|^{-a}$ as $x \to 0$, for some $m_1 \geqslant m_0$, and this allows us to get the desired regularity. We use the result obtained in [3] and [16].

Using the hypothesis on p, it is not difficult to see that $(\lambda_1 + Cv_{k,\alpha}^{p-1}) \in L^r(\Omega)$ for some $r > \frac{N}{2}$ and for all $k > 1$. If $|x|^{-a}w_{m_1,\alpha}(\lambda_1 + Cv_{N,\alpha}^{p-1}) \in L^s(\Omega)$ for some $s > 1$, then from the regularity result obtained in Corollary 5.1 of [3], we obtain that $|x|^{-a}w_{m_1+1,\alpha} \leqslant C$ and then the result follows. If not, then from Theorem 4.2 and using Hölder inequality, we obtain that $|x|^{-a}w_{m_1,\alpha}(\lambda_1 + Cv_{N,\alpha}^{p-1}) \in L^1(\Omega)$. Hence from Corollary 5.1 in [3], we obtain that $|x|^{-a}w_{m_1+1,\alpha} \in L^r(\Omega)$ for all $r > 1$. Therefore using again Hölder inequality and the hypothesis on p, we get the existence of $s > 1$ such that $|x|^{-a}w_{m_1+1,\alpha}(\lambda_1 + Cv_{N+1,\alpha}^{p-1}) \in L^s(\Omega)$ and then we conclude that $|x|^{-a}w_{m_1+2,\alpha} \leqslant C$. Thus $w_{m+2,\alpha} \in H(\Omega)$ and then the first part of the claim follows. The same discussion allows us to show the existence of $m_2 \in \mathbb{N}$ such that $\beta_{m,\alpha} \in L^\theta(\Omega)$ with $\theta > \frac{2N}{N+2}$ for all $m \geqslant m_2$. Hence the claim follows.

Fixed $m \geqslant \max\{m_1+1, m_2\}$, then following the argument of [24], we can easily prove that $h_{m,\alpha}$ is an increasing function in α and that $\beta_{m,\alpha}$ is strictly decreasing with respect to α.

Recall that $\tilde{w_1}$ is the unique solution to problem $L\tilde{w}_1 = \mu$. By iteration, we define $\tilde{w}_k$ as the unique solution to the problem

$$\begin{cases} L\tilde{w}_k = \Big(\lambda_1 + C(\sum_{i=1}^{k-1}\tilde{w}_i)^{p-1}\Big)\tilde{w}_{k-1} \text{ in } \Omega, \\ \tilde{w}_k = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.10}$$

Notice that $w_1 \leqslant \alpha\tilde{w}_1$, hence by induction and using the properties of g we can prove that $w_k \leqslant \alpha\tilde{w}_k$ for all k. In the same way, we have

$$0 \leqslant \beta_{m,\alpha}(x) \leqslant (\lambda_1 + Cv_{m,\alpha}^{p-1})w_{k,\alpha} \leqslant \alpha\tilde{\beta}_{m,\alpha},$$

where $\tilde{\beta}_{m,\alpha} = \Big(\lambda_1 + C(\sum_{i=1}^{m}\tilde{w}_i)^{p-1}\Big)\tilde{w}_m$. Therefore we conclude that

$$\begin{aligned} |h_{m,\alpha}(s,x)| &\leqslant g'(|s|+v_{m,\alpha})|s| \leqslant [(\lambda+\sigma) + C(|s|+v_{m,\alpha})^{p-1}]|s| \\ &\leqslant (\lambda+\sigma)|s| + C_1|s|^p + \alpha^{p-1}C_2(\textstyle\sum_{i=1}^{m}\tilde{v}_i)^{p-1}]|s| \end{aligned}$$

where σ is such that $\lambda+\sigma<\lambda_1$. We set $\zeta(x)=C_2(\sum_{i=1}^m \tilde{v}_i)^{p-1}$, as above we know that $\zeta\in L^r(\Omega)$ for some $r>\frac{N}{2}$, thus

$$|h_{m,\alpha}(s,x)|\leqslant((\lambda+\sigma)+\alpha^{p-1}\zeta(x))|s|+C|s|^p.$$

Choosing α small, since $p>1$, we get the existence of $\bar{u}$, the minimal solution to the problem

$$L\bar{u}=\Big((\lambda+\sigma)+\alpha^{p-1}\zeta(x)\Big)\bar{u}+C_1\bar{u}^p+\alpha\tilde{f}_{m,\alpha},\ \bar{u}\in H(\Omega).$$

It is clear that $\bar{u}$ is a supersolution to problem (4.1). Since 0 is a subsolution, then an iteration argument allows us to prove the existence of a minimal solution. Hence $E\neq\emptyset$.

Step 3. Let $\alpha^*=\sup\{\alpha\in E\}$, then $\alpha^*<\infty$.

Let ϕ be the solution to the problem

$$-\Delta\phi=\lambda_1\phi,\text{ and }\phi>0.$$

It is clear that $\phi\in\mathscr{C}(\bar{\Omega})$. Using ϕ as a test function in (4.1), we obtain that

$$\lambda_1\int_\Omega u\phi\geqslant\int_\Omega g(u)\phi dx+\alpha\int_\Omega\phi d\mu.$$

Since $g(s)\geqslant(\lambda_1+\sigma)s-C$, it follows that $C\int_\Omega\phi dx\geqslant\alpha\int_\Omega\phi d\mu$. Hence $\alpha^*<\infty$.

Step 4. $\alpha^*\in E$. Let $\{\alpha_n\}$ be an increasing sequence such that $\alpha_n\uparrow\alpha^*$ as $n\to\infty$. Consider u_n the minimal solution to (4.1) for $\alpha=\alpha_n$, hence $u_n\leqslant u_m$ if $n\leqslant m$ and then $\{u_n\}$ is an increasing sequence in n. Let $\psi\in\mathscr{C}_0^\infty(\Omega)$ be a regular fixed function. Using Picone type inequality as in [4], it follows that

$$\int_\Omega|\nabla\psi|^2dx\geqslant\lambda\int_\Omega\frac{\psi^2}{|x|^2}dx+\int_\Omega\frac{g(u_n)}{u_n}\psi^2dx.$$

Notice that by the hypothesis *iii)* on g we obtain that $g(s)\geqslant cs^{1+\sigma}-C$, hence we conclude that

$$\int_\Omega|\nabla\psi|^2dx-\lambda\int_\Omega\frac{\psi^2}{|x|^2}dx+\int_\Omega\frac{\phi^2}{u_1^2}\geqslant\int_\Omega u_n^\sigma\psi^2dx,$$

where we have used the fact that $\dfrac{\phi^2}{u_n^2}\leqslant\dfrac{\phi^2}{u_1^2}$ for all $n\geqslant 1$. Since $\{u_n\}$ is a monotone sequence, we get the existence of a measurable function u such that $u_n\uparrow u$ a.e in Ω and $u^\alpha\in L^1_{loc}(\Omega)$. Let $x_0\in\Omega$ be such that $u_n(x_0)\to u(x_0)$ as $n\to\infty$. Using the general extension of the maximum principle obtained in [18], we obtain that

$$C\geqslant u_n(x_0)\geqslant C\Big(\lambda\int_\Omega\frac{u_n}{|x|^2}\delta(x)dx+\int_\Omega g(u_n)\delta(x)dx\Big),$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Hence we conclude that

$$\lambda \frac{u_n}{|x|^2} + g(u_n) \to \lambda \frac{u}{|x|^2} + g(u) \text{ strongly in } L^1_{loc}(\Omega).$$

It is not difficult to show that u is a distributional solution to (4.1) for $\alpha = \alpha^*$. To complete the proof, we need to show that $\frac{u}{|x|^2}$ and $g(u)$ are in $L^1(\Omega)$. Let ϕ_1 the solution to the problem

$$-\Delta\phi_1 = 1, \phi_1 = 0 \text{ on } \Omega.$$

It is clear that $\phi_1 \simeq C\delta(x)$. Using ϕ_1 as a test function in the equation of u_n, there results that

$$\begin{aligned}\int_\Omega u_n dx &= \lambda \int_\Omega \frac{u_n}{|x|^2}\phi_1 dx + \int_\Omega g(u_n)\phi_1 dx + \int_\Omega \phi_1 d\mu \\ &\leqslant C(\lambda \int_\Omega \frac{u_n}{|x|^2}\delta(x)dx + \int_\Omega g(u_n)\delta(x)dx + \int_\Omega \delta(x)d\mu) \leqslant C.\end{aligned}$$

Therefore the Monotone Convergence Theorem allows us to conclude that $u \in L^1(\Omega)$. Let Ω_1 be a regular domain such that $\Omega \subset\subset \Omega_1$ and define ψ_2 as the solution to the problem

$$-\Delta\phi_2 = \chi_\Omega \text{ in } \Omega_1, \phi_2 = 0 \text{ on } \partial\Omega_1.$$

Using ψ_2 as a test function in the u_n-equation, it follows that

$$\int_\Omega u_n(-\Delta\psi_2) \geqslant \lambda \int_\Omega \frac{u_n}{|x|^2}\phi_2 dx + \int_\Omega g(u_n)\phi_2 dx.$$

Using the strong maximum principle we know that $\psi_2 \geqslant c$ in Ω. Hence we conclude that

$$\lambda \int_\Omega \frac{u_n}{|x|^2}dx + \int_\Omega g(u_n)dx \leqslant \int_\Omega u_n dx \leqslant C.$$

Thus $\frac{\lambda u_n}{|x|^2} + g(u_n) \to \frac{\lambda u}{|x|^2} + g(u)$ strongly in $L^1(\Omega)$ and the result follows. □

REMARKS 4.4. Fixed m as above, it is clear that solutions to (4.9) are critical points to the functional

$$J_{m,\alpha}(w) = \frac{1}{2}||w||^2 - \int_\Omega H_{m,\alpha}(x,w)dx - \int_\Omega \beta_{m,\alpha} w dx,$$

where $H_{m,\alpha}(x,s) = \int_0^s h_{m,\alpha}(x,t)dt$. Using the properties of g, we easily get that $J_{m,\alpha}$ is well defined and $J_{m,\alpha} \in \mathscr{C}^1(H(\Omega))$. Notice that for α small, $J_{m,\alpha}$ has a concave-convex geometry. This will be used to prove the existence of a second positive solution.

Let consider the next set $P=\{\alpha>0\,|J_{m,\alpha}$ has a local minimum $u_\alpha\}$. If $\alpha\in P$ and u_α is the local minimum of $J_{m,\alpha}$ in $H(\Omega)$, then $v=0$ is a local minimum of the functional

$$\Phi(v)=\frac{1}{2}(\int\limits_{\Omega}|\nabla v|^2-\Lambda_N\frac{v^2}{|x|^2})dx-\int\limits_{\Omega}F(\lambda,x,v)dx, \tag{4.11}$$

where

$$F(x,v)=\int_0^v f(x,s)ds$$

and

$$f(x,s)=\begin{cases}h_{m,\alpha}(x,s+u_\alpha(x))-h_{m,\alpha}(x,u_\alpha(x)) & \text{if } s\geqslant 0\\ 0, & \text{if } s<0,\end{cases}$$

it is clear that

$$f(x,s)=\begin{cases}g(s+u_\alpha(x)+v_{m,\alpha}(x))-g(u_\alpha(x)+v_{m,\alpha}(x)) & \text{if } s\geqslant 0\\ 0, & \text{if } s<0.\end{cases}$$

Therefore we get the next result.

THEOREM 4.5. *If $\alpha\in P$, then $J_{m,\alpha}$ has a second critical point and then problem* (4.9) *has a second positive solution. As a consequence, problem* (4.1) *has also a second positive solution.*

Proof. We argue by contradiction. Assume that w_α is the only critical point of $J_{m,\alpha}$. Then $v=0$ is a local minimum and the unique critical point of Φ. It is clear that Φ has the mountain pass geometry. Choosing $w_0\in H(\Omega)$ such that $\Phi(w_0)<0$ and consider

$$\Gamma=\left\{\gamma:\,[0,1]\to H_0^1\ \gamma(0)=0,\gamma(1)=w_0\right\}\text{ and } c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi(\gamma(t)).$$

Using the properties of g we obtain that $c>0$. Notice that, in general the functional Φ_λ dos not satisfies the Ambrosetti-Rabinowitz condition. Hence to prove the existence of a critical mountain pass point of Φ we follow closely the argument used in [27], see also [1]. Let $\nu>0$, then define the functional Φ_ν by

$$\Phi_\nu(v)=\frac{1}{2}(\int\limits_{\Omega}|\nabla v|^2-\Lambda_N\frac{v^2}{|x|^2})dx-\nu\int\limits_{\Omega}F(x,v)dx. \tag{4.12}$$

By a continuity argument we get the existence of $\varepsilon>0$ such that for all $\nu\in\mathscr{I}=[1-\varepsilon,1+\varepsilon]$, the family of functional $\{\Phi_\nu\}_{\nu\in\mathscr{I}}$ has the same geometry as Φ, namely

$$c(\nu)=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\nu(\gamma(t))>0.$$

Notice that $\Phi_\nu(w_0)<0$ for all $\nu\in\mathscr{I}$. Using Theorem 1.1 in [27] we obtain that for almost every $\nu\in\mathscr{I}$ there exists a sequence $\{v_k^{(\nu)}\}$ such that: *i*) $\{v_k^{(\nu)}\}$ is bounded in

$H(\Omega)$ *ii)* $\Phi_\nu(v_k^{(\nu)}) \to c(\nu)$ and *iii)* $\Phi'_\nu(v_k^{(\nu)}) \to 0$ in $H^{-1}(\Omega)$. Using the subcritical behavior of g, we obtain that $\{v_k^{(\nu)}\}$ is a bounded Palais-Smale sequence of Φ_ν. Thus, up to a subsequence, $v_k^{(\nu)} \to v^{(\nu)}$ strongly in $H(\Omega)$, where $v^{(\nu)}$ solves

$$\begin{cases} -\Delta v^{(\nu)} - \Lambda_N \dfrac{v^{(\nu)}}{|x|^2} = \nu f(x, v^{(\nu)}) & \text{in } \Omega \\ \qquad\qquad v^{(\nu)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

with $\Phi_\nu(v^{(\nu)}) = c(\nu)$. Let $\{\nu_n\}$ be a decreasing sequence in $\mathscr{I}$ such that $\nu_n \downarrow 1$ as $n \to \infty$ and consider $v^{(\nu_n)}$ the corresponding solution to problem (4.13). We will prove that $\{v^{(\nu_n)}\}$ is bounded in $H(\Omega)$. For the simplicity of notation we set $v_n = v^{(\nu_n)}$.

Let ϕ_1 be the solution to problem

$$\begin{cases} -\Delta \phi_1 - \Lambda_N \dfrac{\phi_1}{|x|^2} = \lambda_1 \phi_1 & \text{in } \Omega \\ \qquad\qquad \phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Using ϕ_1 as a test function in (4.13) we get

$$\nu_n \int_\Omega f(x, v_n)\, \phi_1 \nu_n dx = \lambda_1 \int_\Omega v_n\, \phi_1 dx.$$

Therefore using the definition of f and the convexity hypotheses on g we get the existence of a constant C_1 such that

$$\int_\Omega \phi_1 v_n dx \leqslant C_1 \quad \text{and} \quad \int_\Omega \phi_1 f(x, v_n) dx \leqslant C_1.$$

Let now ϕ_2 be the solution to problem

$$\begin{cases} -\Delta \phi_2 - \Lambda_N \dfrac{\phi_2}{|x|^2} = 1 & \text{in } \Omega \\ \qquad\qquad \phi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

From Proposition 2.5 and using the Hopf Lemma, we get the existence of $c_1, c_2 > 0$ such that $c_1\phi_1 \leqslant \phi_2 \leqslant c_2\phi_1$. Taking ϕ_2 as a test function in (4.13) we obtain that

$$\int_\Omega v_n dx = \nu_n \int_\Omega f(x, v_n)\, \phi_2 dx \leqslant c_2 \nu_n \int_\Omega f(x, v_n)\, \phi_1 dx \leqslant C. \tag{4.16}$$

Hence $\int_\Omega v_n dx \leqslant C$. Since $\Phi_{\nu_n}(v_n) = c(\nu_n) \leqslant c + 1$, then using (4.13) we obtain that

$$\int_\Omega f(x, v_n) v_n - 2F(x, v_n)) dx \leqslant C. \tag{4.17}$$

We prove now the energy estimate. Assume by contradiction that $||v_n||_{H(\Omega)} \to \infty$ as $n \to \infty$. We set $w_n = \dfrac{v_n}{||v_n||_{H(\Omega)}}$, then $||w_n||_{H(\Omega)} = 1$, hence we get the existence of $w_0 \in H(\Omega)$ such that, up to subsequences, $w_n \rightharpoonup w_0$ weakly in $H(\Omega)$ and $w_n \to w_0$ strongly in $L^a(\Omega)$ for all $a < \frac{2N}{N-2}$. Moreover w_n verifies

$$-\Delta w_n - \Lambda_N \frac{w_n}{|x|^2} = \frac{v_n f(x, v_n)}{||v_n||_{H(\Omega)}}.$$

Since $w_n \rightharpoonup w_0$ weakly in $H(\Omega)$ we obtain that

$$\int_\Omega (-\Delta w_0 - \Lambda_N \frac{w_0}{|x|^2})\phi = \lim_{n\to\infty} \int_\Omega \frac{f(x, v_n)}{||v_n||_{H(\Omega)}} \phi \text{ for all } \phi \in \mathscr{C}_0^\infty(\Omega). \tag{4.18}$$

From (4.16) we obtain that $f(x, v_n)$ is bounded in $L^1_{\text{loc}}(\Omega)$. Therefore (4.18) implies $w_0 = 0$. Let $z_n = t_n v_n$ where t_n is defined as

$$t_n = \inf\left\{ t \in [0,1] \mid \Phi_{v_n}(tv_n) = \max_{s\in[0,1]} \Phi_{v_n}(sv_n) \right\}.$$

We prove that $t_n \in (0,1)$ for n large enough. Since $\Phi_{v_n}(0) = 0$, it follows that $t_n \neq 0$ for all v_n. To show that $t \neq 1$ we claim that

$$\lim_{n\to\infty} \Phi_{v_n}(z_n) = +\infty. \tag{4.19}$$

We argue by contradiction. If $\liminf\limits_{n\to\infty} \Phi_{v_n}(z_n) \leqslant M$, we set $u_n = \sqrt{4M} w_n$, then $u_n \rightharpoonup 0$ weakly in $H(\Omega)$, hence $\int_\Omega F(x, u_n) dx$, $\int_\Omega u_n dx \to 0$ as $n \to \infty$. Therefore we obtain that

$$\Phi_{v_n}(u_n) = 2M - \alpha_n \int_\Omega F(x, u_n) dx \geqslant \frac{3}{2} M \text{ as } n \to \infty. \tag{4.20}$$

On the other hand, using the definition of z_n and observing that $u_n = \frac{\sqrt{4M}}{||v_n||_H} v_n$, we obtain that

$$\Phi_{v_n}(u_n) \leqslant \Phi_{v_n}(z_n) \leqslant M,$$

a contradiction with (4.20). Hence (4.19) is proved.

Therefore, taking into account that $\Phi_{v_n}(v_n) = c_{\alpha_n} \leqslant c+1$ and by the claim, we conclude $t_n \neq 1$ for n large enough. As a consequence by the definition of z_n we have $\langle \Phi'_{v_n}(z_n), z_n \rangle = 0$, hence we conclude that

$$\Phi_{v_n}(z_n) = \frac{v_n}{2} \int_\Omega (f(x, z_n) z_n - 2F(x, z_n))\, dx \to \infty \text{ as } n \to \infty.$$

By the fact that the function $l(x,s) = f(x,s)s - 2F(x,s)$ is an increasing function in s, it follows that

$$f(x, z_n) z_n - 2F(x, z_n) \leqslant f(x, v_n) v_n - 2F(x, v_n)$$

and then

$$\int_\Omega f(x,v_n)v_n - 2F(x,v_n))\,dx \to \infty \quad \text{as } n \to \infty,$$

a contradiction with (4.17). As a consequence we conclude that $||v_n||_{H(\Omega)} \leqslant C_1$. Thus using the subcritical behavior of g we conclude that, up to a subsequence, $v_n \to v$ strongly in $H(\Omega)$, where v is a critical point of Φ. Hence problem (4.1) has a second positive solution. □

As a consequence, using the main ideas of [20] and [8], we get the next global existence result.

THEOREM 4.6. *Assume that $\alpha \in (0,\alpha^*)$, then problem* (4.9) *has a second positive solution.*

Proof. We fix $\alpha_0 \in (0,\alpha^*)$, $\alpha_0 < \bar{\alpha}_1 < \alpha^*$. Let w_0, w_1 the minimal solutions of (4.9) with $\alpha = \alpha_0$ and $\alpha = \bar{\alpha}_1$ respectively. It is clear that $w_0 < w_1$. Let

$$M = \{w \in H(\Omega) : 0 \leqslant w \leqslant w_1\}.$$

Notice that M is a convex closed set in H. Since J_{m,α_0} is bounded and lower semi continuous over M, then we get the existence of $\vartheta \in M$ such that

$$J_{m,\alpha_0}(\vartheta) = \inf_{w\in M} J_{m,\alpha_0}(w).$$

Notice that $\vartheta \neq 0$ since $J_{m,\alpha_0}(\bar{w}) < 0$, where $\bar{w}$ solves $L(\bar{w}) = \beta_{m,\alpha_0}(x), \bar{w} \in H(\Omega)$. We conclude that $J_{m,\alpha_0}(\vartheta) < 0$. Using a similar argument as in Theorem 2.4 in [30], we obtain that ϑ is a solution to (4.9). If $\vartheta \neq w_0$, we have done. Suppose that $\vartheta \equiv w_0$. We will prove that in this case ϑ is a local minimum to J_{m,α_0} and then using Theorem 4.5, we conclude.

We argue by contradiction. Assume that ϑ is not a local minimum of J_{m,α_0}, then we get the existence of $\{v_n\} \subset H(\Omega)$ such that $||v_n - \vartheta||_{H(\Omega)} \to 0$ and $J_{m,\alpha_0}(v_n) < J_{m,\alpha_0}(\vartheta)$. Let $w_n = (v_n - w_1)_+$ and $\phi_n = \max\{0,\min\{v_n,w_1\}\}$. It is clear that $\phi_n \in M$ and

$$\phi_n(x) = \begin{cases} 0 & \text{if } v_n(x) \leqslant 0, \\ v_n(x) & \text{if } 0 \leqslant v_n(x) \leqslant w_1(x), \\ w_1(x) & \text{if } w_1(x) \leqslant v_n(x). \end{cases}$$

Let $T_n \equiv \{x \in \Omega : \phi_n(x) = v_n(x)\}$ and $S_n \equiv \text{supp } w_n$. Notice that supp $v_n^+ = T_n \cup S_n$. We claim that $|S_n| \to 0$ as $n \to \infty$, where $|\cdot|$ is the Lebesgue measure. Let $\varepsilon > 0$,

$$E_n = \{x \in \Omega : v_n(x) \geqslant w_1(x) > \vartheta(x) + \delta\},$$
$$F_n = \{x \in \Omega : v_n(x) \geqslant w_1(x) \text{ and } w_1(x) \leqslant \vartheta(x) + \delta\},$$

where δ is a positive constant that we will chose later. Using the fact that

$$0 = |\{x \in \Omega : w_1(x) < \vartheta(x)\}| = |\cap_{j=1}^{\infty} \{x \in \Omega : w_1(x) \leqslant \vartheta(x) + \frac{1}{j}\}|$$
$$= \lim_{j\to\infty} |\{x \in \Omega : w_1(x) \leqslant \vartheta(x) + \frac{1}{j}\}|,$$

we get the existence of $\delta_0 \equiv 1/j_0$ such that if $\delta < \delta_0$,

$$|\{x \in \Omega : w_1(x) \leqslant \vartheta(x) + \delta\}| \leqslant \varepsilon/2.$$

Thus we conclude that $|F_n| \leqslant \varepsilon/2$. Since $||\phi_n - w_0||_{L^2(\Omega)} \to 0$ as $n \to \infty$ we obtain that giving $\eta = \frac{\delta^2 \varepsilon}{2}$, for $n \geqslant n_0$,

$$\frac{\delta^2 \varepsilon}{2} \geqslant \int\limits_\Omega |v_n - \vartheta|^2 dx \geqslant \int_{E_n} |v_n - \vartheta|^2 dx \geqslant \delta^2 |E_n|.$$

Hence $|E_n| \leqslant \frac{\varepsilon}{2}$. As $S_n \subset F_n \cup E_n$, we conclude that $|S_n| \leqslant \varepsilon$ for $n \leqslant n_0$, then $|S_n| \to 0$ as $n \to \infty$.

If $Q(x,w) = H_{m,\alpha_0}(x,w^+) + \beta_{m,\alpha_0}(x) w_+$, then we have

$$\begin{aligned}
J_{m,\alpha_0}(v_n) &= \frac{1}{2} \int\limits_\Omega (|\nabla v_n(x)|^2 - \Lambda_N \frac{v_n^2}{|x|^2}) dx - \int\limits_\Omega Q(x, v_n) dx \\
&= \frac{1}{2} \int_{T_n} (|\nabla \phi_n(x)|^2 - \Lambda_N \frac{\phi_n^2}{|x|^2}) dx - \int_{T_n} Q(x, \phi_n) dx + \frac{1}{2} \int_{S_n} (|\nabla v_n(x)|^2 - \Lambda_N \frac{v_n^2}{|x|^2}) dx \\
&\quad - \int_{S_n} Q(x, v_n) dx + \frac{1}{2} \int\limits_\Omega (|\nabla v_n^-(x)|^2 - \Lambda_N \frac{(v_n^-)^2}{|x|^2}) dx \\
&= \frac{1}{2} \int_{T_n} (|\nabla \phi_n(x)|^2 - \Lambda_N \frac{\phi_n^2}{|x|^2}) dx - \int_{T_n} Q(x, \phi_n) dx + \frac{1}{2} \int_{S_n} (|\nabla (w_n + w_1)|^2 \\
&\quad - \Lambda_N \frac{(w_n + w_1)^2}{|x|^2}) dx - \int_{S_n} Q(x, w_n + w_1) dx + \frac{1}{2} \int\limits_\Omega (|\nabla v_n^-(x)|^2 - \Lambda_N \frac{(v_n^-)^2}{|x|^2}) dx.
\end{aligned}$$

Since

$$\int\limits_\Omega (|\nabla \phi_n(x)|^2 - \Lambda_N \frac{\phi_n^2}{|x|^2}) dx = \int_{T_n} (|\nabla \phi_n(x)|^2 - \Lambda_N \frac{\phi_n^2}{|x|^2}) dx + \int_{S_n} (|\nabla w_1|^2 - \Lambda_N \frac{w_1^2}{|x|^2}) dx,$$

$$\int\limits_\Omega Q(x, \phi_n) dx = \int_{T_n} Q(x, \phi_n) dx + \int_{S_n} Q(x, w_1) dx,$$

then using the fact that w_1 is a supersolution to (4.9) with $\alpha = \alpha_0$, we obtain that

$$\begin{aligned}
J_{m,\alpha_0}(v_n) &= J_{m,\alpha_0}(\phi_n) + \frac{1}{2} \int_{S_n} [(|\nabla (w_n + w_1)|^2 - \Lambda_N \frac{(w_n + w_1)^2}{|x|^2}) \\
&\quad - (|\nabla w_1|^2 - \Lambda_N \frac{w_1^2}{|x|^2})] dx - \int_{S_n} (Q(x, w_n + w_1) - Q(x, w_1)) dx \\
&\quad + \frac{1}{2} \int\limits_\Omega (|\nabla v_n^-(x)|^2 - \Lambda_N \frac{(v_n^-)^2}{|x|^2}) dx = J_{m,\alpha_0}(u_n) + \frac{1}{2} ||w_n||_H^2 + \frac{1}{2} ||(v_n)_-||_H^2 \\
&\quad - \int\limits_\Omega \Big\{ Q(x, w_n + w_1) - Q(x, w_1) - Q_u(x, w_1) w_n \Big\} dx
\end{aligned}$$

$$\geqslant J_{m,\alpha_0}(\vartheta)+\frac{1}{2}||w_n||_H^2+\frac{1}{2}||(v_n)_-||_H^2$$
$$-\int\limits_\Omega \Big\{Q(x,w_n+w_1)-Q(x,w_1)-Q_u(x,w_1)w_n\Big\}dx.$$

Since

$$Q(x,w_n+w_1)-Q(x,w_1)-Q_u(x,w_1)w_n$$
$$=H_{m,\alpha_0}(x,(w_n+w_1)^+)-H_{m,\alpha_0}(x,w_1)-h_{m,\alpha_0}(x,w_1)w_n,$$

using the properties of g, in particular the point *iii*), it follows that

$$\int\limits_\Omega \Big\{Q(x,w_n+w_1)-Q(x,w_1)-Q_u(x,w_1)w_n\Big\}dx=o(1)||w_n||^2_{H(\Omega)}.$$

Thus

$$J_{m,\alpha_0}(v_n)\geqslant J_{m,\alpha_0}(\vartheta)+\frac{1}{2}||w_n||^2_{H(\Omega)}(1-o(1))+\frac{1}{2}||(v_n)_-||^2_{H(\Omega)}$$
$$\equiv J_{m,\alpha_0}(\vartheta)+\frac{1}{2}||w_n||^2_{H(\Omega)}(1-o(1))+o(1).$$

Therefore we conclude that $J_{m,\alpha_0}(\vartheta)>J_{m,\alpha_0}(v_n)\geqslant J_{m,\alpha_0}(\vartheta)$ for $n>n_0$, which is a contradiction. Hence the result follows. □

4.2. The asymptotic linear case

In this subsection we will consider the resonance problem,

$$\begin{cases} Lu=f(x,u)+\mu \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega,\end{cases} \tag{4.21}$$

where $f(x,s)$ is such that $f(x,0)=0$ and

1) $(\lambda_k+\delta)|x|^{-2\beta}<\dfrac{f(x,t)-f(x,s)}{t-s}<(\lambda_{k+1}-\delta)|x|^{-2\beta}$ for all $t,s\in\mathbb{R},t\neq s,k\geqslant 0$,

2) $\lim\limits_{s\to-\infty}\dfrac{f(x,s)}{s}=\theta_-(x),\lim\limits_{s\to\infty}\dfrac{f(x,s)}{s}=\theta_+(x)$.

Recall that $Lu\equiv-\Delta u-\Lambda_N\dfrac{u}{|x|^2}$. Then we have the next existence result.

THEOREM 4.7. *Assume that f satisfies the hypotheses 1) and 2), then problem* (4.9) *has a solution for all $\mu\in\mathcal{M}_1(\Omega)$. In addition, if μ is a positive measure and $k=1$, then* (4.9) *has a minimal positive solution.*

Proof. From the above hypotheses on f, we get easily that

$$(\lambda_k+\delta)|x|^{-2\beta}\leqslant \theta_-(x),\theta_+(x)\leqslant (\lambda_{k+1}-\delta)|x|^{-2\beta}.$$

We follow by approximation. Let μ be a measure such that $\mu\in\mathcal{M}_1(\Omega)$, then we get the existence of a sequence $\{f_n\}\in L^\infty(\Omega)$ such that $||f_n||_{L^1}\leqslant C$, $f_n\to\mu$ in $\mathcal{M}(\Omega)$ and $\int\limits_\Omega |f_n||x|^{-a}dx\leqslant C$ for all n. Consider $w_n\in H(\Omega)$ the unique solution to the problem

$$\begin{cases} Lw_n = f(x,w_n)+f_n \text{ in } \Omega,\\ \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.22}$$

The existence and uniqueness of w_n follow by the hypothesis $1)$ on f. We claim that $\{w_n\}$ is bounded in $L^1(\Omega)$. We argue by contradiction. Assume that $||w_n||_{L^1}\to\infty$ and let $v_n=\frac{w_n}{||w_n||_{L^1}}$, then $||v_n||_1=1$ and v_n solves $Lv_n=\dfrac{f(x,w_n)}{w_n}v_n+\frac{f_n}{||w_n||}$ in Ω. We set

$$D_n(x)=\begin{cases} |x|^{2\beta}\dfrac{f(x,w_n(x))}{w_n(x)} & \text{if } \ w_n(x)\neq 0,\\ \\ f_s(x,0)|x|^{2\beta} & \text{if } \ w_n(x)=0. \end{cases}$$

It is clear that $\{D_n\}$ is bounded in $L^\infty(\Omega)$ and that $\lambda_k+\delta\leqslant D_n(x)\leqslant\lambda_{k+1}-\delta$ for all n. Hence we get the existence of $D_0(x)\in L^\infty(\Omega)$ such that $D_n\to D_0$ in L^∞ weak$*$ topology and $\lambda_k+\delta\leqslant D_0(x)\leqslant\lambda_{k+1}-\delta$. Then v_n is a solution to the problem

$$Lv_n=|x|^{-2\beta}D_n(x)v_n+\frac{f_n}{||w_n||}. \tag{4.23}$$

Since $\lambda_k+\delta\leqslant D_n(x)\leqslant\lambda_{k+1}-\delta$ for all n, we get the existence of $\phi_n\in H(\Omega)$, the unique solution to the problem

$$L\phi_n=|x|^{-2\beta}D_n(x)\phi_n+\frac{sign(v_n(x))}{|x|^2}.$$

Using the fact that $\beta<1$ and by the hypothesis on D_n, there results that $|\phi_n(x)|\leqslant C_0|x|^{-\frac{N-2}{2}}$ for all $n\in\mathbb{N}$. Using ϕ_n as a test function in (4.23), we obtain

$$\int\limits_\Omega\frac{|v_n|}{|x|^2}dx=\int\limits_\Omega\frac{f_n}{||w_n||_{L^1}}\phi_n dx.$$

It is clear that

$$|\int\limits_\Omega f_n\phi_n dx|\leqslant\int\limits_\Omega c_1|f_n||x|^{-\frac{N-2}{2}}dx+c_2\int\limits_\Omega |f_n|dx\leqslant C.$$

Therefore we have that $\int\limits_\Omega\dfrac{|v_n|}{|x|^2}dx\leqslant\dfrac{C|\Omega|}{||w_n||_{L^1}}\to 0$ as $n\to\infty$, a contradiction with the main normalization. Hence we conclude that $||w_n||_{L^1}\leqslant C$. Using the same test function as above, we conclude that $\{\frac{w_n}{|x|^2}\}$ is bounded in $L^1(\Omega)$. Hence using the classical theory of renormalized solutions we obtain that $\{w_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for all $q<\frac{N}{N-1}$, and then $w_n \rightharpoonup w_0$ weakly in $W_0^{1,q}(\Omega)$. Passing to the limit as $n\to\infty$, we obtain that w solves

$$Lw_0 = f(x,w_0)+\mu, w_0\in W_0^{1,q}(\Omega), \text{ for all } q<\frac{N}{N-1}.$$

Hence the result follows.

4.3. The antimaximum Principle

In this subsection we deal with the case $f(x,s)=\lambda|x|^{-2\beta}s$, namely we consider the problem

$$\begin{cases} Lu = \lambda|x|^{-2\beta}u+h(x) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{4.24}$$

where $h(x)\in L^r_k(\Omega)\subset H'(\Omega)$ is such that $h(x)\gneqq 0$. From the Fredholm Alternative Theorem, we get that:

i) If $\lambda\neq\lambda_i, \forall i$, being λ_i the eigenvectors associated to problem (2.2), then (4.24) has an unique solution for all h in $H'(\Omega)$.

ii) If λ is an eigenvalue of (2.2), then (4.24) has a solution if and only if $\int_\Omega h\phi=0$ for any eigenvector ϕ associated to λ, i.e, $h\in \text{Ker }(L-\lambda Id)^\perp$.

If $h(x)\geqslant 0, h(x)\not\equiv 0$, then $h\notin \text{Ker }(L-\lambda_1 Id)^\perp$ and there is no solution in $H(\Omega)$ to (4.24) with $\lambda=\lambda_1$.

THEOREM 4.8. (Anti-maximum Principle) *Assume that $h(x)$ is a nonnegative function such that $h\in L^r_k(\Omega), k=(N-2)(2-r)/(2r)$ and $r>N/2$. Suppose that $\lambda\neq\lambda_i,\forall i$ and let $u\in H(\Omega)$ be the unique solution to* (4.24)*, then*

i) if $\lambda<\lambda_1$, it follows that $u>0$ in Ω,

ii) there exists $\varepsilon(h)>0$ such that if $\lambda_1<\lambda<\lambda_1+\varepsilon(h)$, then $u<0$ in Ω.

Proof. Assume that $\lambda<\lambda_1$, then to prove $i)$ we take u^- as a test function in (4.24). Hence,

$$\langle Lu,u^-\rangle = -\Big(\int_\Omega |\nabla u_-|^2dx-\Lambda_N\int_\Omega \frac{u_-^2}{|x|^2}dx\Big) = -\lambda\int_\Omega |x|^{-2\beta}u_-^2dx+\int_\Omega h(x)u_-dx,$$

so $||u_-||^2_{H(\Omega)}\leqslant\lambda\int_\Omega |x|^{-2\beta}u_-^2dx$. Since $\lambda<\lambda_1$, we can conclude that $u^-=0$, so the proof is finished in this case.

We prove now $ii)$. We follow an argument by contradiction. Assume that for all $\varepsilon_n > 0$, there exists $u_n \in H(\Omega)$, a solution to the problem

$$\begin{cases} Lu_n = \lambda_n |x|^{-2\beta} u_n + h(x) \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\lambda_1 < \lambda_n < \lambda_1 + \varepsilon_n$ and $u_n(x_n) \geqslant 0$ for some points $x_n \in \Omega$. Without loss of generality, we can assume that $\varepsilon_n \to 0$ and $x_n \to \bar{x} \in \bar{\Omega}$, as $n \to \infty$.

Two cases are possible.

Case 1: If we suppose that $\{u_n\}$ is bounded in $H(\Omega)$, then we conclude that for some subsequence, $u_n \rightharpoonup u_0$ in $H(\Omega)$ and $u_n \to u_0$ in $L^2_\beta(\Omega)$. Therefore, u_0 is a solution to

$$\begin{cases} Lu_0 = \lambda_1 |x|^{-2\beta} u_0 + h(x) \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.25}$$

In fact, notice that $u_n \to u_0$ in $H(\Omega)$. Getting ϕ_1 as a test function in (4.25), it follows that:

$$\langle Lu_0, \phi_1 \rangle = \lambda_1 \int_\Omega |x|^{-2\beta} \phi_1 u_0 dx + \int_\Omega h\phi_1 dx,$$

so $\int_\Omega h\phi_1 dx = 0$, but $h \not\equiv 0$, $h \geqslant 0$, then we reach a contradiction.

Case 2: We suppose that $\|u_n\|_H \to \infty$. We consider $v_n = u_n / \|u_n\|_H$, then $v_n(x_n) \geqslant 0$. It follows that $v_n > 0$ in $\Omega_n \subset \Omega$ and since $\|v_n\|_H = 1$, then for some subsequence, $v_n \rightharpoonup v_0$ in $H(\Omega)$ and $v_n \to v_0$ in $L^2_\beta(\Omega)$. Furthermore,

$$Lv_n = \lambda_n |x|^{-2\beta} v_n + \frac{h}{\|u_n\|_H}, \tag{4.26}$$

and since $\dfrac{h}{\|u_n\|_H} \to 0$ in $H'(\Omega)$, then

$$\begin{cases} Lv_0 = \lambda_1 v_0 \text{ in } \Omega, \\ v_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

In fact, notice that $v_n \to v_0$ in $H(\Omega)$. From the simplicity of the first eigenvalue, we conclude that $v_0 = \alpha\phi_1$.

a) If $v_0 = 0$, then $v_n \rightharpoonup 0$ in $H(\Omega)$ and $v_n \to 0$ in $L^2_\beta(\Omega)$. Therefore,

$$1 = \|v_n\|_H^2 = \lambda_n \int_\Omega |x|^{-2\beta} v_n^2 dx + \int_\Omega \frac{v_n}{\|u_n\|_H} dx \to 0 \text{ in } H(\Omega),$$

a contradiction.

b) If $v_0 > 0$, then $v_n \rightharpoonup v_0 > 0$ in $H(\Omega)$, and $v_n \to v_0 > 0$ in $L^2_\beta(\Omega)$. We choose v_0 as a test function in (4.26) and then

$$(\lambda_1 - \lambda_n)\|u_n\|_H \int_\Omega |x|^{-2\beta} v_n v_0 dx = \int_\Omega h v_0 > 0.$$

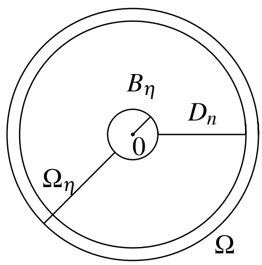
Since $(\lambda_1-\lambda_n)<0$ and thanks to the strong convergence in $L^2_\beta(\Omega)$, we conclude that

$$\int_\Omega |x|^{-2\beta} v_0^2 dx \leqslant 0,$$

so $v_0=0$, which is a contradiction.

c) If $v_0<0$, for every $\varepsilon>0$, we consider $\Omega_\eta=\Omega\setminus B_\eta(0)$, then $v_n\to v_0$ in $C^{1,\alpha}(\Omega_\eta)$ and $\dfrac{\partial v_n}{\partial \nu}\to\dfrac{\partial v_0}{\partial \nu}$ in $C^\alpha(\Omega_\eta)$. If $\bar{x}\in\Omega_\eta$, then $0\leqslant v_n(x_n)\to v(\bar{x})<0$, which is a contradiction. Hence we conclude in this case.

Assume that $\bar{x}\in\partial\Omega$. Let denote $\Omega_\eta\setminus D_n=\left\{x\in\Omega_\eta,\operatorname{dist}(x,\partial\Omega)<\dfrac{\eta}{n}\right\}$.



By Hopf Maximum Principle, there results that $\dfrac{\partial v_0}{\partial \nu}>0$ in $\partial\Omega$. Hence, by the continuity of $\dfrac{\partial v_n}{\partial \nu}$, we get the existence of $c>0$ and $n_0\in\mathbb{N}$ such that $\forall n\geqslant n_0$, we have $\dfrac{\partial v_n}{\partial \nu}>\dfrac{c}{2}>0$ in $\Omega_\eta\setminus D_n$.

Since $v_n=0$ on $\partial\Omega$, $v_n<0$ on ∂D_n and $v_n(x_n)\geqslant 0$, we get the existence of $\{P_n\}\in\Omega_\eta\setminus D_n$ such that $v_n(P_n)\geqslant 0$ and $\nabla v_n(P_n)=0$, which is a contradiction with $\dfrac{\partial v_n}{\partial \nu}>\dfrac{c}{2}$.

We consider now the case where $x_n\in B_\eta(0)$, then recall that $v_n(x_n)\geqslant 0$. We use the change of variables in (2.3) $w_n=|x|^a v_n$, then we obtain an equivalent problem in $W_0^{1,2}(|x|^{-(N-2)}dx)\cap L^\infty(\Omega)\cap C^1(\Omega)$,

$$\begin{cases} -\operatorname{div}(|x|^{-(N-2)}\nabla w_n)=\lambda_n|x|^{-N-2}w_n+|x|^{-\frac{N-2}{2}}\dfrac{h(x)}{\|u_n\|_H} \text{ in } \Omega,\\ w_n\geqslant 0 \text{ in } \Omega_n.\end{cases}$$

It follows that

$$-\operatorname{div}(|x|^{-(N-2)}\nabla w_1)=\lambda_1|x|^{-N-2}w_1,$$

where $w_1=|x|^a v_1<0$ is the first eigenvector associated to the eigenvalue problem in the space $W_0^{1,2}(|x|^{-(N-2)}dx)$. Moreover, $w_n\to w_1$ in $\mathscr{C}^{1,\alpha}(\Omega)$, in particular $w_n(x_n)\to w_1(x_n)<0$. Since $w_n(x_n)=|x_n|^a v_n(x_n)\geqslant 0$, we reach a contradiction. Hence the result follows. $\square$

5. Appendix

5.1. Semilinear problems

We consider the following problem in $H_\gamma(\Omega)$,

$$(P) \equiv \begin{cases} Lu = f(x,u) + |x|^{-2\beta} h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\beta < 1$, $h(x) \in L^r_k(\Omega) \subset H'(\Omega)$, $k = (N-2)(2-r)/(2r)$ with $r > N/2$, and $f(u,x) = |x|^{-2\beta} g(u)$ with $g \in \mathscr{C}^2(I\!R)$ is such that

1) $g(0) = 0$,

2) $g''(s) > 0$ in $I\!R$, namely, $g(s)$ is a convex function,

3) $\lim_{s\to-\infty} g'(s) = \delta'$, $\lim_{s\to+\infty} g'(s) = \delta''$ and $0 < \delta' < \lambda_1 < \delta'' < \lambda_2$.

REMARK 5.1. Since $g(s) \in C^2(I\!R)$ and $g'(s)$ is an increasing function, then

$$\lim_{s\to-\infty} g'(s) = \delta' \text{ and } \lim_{s\to\infty} g'(s) = \delta''$$

implies $0 < \delta' \leqslant g'(s) \leqslant \delta''$. Therefore, $g(s)$ is strictly increasing function and since $g(0) = 0$ we state that $sg(s) > 0$ for $s > 0$. Furthermore, by integration it follows easily that $\delta' s \leqslant g(s) \leqslant \delta'' s$ in $[0,s]$ and $\delta'' s \leqslant g(s) \leqslant \delta' s$ in $[s,0]$.

We begin by proving the next result.

THEOREM 5.2. *Assume that the above hypotheses on g hold, then problem has at most two positive solutions u_1, u_2 such that $u_1 < u_2$. Moveover if u is a solution, then $v = |x|^a u \in \mathscr{C}^\alpha(\bar{\Omega})$.*

Proof. Assume that problem (P) has two solutions u_1 and u_2. Consider $w = u_1 - u_2$, then w solves

$$Lw = f(x,u_1) - f(x,u_2) = |x|^{-2\beta} \frac{g(u_1) - g(u_2)}{u_1 - u_2} w \equiv \eta(x) w,$$

where $\eta(x) = \dfrac{g(u_1) - g(u_2)}{u_1 - u_2}$ if $u_1(x) \neq u_2(x)$ and $\eta(x) = g'(u_1(x))$ for $u_1(x) = u_2(x)$. From the hypotheses on g we conclude that $0 \leqslant \eta(x) < \lambda_2$. Since $w \neq 0$, then $\lambda_i(\eta(x)) = 1$ for some i. It is clear that $1 = \lambda_i(\eta(x)) > \lambda_i(\lambda_2) = \frac{\lambda_i}{\lambda_1}$. Hence it follows that $i = 1$ and then w has a fixed sign, therefore $u_1 < u_2$ in Ω.

Assume by contradiction that problem (P) has three solutions $u_1 < u_2 < u_3$. Consider $w_1 = u_2 - u_1$ and $w_2 = u_3 - u_2$, then

$$L(w_1) = \eta_1(x) w_1 \text{ and } L(w_2) = \eta_2(x) w_2.$$

Using the convexity of g it follows that $\eta_1(x) < \eta_2(x)$, hence we get a contradiction with the fact that $\lambda_1(\eta_1(x)) = \lambda_1(\eta_2(x))$. □

The next lemma will be systematically used in this section.

LEMMA 5.3. *Let z be a real function such that $|x|^a z \in \mathscr{C}^\alpha(\bar{\Omega})$, then for any h such that $|x|^a h \in \mathscr{C}^\alpha(\bar{\Omega})$, problem* (5.1) *has a subsolution u_1 such that $u \leqslant z$ in Ω. Moreover, $|x|^a u_1 \in \mathscr{C}^\alpha(\bar{\Omega})$.*

Proof. Let u any solution to problem (5.1). By setting $v(x) = |x|^a u$, $a = \frac{N-2}{2}$, it follows that

$$\begin{cases} -\operatorname{div}(|x|^{-(N-2)}\nabla v) = |x|^{-a-2\beta} g(|x|^{-a}v) + |x|^{-a-2\beta} h(x) \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.2}$$

It is clear that $v \in \mathscr{C}^\alpha(\bar{\Omega})$. To prove the lemma we have to show that for all $z_1(x) = |x|^a z(x)$ such that $z_1 \in \mathscr{C}^\alpha(\bar{\Omega})$, there exists a subsolution v_1 of (5.2) such that $v_1 \in \mathscr{C}^\alpha(\bar{\Omega})$ and $v_1 \leqslant z_1$.

Let $\bar{\lambda}$ and ε_0 fixed constants such that

$$\lim_{s \to -\infty} \frac{g(s)}{s} \leqslant \underline{\lambda} - \varepsilon_0 < \underline{\lambda} < \lambda_1. \tag{5.3}$$

For fixed $k > 0$, consider φ as the solution to the problem

$$-\operatorname{div}(|x|^{-(N-2)}\nabla \varphi) = \underline{\lambda}|x|^{-2a-2\beta}\varphi - \underline{\lambda}|x|^{-2a-2\beta}k \text{ in } \Omega, \varphi \in W_0^{1,2}(|x|^{-2a}dx, \Omega).$$

It is clear that $\varphi \in \mathscr{C}^\alpha(\bar{\Omega})$ and $\varphi < 0$ in Ω. We set $\underline{v} = \varphi - k$, then $\underline{v}$ solves

$$-\operatorname{div}(|x|^{-(N-2)}\nabla \underline{v}) = \underline{\lambda}|x|^{-2a-2\beta}\varphi - \underline{\lambda}|x|^{-2a-2\beta}k = \underline{\lambda}|x|^{-2a-2\beta}\underline{v} \text{ in } \Omega,\ \underline{v} = -k \text{ on } \partial\Omega.$$

Since $\underline{\lambda} < \lambda_1$, using the strong maximum principle we conclude that $\underline{v} \leqslant -k$. Since $z_1, |x|^a h \in \mathscr{C}^\alpha(\bar{\Omega})$, then we can choose k large enough such that

$$\underline{v} \leqslant z_1 \text{ and } |||x|^a h||_{\mathscr{C}^\alpha(\bar{\Omega})} \leqslant \varepsilon_0 k.$$

Using the main properties of g we get $\dfrac{g(|x|^{-a}\underline{v})}{|x|^{-a}\underline{v}} < \underline{\lambda} - \varepsilon_0$, hence we conclude that $g(|x|^{-a}\underline{v}) > \underline{\lambda}|x|^{-a}\underline{v} + \varepsilon_0 k|x|^{-a}$ and then

$$|x|^{-a-2\beta} g(|x|^{-a}\underline{v}) > \underline{\lambda}|x|^{-2a-2\beta}\underline{v} + \varepsilon_0 k|x|^{-2a-2\beta}.$$

Thus

$$\begin{aligned} -\operatorname{div}(|x|^{-(N-2)}\nabla \underline{v}) &= \underline{\lambda}|x|^{-2a-2\beta}\underline{v} \leqslant |x|^{-a-2\beta} g(|x|^{-a}\underline{v}) - \varepsilon_0 k|x|^{-2a-2\beta} \\ &\leqslant |x|^{-a-2\beta} g(|x|^{-a}\underline{v}) + |x|^{-a}h \text{ in } \Omega, \end{aligned}$$

and then the result follows. □

In the sequel, we will assume that $h \in \mathscr{C}^{\alpha}(\bar{\Omega})$. Define

$$N_1 = \{h \in \mathscr{C}^{\alpha}(\bar{\Omega}) : \text{ problem (5.1) has exactly one solution}\},$$
$$N_2 = \{h \in \mathscr{C}^{\alpha}(\bar{\Omega}) : \text{ problem (5.1) has two solutions}\},$$
$$N_3 = \{h \in \mathscr{C}^{\alpha}(\bar{\Omega}) : \text{ problem (5.1) has no solution}\}.$$

Let $N = N_1 \cup N_2$. We begin by proving the next lemma.

LEMMA 5.4. *Let* $\mathbf{M}$ *be a bounded set of* $\mathscr{C}^{\alpha}(\overline{\Omega})$*, then there exists* $\rho > 0$ *such that if* u *is a solution to problem* (5.1) *with* $h \in \mathbf{M}$*, then* $|||x|^a u||_{\mathscr{C}^{\alpha}(\overline{\Omega})} \leqslant \rho$.

Proof. As above, by setting $v = |x|^a u$, problem (5.1) is equivalent to the problem

$$-\operatorname{div}(|x|^{-(N-2)}\nabla v) = |x|^{-a-2\beta} g(|x|^{-a}v) + |x|^{-a-2\beta} h \text{ in } \Omega. \tag{5.4}$$

Therefore we have just to show that $||v||_{\mathscr{C}^{\alpha}(\overline{\Omega})} \leqslant \rho$ for all $h \in \mathbf{M}$. We claim the existence of $\underline{v} \in \mathscr{C}^{\alpha}(\overline{\Omega})$ such that for all solution to (5.4) with $h \in \mathbf{M}$, we have $v \geqslant \underline{v}$. Fixed $\underline{\lambda}$ as in (5.3), then

$$g(|x|^{-a}s) \geqslant \underline{\lambda}|x|^{-a}s - C|x|^{-a}.$$

Let $C_1 = \sup_{h \in \mathbf{M}} ||h||_{\mathscr{C}^{\alpha}(\overline{\Omega})}$. Thus

$$|x|^{-a-2\beta} g(|x|^{-a}v) + |x|^{-a-2\beta} h(x) \geqslant \underline{\lambda}|x|^{-2a-2\beta} v - C|x|^{-2a}(|x|^{-2\beta} + C_1).$$

Consider now $\widehat{v}$ the unique solution to the problem

$$-\operatorname{div}(|x|^{-(N-2)}\nabla \widehat{v}) = \underline{\lambda}|x|^{-2a-2\beta}\widehat{v} - C|x|^{-2a}(|x|^{-2\beta} + C_1).$$

Then $\widehat{v} \in \mathscr{C}^{\alpha}(\bar{\Omega})$ and since $\underline{\lambda} < \lambda_1$, it follows that $\widehat{v} \leqslant v$, where v is any solution to (5.4). Hence the claim follows. To prove the lemma we argue by contradiction. Suppose the existence of a sequence $\{h_n\} \subset \mathbf{M}$ with $||v_n||_{\mathscr{C}^{\alpha}(\overline{\Omega})} \to \infty$. We set $w_n = \dfrac{v_n}{||v_n||_{\mathscr{C}^{\alpha}(\overline{\Omega})}}$, then

$$-\operatorname{div}(|x|^{-(N-2)}\nabla w_n) = \frac{|x|^{-a-2\beta} g(|x|^{-a}v_n)}{||v_n||_{\mathscr{C}^{\alpha}(\overline{\Omega})}} + \frac{|x|^{-a}h}{||v_n||_{\mathscr{C}^{\alpha}(\overline{\Omega})}}. \tag{5.5}$$

Using the linear behavior of g we obtain that $\{w_n\}$ is bounded in

$$\mathscr{C}^{\alpha}(\overline{\Omega}) \cap W_0^{1,2}(|x|^{-(N-2)}dx, \Omega).$$

Hence classical regularity theory allows us to prove that $w_n \to w$ strongly in $\mathscr{C}^{\alpha}(\overline{\Omega})$. Thus $||w||_{\mathscr{C}^{\alpha}(\overline{\Omega})} = 1$. Since $v_n \geqslant \widehat{v}$, then $w \geqslant 0$. Consider φ_1 the first eigenfunction defined in Theorem 2.3. Using φ_1 as a test function in (5.5), we obtain that

$$\lambda_1 \int_\Omega |x|^{-2a-2\beta}\varphi_1 w_n dx = \int_\Omega \frac{|x|^{-a-2\beta} g(|x|^{-a} v_n)\varphi_1}{||v_n||_{\mathscr{C}^\alpha(\overline{\Omega})}} dx + \int_\Omega \frac{|x|^{-a} h\varphi_1}{||v_n||_{\mathscr{C}^\alpha(\overline{\Omega})}}$$
$$\geqslant \bar{\lambda} \int_\Omega |x|^{-2a-2\beta}\varphi_1 w_n dx + o(1).$$

Letting $n \to \infty$, it follows that

$$\lambda_1 \int_\Omega |x|^{-2a-2\beta}\varphi_1 w_n dx \geqslant \bar{\lambda} \int_\Omega |x|^{-2a-2\beta}\varphi_1 w dx,$$

which is a contradiction with the fact that $\bar{\lambda} > \lambda_1$ and that $w \gneqq 0$. Hence we get the desired result.

LEMMA 5.5. *Under the same hypotheses as in the previous lemmas, we obtain that N is a non bounded convex set with $N - U = N$, where $U = \{z \in \mathscr{C}^\alpha(\overline{\Omega}) : z \geqslant 0\}$.*

Proof. We begin by proving that N is a convex set. Let $h_1, h_2 \in N$ and let u_1, u_2, the corresponding solution to h_1 and h_2. Let $h = th_1 + (1-t)h_2$ where $0 \leqslant t \leqslant 1$, and define $\widetilde{u} = tu_1 + (1-t)u_2$. Since $f(x,\widetilde{u}) \leqslant |x|^{-2\beta}(t(g(u_1) + (1-t)g(u_2)) = tf(x,u_1) + (1-t)f(x,u_2)$, we conclude that

$$L(\widetilde{u}) = tL(u_1) + (1-t)L(u_2) \geqslant f(x,\widetilde{u}) + h.$$

Therefore $\widetilde{u}$ is a supersolution to problem (5.1). It is clear that $|x|^a u \in \mathscr{C}^\alpha(\overline{\Omega})$, then using the previous lemma, we get the existence of a subsolution $\widehat{u}$ to problem (5.1) with $\widehat{u} \leqslant \widetilde{u}$. The existence result follows using an iteration argument. It is clear that N is non empty. To see that, we consider $\phi \in \mathscr{C}_0^\infty(\Omega)$ such that $\phi = 0$ in $B_\eta(0) \subset\subset \Omega$. By setting $h = L\phi - f(x,\phi)$, it follows that $h \in N$, thus $N \neq \emptyset$. We prove now that $N - U = N$. Let $h^* \in N - U$, then $h^* = h_1 - z \leqslant h_1$ with $h_1 \in N$. Let u_1 be a solution to (5.1) with $h = h_1$. It is clear that u_1 is a supersolution to (5.1) with $h = h^*$. Using Lemma 5.3 we get the existence of a subsolution u_2 such that $u_2 \geqslant u_1$. Hence an iteration argument allows us to prove the existence of a solution to problem (5.1) with $h = h^*$. Thus $h^* \in N$. Therefore we conclude that N is not bounded. $\square$

As a consequence we get easily that $N_3 = \complement N$ and $N_3 - U = N_3$. We prove now the main result of this section.

THEOREM 5.6. *Under the above hypotheses on g, we have that N is a closed convex set, $N_2 = Int(N)$, $N_1 = \partial N$. N_3 is a convex open set and it is not bounded.*

Proof. We begin by proving that $N_3 \neq \emptyset$. We claim the existence of $C_0 > 0$ such that if $h \in \mathscr{C}^\alpha(\overline{\Omega})$ with $h(x) \geqslant C_0$, then problem (5.1) has no solution. To prove the claim, we use the main properties of g. Notice that

$$g(s) > \bar{\lambda} s - C_1 \text{ for all } s \in \mathbb{R},$$

$$g(s) > \underline{\lambda} s - C_1 \text{ for all } s \in I\!R,$$

where $\underline{\lambda} < \lambda_1 < \overline{\lambda}$. We set $C_0 = C_1$. Let $h \in \mathscr{C}^\alpha(\overline{\Omega})$ with $h(x) \geqslant C_0$. Assume by contradiction that $h \in N$. Let u be a solution to problem (5.1) corresponding to the above h. Then

$$Lu = f(x,u) + |x|^{-2\beta} h, u \in H(\Omega). \tag{5.6}$$

Using u_- as a test function in the above equation, we get

$$\begin{aligned} -\Big(\int_\Omega |\nabla u_-|^2 dx - \Lambda_N \int_\Omega \frac{u_-^2}{|x|^2} dx\Big) &= \int_\Omega g(x,u) u_- dx + \int_\Omega |x|^{-2\beta} h u_- dx \\ &\geqslant -\underline{\lambda} \int_\Omega |x|^{-2\beta} u_-^2 dx + \int_\Omega |x|^{-2\beta} u_-(h(x) - C_0) dx. \end{aligned}$$

Since $h(x) \geqslant C_0$, there results that

$$\int_\Omega |\nabla u_-|^2 dx - \Lambda_N \int_\Omega \frac{u_-^2}{|x|^2} dx \leqslant \underline{\lambda} \int_\Omega |x|^{-2\beta} u_-^2 dx.$$

Using the fact that $\underline{\lambda} < \lambda_1$, we conclude that $u_- \equiv 0$ and then $u \geqslant 0$. Taking φ_1 (the first eigenfunction defined in Theorem 2.3) as a test function in (5.6), it follows that

$$\begin{aligned} \lambda_1 \int_\Omega u \phi_1 |x|^{-2\beta} dx &= \int_\Omega |x|^{-2\beta} g(u) \phi_1 dx + \int_\Omega |x|^{-2\beta} h(x) \phi_1 dx \\ &\geqslant \overline{\lambda} \int_\Omega u \phi_1 |x|^{-2\beta} dx + \int_\Omega |x|^{-2\beta} u(h(x) - C) dx \geqslant \overline{\lambda} \int_\Omega u \phi_1 |x|^{-2\beta} dx. \end{aligned}$$

Since $u \gneqq 0$, we reach a contradiction. Thus $h \in N_3$ and then $N_3 \neq \emptyset$.

To complete the proof, we use classical Leray-Schauder topological degree, see [14].

For $h \in \mathscr{C}^\alpha(\overline{\Omega})$, we define $T(f) = w \in \mathscr{C}^\alpha(\overline{\Omega})$, the solution to the problem

$$-\operatorname{div}(|x|^{-(N-2)} \nabla w) = |x|^{-a-2\beta} h,\ w \in W_0^{1,2}(|x|^{-(N-2)} dx, \Omega).$$

It is clear that T is well defined as a linear operator from $\mathscr{C}^\alpha(\overline{\Omega})$ into $\mathscr{C}^\alpha(\overline{\Omega})$. We set $G(w) = T(g(|x|^{-a} w))$. It is not difficult to see that G is a compact operator from E in itself, where

$$E = \{w \in \mathscr{C}^\alpha(\overline{\Omega}) :\ w = 0 \text{ on } \partial\Omega\}.$$

Define now $\Psi(w) = w - G(w)$. Solving problem (5.2), and then problem (5.1), is equivalent to solve $\Psi(w) = T(h)$. Using Lemma 5.4, we know that for all $C > 0$, there exists $R(C)$ such that if $\Psi(w) = T(h)$ with $||h||_{\mathscr{C}^\alpha(\overline{\Omega})} \leqslant C$, then $||w||_{\mathscr{C}^\alpha(\overline{\Omega})} < R(C)$. Hence the topological degree $\deg(\Psi, B_R, Sh)$ is well defined and it is independent of R once we have $R \geqslant R(||h||_{\mathscr{C}^\alpha(\overline{\Omega})})$.

In the same way, it follows that $\deg(\Psi, B_R, Sh)$ is independent of h once we have $||h||_{\mathscr{C}^\alpha(\overline{\Omega})} \leqslant C_2$ and $R > R(C_2)$. Since $N_3 \neq \emptyset$, then for $h^* \in N_3$, we get $\deg(\Psi, B_R, Sh^*) = 0$. Hence there results that

$$\deg(\Psi, B_R, Sh) = 0,\ \forall h \in \mathscr{C}^\alpha(\overline{\Omega}),\ \forall R \geqslant R(||h||_{\mathscr{C}^\alpha(\overline{\Omega})}.$$

We prove now that $N_1 \subset \partial N$. We argue by contradiction. Let $h^* \in N_1$ with $h^* \in \text{int}(N)$, then for ε small, $h^* + \varepsilon \in \text{int}(N)$ and then we get the existence of a solution w_ε to the problem

$$\begin{aligned} &-\text{div}(|x|^{-(N-2)}\nabla w_\varepsilon) = |x|^{-a-2\beta} g(|x|^{-a} w_\varepsilon) + |x|^{-a-2\beta}(h^* + \varepsilon), \\ &w_\varepsilon \in W_0^{1,2}(|x|^{-(N-2)}dx, \Omega). \end{aligned}$$

Let w^* be the unique solution to (5.2) with $h = h^*$. It is clear that w_ε is a supersolution to the corresponding equation with h^*. Since $h^* \in N_1$, then necessary $w^* \leqslant w_\varepsilon$, otherwise we get the existence of second solution to problem (5.2) with $h = h^*$, which is a contradiction with the fact that $h^* \in N_1$. Hence $w^* \leqslant w_\varepsilon$ and by the strong maximum principle it follows that $w^* < w_\varepsilon$ in Ω. Set $\varpi = w_\varepsilon - w^*$, then ϖ solves

$$-\text{div}(|x|^{-(N-2)}\nabla \varpi) = |x|^{-2a-2\beta}\theta(x)\varpi + \varepsilon |x|^{-a-2\beta},$$

where

$$\theta(x) = \frac{g(|x|^{-a}w_\varepsilon) - g(|x|^{-a}w^*)}{|x|^{-a}(w_\varepsilon - w^*)}.$$

Hence we conclude that $\lambda_1(\theta(x)) > 1$. On the other hand, using the convexity of g, we obtain that $g'(|x|^{-a}w^*) < \theta(x)$ in Ω. Hence $1 < \lambda_1(\theta) < \lambda_1(g'(|x|^{-a}w^*))$. Therefore $I - G'(w^*)$ is invertible and then $G'(w^*)$ has no eigenvalue in $[0,1]$, thus $i(\Psi, B_R, Sh^*) = +1$, the topological index.

Since the global topological degree is 0, we conclude that problem (5.2) has a second solution corresponding to $h = h^*$, a contradiction with the fact that $h^* \in N_1$. Hence $N_1 \subset \partial N$.

We prove now that $N_2 \subset \text{Int}(N)$. Let $h_1 \in N_2$, then problem (5.1), with $h = h_1$, has two solutions u_1, u_2 such that $u_1 < u_2$ in Ω. Setting $w_i = |x|^a u_i$, $i = 1, 2$, we get the existence of two solutions $w_1 < w_2$ to problem (5.2). Let $w^* = w_2 - w_1$, then

$$-\text{div}(|x|^{-(N-2)}\nabla w^*) = |x|^{-2a-2\beta}\rho(x)w^*,$$

where

$$\rho(x) = \frac{g(|x|^{-a}w_2) - g(|x|^{-a}w_1)}{|x|^{-a}(w_2 - w_1)}.$$

Hence $\lambda_1(\rho(x)) = 1$. Using the properties of g it follows that

$$g'(|x|^{-a}w_2(x)) < \rho(x) < g'(|x|^{-a}w_1(x)) \text{ in } \Omega.$$

Thus we conclude

$$\lambda_1(g'(|x|^{-a}w_2(x))) < 1 < \lambda_1(g'(|x|^{-a}w_1(x))).$$

Now, since $g'(|x|^{-a}w_2(x)) < \lambda_2$ there results that $\lambda_2(g'(|x|^{-a}w_2(x))) > 1$. Hence $\Phi'(w_1)$ and $\Psi(w_2)$ are invertible. Therefore using the local Inversion Theorem, we conclude that $h^* \in \mathrm{Int}(N_2)$. Therefore, $N_2 \subset \mathrm{Int}(N)$.

Since $N = N_1 \cup N_2$, $N_1 \subset \partial N$ and $N_2 \subset \mathrm{Int}N$, we conclude that $N_2 = \mathrm{Int}N$ and $N_1 = \partial N$ and the result follows.

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Boumediene Abdellaoui
Dpartement de Mathmatiques
Universit Aboubekr Belkaïd
Tlemcen, Tlemcen 13000
Algeria
e-mail: boumediene.abdellaoui@uam.es

Ana Primo
Departamento de Matemáticas
U. Autonoma de Madrid
28049 Madrid
Spain
e-mail: ana.primo@uam.es

Tarik Mohamed Touaoula
Dpartement de Mathmatiques
Universit Aboubekr Belkaïd
Tlemcen, Tlemcen 13000
Algeria
e-mail: t_touaoula@yahoo.fr