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BLOW–UP PROPERTIES FOR PARABOLIC SYSTEMS WITH LOCALIZED NONLINEAR SOURCE

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Abstract. This paper deals with blow-up properties of solutions to a semilinear parabolic system with nonlinear localized source involved a product with local terms

$$u_t = \Delta u + \exp\{mu(x,t) + nv(x_0,t)\}, \quad v_t = \Delta v + \exp\{pu(x_0,t) + qv(x,t)\}$$

with homogeneous Dirichlet boundary conditions. We investigate the influence of localized sources and local terms on blow-up properties for this system, and prove that: (*i*) when $m,\ q \leqslant 0$ this system possesses uniform blow-up profiles, in other words, the localized terms play a leading role in the blow-up profile for this case; (*ii*) when $m,\ q > 0$, this system presents single point blow-up patterns, or say that local terms dominate localized terms in the blow-up profile. Moreover, the blow-up rate estimates in time and space are obtained, respectively.

1. Introduction

In this paper, we deal with the following problem

$$\begin{cases} u_t = \Delta u + \exp\{mu(x,t) + nv(x_0,t)\}, & x \in \Omega,\ t > 0, \\ v_t = \Delta v + \exp\{pu(x_0,t) + qv(x,t)\}, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p,\ n \geqslant 0$, $q,\ m \in \mathbb{R}$, $pn \neq 0$ and $x_0 \in \Omega$ is a fixed point, the initial data $u_0(x),\ v_0(x) \in C_0(\Omega)$ are non-negative and nontrivial functions.

Problem (1.1) describes a physical phenomenon where the reaction in a dynamic system is driven by the temperature at a single point involved a product with local terms see [1, 12]. Using the methods used in [4, 17] we know that (1.1) has a local non-negative solution, and that the comparison principle is true.

The blow-up properties of solution to the following single equation

$$\begin{cases} u_t = \Delta u + f(u(x_0(t),t)), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega \end{cases} \tag{1.2}$$

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have been discussed by many authors, see [2, 3, 17, 18, 19] and the references therein. In particular, Souplet [18] proved that if $f(u)=u^p$ with $p>1$, then

$$\lim_{t\to T}(T-t)^{1/(p-1)}u(x,t)=\lim_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_\infty=(p-1)^{-1/(p-1)} \tag{1.3}$$

uniformly on the compact subset of Ω, and if $f(u)=e^u$, then

$$\lim_{t\to T}|\ln(T-t)|^{-1}u(x,t)=\lim_{t\to T}|\ln(T-t)|^{-1}\|u(\cdot,t)\|_\infty=1 \tag{1.4}$$

uniformly on the compact subset of Ω, where T is the blow-up time of u.

In [11], Lin et. al. studied the blow-up properties of solutions to the parabolic system

$$\begin{array}{ll} u_t=\Delta u+e^{v(x_0,t)},\;\; v_t=\Delta v+e^{u(x_0,t)} & x\in\Omega,\ t>0,\\ u=v=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x)\geqslant 0,\;\; v(x,0)=v_0(x)\geqslant 0,\ x\in\Omega. & \end{array} \tag{1.5}$$

They first proved that the solution (u,v) of (1.5) blows up in finite T. And for a special case $\Omega=B(0,R)$ and $x_0=0$, they obtain the blow-up rate estimates

$$\begin{array}{l} -\ln(T-t)-v_0(0)\leqslant\sup u(\cdot,t)\leqslant C(1-\ln(T-t)),\quad t\in[0,T),\\ -\ln(T-t)-u_0(0)\leqslant\sup v(\cdot,t)\leqslant C(1-\ln(T-t)),\quad t\in[0,T). \end{array} \tag{1.6}$$

In [8], Li and Wang considered the following system

$$\begin{array}{ll} u_t=\Delta u+\exp\{mu(x_0,t)+nv(x_0,t)\}, & x\in\Omega,\ t>0,\\ v_t=\Delta v+\exp\{pu(x_0,t)+qv(x_0,t)\}, & x\in\Omega,\ t>0 \end{array} \tag{1.7}$$

with null Dirichlet boundary conditions and $m,\ n,\ p,\ q$ are positive constants. They obtain the necessary conditions and a sufficient condition for which two components blow up simultaneously and establish the uniform blow-up profiles in the interior. Their main results were stated as follows.

THEOREM A. (1) *Suppose that the initial data* $(u_0(x),v_0(x))$ *satisfies* $\Delta u_0(x)+\exp\{mu_0(x_0)+nv_0(x_0)\}\geqslant 0,\ \Delta v_0(x)+\exp\{pu_0(x_0)+qv_0(x_0)\}\geqslant 0$*: if u and v blow up simultaneously, then $p\geqslant m$ and $n\geqslant q$, or $p<m$ and $n<q$; if $p\geqslant m$ and $n\geqslant q$, then u and v blow up simultaneously.*

(2) *Under the assumptions of* (1)*, the following statements hold on any compact subset of* Ω*:*

(*i*) *If $p>m$ and $n>q$, then*

$$\lim_{t\to T}u(x,t)|\ln(T-t)|^{-1}=\frac{n-q}{np-mq},\quad \lim_{t\to T}v(x,t)|\ln(T-t)|^{-1}=\frac{p-m}{np-mq}.$$

(*ii*) *If $p>m$ and $n=q$, then*

$$\lim_{t\to T}u(x,t)\{\ln(|\ln(T-t)|)\}^{-1}=\frac{1}{p-m},\quad \lim_{t\to T}v(x,t)|\ln(T-t)|^{-1}=\frac{1}{q}.$$

(*iii*) *If* $p = m$ *and* $n > q$, *then*

$$\lim_{t\to T} u(x,t)|\ln(T-t)|^{-1} = \frac{1}{m}, \ \lim_{t\to T} v(x,t)\{\ln(|\ln(T-t)|)\}^{-1} = \frac{1}{n-q}.$$

(*iv*) *If* $p = m$ *and* $n = q$, *then*

$$\lim_{t\to T} u(x,t)|\ln(T-t)|^{-1} = \frac{1}{m+n}, \ \lim_{t\to T} v(x,t)|\ln(T-t)|^{-1} = \frac{1}{p+q}.$$

(*v*) *If* $p < m$ *and* $n < q$, *then*

$$\lim_{t\to T} u(x,t)|\ln(T-t)|^{-1} = \frac{q-n}{mq-np}, \ \lim_{t\to T} v(x,t)|\ln(T-t)|^{-1} = \frac{m-p}{mq-np}.$$

In [23], Zhao consider the following local problem

$$\begin{array}{ll} u_t = \Delta u + \lambda \exp\{mu(x,t) + nv(x,t)\}, & x \in \Omega, \ t > 0, \\ v_t = \Delta v + \mu \exp\{pu(x,t) + qv(x,t)\}, & x \in \Omega, \ t > 0, \\ u = v = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x), & x \in \Omega, \end{array} \tag{1.8}$$

where $\Omega = B(0;R) = \{x \in \mathbb{R}^N : |x| < R\}$, $u_0(x)$ and $v_0(x)$ are continuous nonnegative functions on $\overline{\Omega}$ vanishing on $\partial\Omega$; constants $\lambda, \mu > 0$ p, $n \geqslant 0$, q, $m \in \mathbb{R}$ and $pn \neq 0$. Their main results are states as follows.

THEOREM B. (1) *If* $(n-q)(np-mq) > 0$ *or* $(p-m)(np-mq) > 0$, *then the solution of problem (1.8) blows up in finite time.*

(2) *If* $(n-q)(np-mq) < 0$ *and* $(p-m)(np-mq) < 0$, *then the solution of problem (1.8) exists globally.*

In this paper, combining [8] and [23], we shall explore the influence of localized terms and local terms in the blow-up properties of system (1.1). The main methods of this paper is to extend Souplet's method [18] to problem (1.1) and establish the uniform blow-up profiles in the interior. Our main results are stated as follows.

THEOREM 1.1. *If* $(n-q)(np-mq) > 0$ *or* $(p-m)(np-mq) > 0$, *then the solution of problem (1.1) blows up in finite time.*

THEOREM 1.2. *Assume that* (u,v) *is a classical solution of (1.1), which blows up in finite time* T. *If* m, $q \leqslant 0$, *then the following statements hold uniformly on any compact subset of* Ω.

(*i*) *If* m, $q < 0$ *and* $pn - qm > 0$, *then*

$$\begin{array}{l} \lim_{t\to T} (T-t)^{\theta} e^{u(x,t)} = \theta^{\theta} (\sigma/\theta)^{n/(pn-qm)}, \\ \lim_{t\to T} (T-t)^{\sigma} e^{v(x,t)} = \sigma^{\sigma} (\theta/\sigma)^{p/(pn-qm)}, \end{array}$$

where $\sigma = (p-m)/(pn-qm)$, $\theta = (n-q)/(pn-qm)$.

(*ii*) *If* $m=0$ *and* $q<0$, *then*

$$\lim_{t\to T}|\ln(T-t)|^{-1}u(x,t)=(n-q)/(np),\ \lim_{t\to T}|\ln(T-t)|^{-1}v(x,t)=1/n.$$

(*iii*) *If* $m=q=0$, *then*

$$\lim_{t\to T}|\ln(T-t)|^{-1}u(x,t)=1/p,\ \lim_{t\to T}|\ln(T-t)|^{-1}v(x,t)=1/n.$$

(*iv*) *If* $m<0$ *and* $q=0$, *then*

$$\lim_{t\to T}|\ln(T-t)|^{-1}u(x,t)=1/p,\ \lim_{t\to T}|\ln(T-t)|^{-1}v(x,t)=(p-m)/(np).$$

REMARK 1.1. When $m\leqslant 0$ and $q\leqslant 0$, Theorem 1.2 shows that the localized terms $e^{pu(x_0,t)}$ and $e^{nv(x_0,t)}$ play a leading role in the blow-up profile.

REMARK 1.2. In the case when $m>0$ and $q\leqslant 0$, or $m\leqslant 0$ and $q>0$, we do not know how to deal with the blow-up properties of system (1.1).

Next we focus on the case m, $q>0$. Let us first introduce the following assumptions:

(H1) m, $q>0$ and $\Omega=B(0;R)$, $x_0=0$;

(H2) initial data $u_0(x)$, $v_0(x):\overline{B}(0;R)\to\mathbb{R}^1$ are nonnegative nontrivial, radially symmetric non-increasing functions and vanish on $\partial B(0;R)$;

(H3) the initial data $u_0(x)$ and $v_0(x)$ satisfy $\Delta u_0(x)+\exp\{mu_0(x)+nv_0(0)\}\geqslant 0$ and $\Delta v_0(x)+\exp\{pu_0(0)+qv_0(x)\}\geqslant 0$ in Ω.

REMARK 1.3. Under the assumption (H2) and $\Omega=B(0;R)$, the solution (u,v) of problem (1.1) is radially symmetric and non-increasing in x (see [5]). Therefore, $u(x,t)=u(r,t)$, $v(x,t)=v(r,t)$ and $u(0,t)=\max_{\overline{\Omega}}u(\cdot,t)$, $v(0,t)=\max_{\overline{\Omega}}v(\cdot,t)$.

THEOREM 1.3. *Let* $p\geqslant m>0$ *and* $n\geqslant q>0$, *and assumptions (H1)-(H3) hold. Let* (u,v) *be a classical solution to problem (1.1) in* $B(0;R)\times(0,T)$, *which blows up in finite time* T. *Then* u *and* v *must blow up simultaneously.*

THEOREM 1.4. *Under assumptions (H1)-(H3), if* (u,v) *is a classical solution to problem (1.1) in* $B(0;R)\times(0,T)$ *and* u, v *blow up simultaneously in finite time* T, *then the parameters* m, n, p *and* q *must satisfy: (a)* $p\geqslant m$ *and* $n\geqslant q$, *or (b)* $p<m$ *and* $n<q$.

THEOREM 1.5. *Let the assumptions (H1) and (H2) be satisfied. If* (u,v) *is a classical solution of problem (1.1) which blows up in finite time* T, *then* $x=0$ *is the only blow-up point.*

REMARK 1.4. When $m>0$ and $q>0$, Theorem 1.5 illustrates that the local terms $e^{mu(x,t)}$ and $e^{qv(x,t)}$ dominate the localized terms $e^{pu(x_0,t)}$ and $e^{nv(x_0,t)}$ in the blow-up profile.

When u and v blow up simultaneously, we may estimate the blow-up rate as follows.

THEOREM 1.6. *Under the condition of Theorem 1.4, there exist constants* $0 < c \leqslant C$ *such that the following statements hold for all* $0 \leqslant t < T$.

(*i*) *If* $p > m$ *and* $n > q$*, or* $p < m$ *and* $n < q$*, then*

$$\ln c(T-t)^{-\theta} \leqslant u(0,t) \leqslant \ln C(T-t)^{-\theta},$$

$$\ln c(T-t)^{-\sigma} \leqslant v(0,t) \leqslant \ln C(T-t)^{-\sigma},$$

where θ *and* σ *are defined in Theorem 1.2.*

(*ii*) *If* $p > m$ *and* $n = q$*, then*

$$c|\ln(T-t)| \leqslant \exp\{(p-m)u(0,t)\} \leqslant C|\ln(T-t)|,$$

$$c(T-t)^{-1} \leqslant \exp\{qv(0,t)\}\{v(0,t)\}^{\frac{p}{p-m}} \leqslant C(T-t)^{-1}.$$

(*iii*) *If* $p = m$ *and* $n > q$*, then,*

$$c(T-t)^{-1} \leqslant \exp\{mu(0,t)\}\{u(0,t)\}^{\frac{n}{n-q}} \leqslant C(T-t)^{-1},$$

$$c|\ln(T-t)| \leqslant \exp\{(n-q)v(0,t)\} \leqslant C|\ln(T-t)|.$$

(*iv*) *If* $p = m$ *and* $n = q$*, then*

$$c|\ln(T-t)| \leqslant u(0,t) \leqslant C|\ln(T-t)|,$$

$$c|\ln(T-t)| \leqslant v(0,t) \leqslant C|\ln(T-t)|.$$

REMARK 1.5. If $m = q$, $n = p$ and $u_0(x) = v_0(x)$, then the system (1.1) turns to a single equation. From Theorems 1.2-1.5, we draw a complete conclusion on the blow-up profiles. More precisely, the problem possesses uniformly blow-up profile if and only if $m \leqslant 0$.

Furthermore, for problem (1.1) with suitable initial data, its blow-up rate in space can be evaluated as follows.

THEOREM 1.7. *Let (H1) and (H2) be satisfied. Suppose further that there exists some constant* $c > 0$ *such that* $u_0'(r) \leqslant -cr$ *and* $v_0'(r) \leqslant -cr$ *in* $[0,R]$*. If the classical solution* (u,v) *of problem (1.1) blows up in finite time* T*, then*

$$u(r,t) \leqslant \ln(Cr^{-\alpha}), \;\; v(r,t) \leqslant \ln(Cr^{-\beta}), \quad (r,t) \in (0,R] \times [0,T)$$

holds for some constant $C > 0$ *and for any* $\alpha > 2/m$, $\beta > 2/q$.

REMARK 1.6. For the following coupled equations with the same initial and boundary condition as system (1.1)

$$u_t = \Delta u + \exp\{mu(0,t) + nv(x,t)\},\; v_t = \Delta v + \exp\{pu(x,t) + qv(0,t)\},\; x \in B(0;R),\; t > 0,$$

the assertion of Theorems 1.3-1.4 and 1.6 are still true only if we keep assumption (H2) and replace assumptions (H1) and (H3) by (A1) and (A3) respectively, (A1) The initial data $u_0(x)$ and $v_0(x)$ satisfy $\Delta u_0(x)+\exp\{mu_0(0)+nv_0(x)\}\geqslant 0$ and $\Delta v_0(x)+\exp\{pu_0(x)+qv(0)\}\geqslant 0$ in $B(0;R)$.

(A3) $n>0$ and $p>0$.

REMARK 1.7. A simple modification of our proofs, we can get the similar results (that is, replacing p with $p|\Omega|$ and q with $q|\Omega|$ in Theorems 1.2-1.7, where $|\Omega|$ is the measure of Ω) to the following nonlocal semilinear parabolic systems

$$\begin{cases} u_t=\Delta u+\exp\{mu(x,t)+n\int_\Omega v(x,t)\}, & x\in\Omega,\ t>0,\\ v_t=\Delta v+\exp\{p\int_\Omega u(x,t)+qv(x,t)\}, & x\in\Omega,\ t>0,\\ u=v=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.9}$$

Similarly, as Remark 1.6, we can consider the following problem

$$\begin{cases} u_t=\Delta u+\exp\{m\int_\Omega u(x,t)+nv(x,t)\}, & x\in\Omega,\ t>0,\\ v_t=\Delta v+\exp\{pu(x,t)+q\int_\Omega v(x,t)\}, & x\in\Omega,\ t>0,\\ u=v=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.10}$$

There are many known results about blow-up properties for parabolic equations, we refer to [6, 7, 8, 9, 10, 11, 13, 15, 16, 24] and the references therein. We remark a recent paper [22], in which, Xiang et. al. considered the following Cauchy problem with moving source

$$\begin{cases} u_t=\Delta u+\lambda\exp\{mu(x_0(t),t)+nv(x_0(t),t)\}, & x\in\mathbb{R}^N,\ t>0,\\ v_t=\Delta v+\mu\exp\{pu(x_0(t),t)+qv(x_0(t),t)\}, & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.11}$$

where $x_0:\mathbb{R}^+\to\mathbb{R}^N$ is Hölder continuous, and $\lambda,\ \mu>0$, $m,\ n,\ p,\ q$ are constants with $pn>0$. They first give the blow-up criterion, and then deal with the possibilities of simultaneous blow-up or non-simultaneous blow-up under some suitable assumptions. Moreover, when simultaneous blow-up occurs, they also establish the precise blow-up rate estimates.

This paper is organized as follows. In the next section, we consider the uniform blow-up profile to problem (1.1) and prove Theorems 1.1 and 1.2. In Section 3, we prove Theorems 1.3-1.7.

2. Proofs of Theorems 1.1 and 1.2

In this section, we investigate the blow-up profiles of (1.1) and prove Theorem 1.1 and 1.2.

2.1. Proof of Theorem 1.1

We use the results of [23] (see Theorem B) and comparison principle (see [14] and [15]) to prove Theorem 1.1. Without loss of generality we may assume that

$u_0(x),\ v_0(x) > 0$ in Ω. And hence $u,\ v > 0$ in $\Omega \times [0,T)$, T being the maximal existence time of (u,v).

Let $B(x_0,d)$ be the ball centered at x_0 with radius $d > 0$ such that $B(x_0,d) \subset \Omega$. Let $(\underline{u},\underline{v})$ be the solution of the auxiliary problem

$$\begin{array}{ll} \underline{u}_t = \Delta\underline{u} + \exp\{m\underline{u}(x,t) + n\underline{v}(x,t)\}, & x \in B(x_0,d),\ t > 0, \\ \underline{v}_t = \Delta\underline{v} + \exp\{p\underline{u}(x,t) + q\underline{v}(x,t)\}, & x \in B(x_0,d),\ t > 0, \\ \underline{u} = \underline{v} = 0, & x \in \partial B(x_0,d),\ t > 0, \\ \underline{u}(x,0) = \underline{u}_0(x),\ \ \underline{v}(x,0) = \underline{v}_0(x), & x \in B(x_0,d), \end{array} \tag{2.1}$$

where $\underline{u}_0(x)$ and $\underline{v}_0(x)$ are non-negative smooth symmetric, radially non-increasing functions which are less than $u_0(x)$ and $v_0(x)$ on $B(x_0,d)$ respectively. Then $\underline{u}(\cdot,t)$ and $\underline{v}(\cdot,t)$ are radially symmetric non-increasing. By the comparison principle, $u \geqslant \underline{u}$, $v \geqslant \underline{v}$ as long as (u,v) and $(\underline{u},\underline{v})$ exist. By Theorem B, $(\underline{u},\underline{v})$ blows up in finite time, and so does (u,v). The proof of Theorem 1.1 is complete. □

2.2. Proof of Theorem 1.2

In what follows we intend to verify Theorem 1.2. For convenience, denote

$$f(t) = e^{pu(x_0,t)},\ F(t) = \int_0^t f(s)ds,\ g(t) = e^{nv(x_0,t)},\ G(t) = \int_0^t g(s)ds. \tag{2.2}$$

Before we prove Theorem 1.2, we claim that if (u,v) is a classical solutions of system (1.1) which blows up in finite time T, that is

$$\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \to \infty \ \text{ as } \ t \to T, \tag{2.3}$$

then u and v blow up simultaneously if $m \leqslant 0$ and $q \leqslant 0$. In fact, we have

LEMMA 2.1. *Let f, F, g, and G be the functions defined in (2.2). Assume that (u,v) is a classic solution of problem (1.1), which blows up in finite time T. Let $m \leqslant 0$ and $q \leqslant 0$, then*

$$\lim_{t\to T} g(t) = \lim_{t\to T} G(t) = \infty,\ \ \lim_{t\to T} f(t) = \lim_{t\to T} F(t) = \infty. \tag{2.4}$$

Moreover, u and v blow up simultaneously.

Proof. Since (u,v) blows up in finite time T, it can be deduced that

$$\|u(\cdot,t)\|_\infty \to \infty \ \text{ or } \ \|v(\cdot,t)\|_\infty \to \infty \ \text{ as } t \to T.$$

Without loss of generality we may assume $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to T$. Suppose in the contrary, that $\lim_{t\to T} g(t) < \infty$. So, from the equation of u in system (1.1), we know that u exists globally, since $m \leqslant 0$, this is a contradiction. Therefore, $\lim_{t\to T} g(t) = \infty$.

Combining $\lim_{t\to T} g(t) = \infty$ and $g(t) = e^{nv(x_0,t)}$ yields that $v(x_0,t) \to \infty$ as $t \to T$. Namely, u and v blow up simultaneously.

Next, we infer that $\lim_{t\to T} G(t) = \infty$. Set $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, then $U(t)$ is Lipschitz continuous and

$$U'(t) \leqslant e^{mU(t)} g(t) \text{ a.e. in } [0,T). \tag{2.5}$$

By integrating (2.5) we get

$$\begin{array}{ll} -\frac{1}{m}e^{-mU(t)} \leqslant \int_0^t g(s) - \frac{1}{m}e^{-mU(0)} = G(t) - \frac{1}{m}e^{-mU(0)} & \text{if } m<0, \\ U(t) \leqslant \int_0^t g(s) + U(0) = G(t) + U(0) & \text{if } m=0 \end{array}$$

from $\lim_{t\to T} U(t) = \infty$, it follows that $\lim_{t\to T} G(t) = \infty$.

Furthermore, because if $\lim_{t\to T} \|v(\cdot,t)\| = \infty$ which was showed above, applying similar arguments as above to the equation of v in system (1.1), it is reasonable that $\lim_{t\to T} f(t) = \infty$ and $\lim_{t\to T} F(t) = \infty$. The proof of Lemma 2.1 is complete. □

LEMMA 2.2. *Let f, F, g, and G be the functions defined in (2.2). Under the conditions of Theorem 1.2, the following statements hold uniformly on any compact subset of Ω.*

(*i*) *If $m<0$ and $q<0$ then $u(x,t) \sim -\frac{1}{m}\ln[-mG(t)]$, $v(x,t) \sim -\frac{1}{q}\ln[-qF(t)]$.*

(*ii*) *If $m=0$ and $q<0$ then $u(x,t) \sim G(t)$, $v(x,t) \sim -\frac{1}{q}\ln[-qF(t)]$.*

(*iii*) *If $m=q=0$ then $u(x,t) \sim G(t)$, $v(x,t) \sim F(t)$.*

(*iv*) *If $m<0$ and $q=0$ then $u(x,t) \sim -\frac{1}{m}\ln[-mG(t)]$, $v(x,t) \sim F(t)$,*

here the notation $u \sim v$ means $u/v \to 1$ as $t \to T$.

Proof. (*i*) Let $m<0$ and $q<0$. Direct computations demonstrate

$$\begin{array}{l} -\frac{1}{m}\frac{d}{dt}e^{-mu} = -\frac{1}{m}\Delta e^{-mu} + \frac{1}{m}e^{mu}|\nabla e^{-mu}|^2 + g(t), \\ -\frac{1}{q}\frac{d}{dt}e^{-qv} = -\frac{1}{q}\Delta e^{-qv} + \frac{1}{q}e^{qv}|\nabla e^{-qv}|^2 + f(t). \end{array} \tag{2.6}$$

Since $e^{mu} \leqslant 1$ and $e^{qv} \leqslant 1$ for $u,\ v \geqslant 0$ and $m,\ q<0$, we have

$$\begin{array}{l} -\frac{1}{m}\frac{d}{dt}e^{-mu} \geqslant -\frac{1}{m}\Delta e^{-mu} + m|\nabla(-\frac{1}{m}e^{-mu})|^2 + g(t), \\ -\frac{1}{q}\frac{d}{dt}e^{-qv} \geqslant -\frac{1}{q}\Delta e^{-qv} + q|\nabla(-\frac{1}{q}e^{-qv})|^2 + f(t). \end{array} \tag{2.7}$$

Consequently, $(-\frac{1}{m}e^{-mu}, -\frac{1}{q}e^{-qv})$ is a super-solution of the problem below

$$\begin{array}{ll} w_t = \Delta w + m|\nabla w|^2 + g(t),\ \ z_t = \Delta z + q|\nabla z|^2 + f(t), & (x,t) \in \Omega \times (0,T), \\ w = z = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,0) = -\frac{1}{m}e^{-mu_0(x)},\ \ z(x,0) = -\frac{1}{q}e^{-qv_0(x)}, & x \in \Omega, \end{array} \tag{2.8}$$

where $g(t) = e^{nv(x_0,t)}$, $f(t) = e^{pu(x_0,t)}$. A simple modification of Theorem 4.1 in [18] asserts that uniformly on any compact subset of Ω hold

$$\lim_{t\to T}\frac{w(x,t)}{G(t)} = \lim_{t\to T}\frac{\|w(\cdot,t)\|_\infty}{G(t)} = 1,\ \ \lim_{t\to T}\frac{z(x,t)}{F(t)} = \lim_{t\to T}\frac{\|z(\cdot,t)\|_\infty}{F(t)} = 1. \tag{2.9}$$

By comparison methods, we obtain that

$$-\frac{1}{m}e^{-mu} \geqslant w(x,t), \quad -\frac{1}{q}e^{-qv} \geqslant z(x,t), \quad (x,t) \in \Omega \times [0,T). \tag{2.10}$$

Hence, from (2.9) it follows that, uniformly on any compact subset of Ω holds

$$\begin{array}{ll} \liminf_{t\to T} \frac{e^{-mu(x,t)}}{-mG(t)} \geqslant 1, & \liminf_{t\to T} \frac{e^{-qv(x,t)}}{-qF(t)} \geqslant 1, \\ \liminf_{t\to T} \frac{\|e^{-mu(\cdot,t)}\|_\infty}{-mG(t)} \geqslant 1, & \liminf_{t\to T} \frac{\|e^{-qv(\cdot,t)}\|_\infty}{-qF(t)} \geqslant 1. \end{array} \tag{2.11}$$

On the other hand, we know that $U'(t) \leqslant e^{mU(t)}g(t)$ and $V'(t) \leqslant e^{qV(t)}f(t)$ a.e. in $[0,T)$, where $U(t) = \max_{x\in\overline{\Omega}} u(x,t)$, $V(t) = \max_{x\in\overline{\Omega}} v(x,t)$. In view of $\lim_{t\to T} F(t) = \lim_{t\to T} G(t) = \infty$ and $m,\ q < 0$, we see that

$$\limsup_{t\to T} \frac{e^{-mU(t)}}{-mG(t)} \leqslant 1, \ \limsup_{t\to T} \frac{e^{-qV(t)}}{-qF(t)} \leqslant 1. \tag{2.12}$$

So, (2.11) and (2.12) guarantee that, uniformly in any compact subset of Ω

$$\lim_{t\to T} \frac{e^{-mu}}{-mG(t)} = \lim_{t\to T} \frac{\|e^{-mu(\cdot,t)}\|_\infty}{-mG(t)} = 1, \ \lim_{t\to T} \frac{e^{-qv}}{-qF(t)} = \lim_{t\to T} \frac{\|e^{-qv(\cdot,t)}\|_\infty}{-qF(t)} = 1. \tag{2.13}$$

(*ii*) Let $m = 0$ and $q < 1$. Analogous to case (*i*), we find that $(u, -\frac{1}{q}e^{-qv})$ is a super-solution of (2.8) with $w(x,0) = u_0(x)$, $z(x,0) = -\frac{1}{q}e^{-qv_0(x)}$. Proceeding as case (*i*) we arrive at the corresponding conclusion.

Cases (*iii*) and (*iv*) can be treated similarly. The proof of Lemma 2.2 is complete. □

LEMMA 2.3. *Let f, F, g, and G be the functions defined in (2.2). Under the assumption of Theorem 1.2, for any given constants δ, ε and τ satisfying $0 < \delta,\ \varepsilon < 1$ and $\tau > 1$, there exists $\widetilde{T}$ such that for all $t \in [\widetilde{T}, T)$, the following statements hold.*

(*i*) *If $m < 0$ and $q < 0$ then*

$$\begin{aligned} \varepsilon\delta^{-\frac{n}{q}}(p-m)[-qF(t)]^{\frac{n-q}{-q}} &\leqslant \tau^{-\frac{p}{m}}(n-q)[-mG(t)]^{\frac{p-m}{-m}}, \\ \varepsilon\delta^{-\frac{p}{m}}(n-q)[-mG(t)]^{\frac{p-m}{-m}} &\leqslant \tau^{-\frac{n}{q}}(p-m)[-qF(t)]^{\frac{n-q}{-q}}. \end{aligned}$$

(*ii*) *If $m = 0$ and $q < 0$ then*

$$\varepsilon\delta^{-\frac{n}{q}}p[-qF(t)]^{\frac{n-q}{-q}} \leqslant \tau(n-q)e^{pG(t)}, \ \ \delta(n-q)e^{pG(t)} \leqslant \varepsilon^{-1}\tau^{-\frac{n}{q}}p[-qF(t)]^{\frac{n-q}{-q}}.$$

(*iii*) *If $m = q = 0$ then*

$$\varepsilon\delta pe^{nF(t)} \leqslant \tau ne^{pG(t)}, \quad \delta ne^{pG(t)} \leqslant \varepsilon^{-1}\tau pe^{nF(t)}.$$

(*iv*) *If $m < 0$ and $q = 0$ then*

$$\delta(p-m)e^{nF(t)} \leqslant \varepsilon^{-1}\tau^{-\frac{p}{m}}n[-mG(t)]^{\frac{p-m}{-m}}, \ \ \varepsilon\delta^{-\frac{p}{m}}n[-mG(t)]^{\frac{p-m}{-m}} \leqslant \tau(p-m)e^{nF(t)}.$$

Proof. (i) If $m<0$ and $q<0$. Observe that $F'(t)=f(t)=e^{pu(x_0,t)}$ and $G'(t)=g(t)=e^{nv(x_0,t)}$. By (i) of Lemma 2.2, we know that for chosen positive constants $\delta<1<\tau$, there exists $t_0<T$ such that

$$\begin{aligned} &[-\delta mG(t)]^{p/(-m)} \leqslant F'(t) \leqslant [-\tau mG(t)]^{p/(-m)}, \quad t\in[t_0,T),\\ &[-\delta qF(t)]^{n/(-q)} \leqslant G'(t) \leqslant [-\tau qF(t)]^{n/(-q)}, \qquad t\in[t_0,T). \end{aligned} \tag{2.14}$$

And thus,

$$\frac{[-\delta mG(t)]^{p/(-m)}}{[-\tau qF(t)]^{n/(-q)}} \leqslant \frac{dF}{dG} \leqslant \frac{[-\tau mG(t)]^{p/(-m)}}{[-\delta qF(t)]^{n/(-q)}}, \ t\in[t_0,T). \tag{2.15}$$

In view of the right-hand side of (2.15)

$$[-\delta qF(t)]^{n/(-q)}dF \leqslant [-\tau mG(t)]^{p/(-m)}dG, \ t\in[t_0,T). \tag{2.16}$$

Integrating the above inequalities yields that for $t_0\leqslant t<T$,

$$\begin{aligned} \frac{q(-\delta q)^{n/(-q)}}{q-n}F^{\frac{q-n}{q}}(s)\Big|_{t_0}^{t} &\leqslant \frac{m(-\tau m)^{p/(-m)}}{m-p}G^{\frac{m-p}{m}}(s)\Big|_{t_0}^{t}\\ &\leqslant \frac{m(-\tau m)^{p/(-m)}}{m-p}G^{\frac{m-p}{m}}(t). \end{aligned} \tag{2.16}$$

Due to $\lim_{t\to T}F(t)=\infty$ and $q<0$, for given constant $0<\varepsilon<1$, there exists $\widetilde{t_0}: t_0\leqslant \widetilde{t_0}<T$ such that $F^{(q-n)/q}(t_0)\leqslant(1-\varepsilon)F^{(q-n)/q}(t)$ for all $t\in[\widetilde{t_0},T)$. Hence from (2.16) it can be deduced that for $\widetilde{t_0}\leqslant t<T$

$$\varepsilon\delta^{-\frac{n}{q}}(p-m)[-qF(t)]^{\frac{q-n}{q}} \leqslant \tau^{-\frac{p}{m}}(n-q)[-mG(t)]^{\frac{m-p}{m}}. \tag{2.17}$$

Application of similar analysis as above to the left-hand side of (2.15) guarantees that there exists $t_0^*<T$ such that for $t_0^*\leqslant t<T$,

$$\varepsilon\delta^{-\frac{p}{m}}(n-q)[-mG(t)]^{\frac{m-p}{m}} \leqslant \tau^{-\frac{n}{q}}(p-m)[-qF(t)]^{\frac{q-n}{q}}. \tag{2.18}$$

Set $\widetilde{T}=\max\{\widetilde{t_0},t_0^*\}$, then (2.17) and (2.18) ensures (i) of Lemma 2.3. Analogous of case (i), we can draw other conclusions of Lemma 2.3. The proof of Lemma 2.3 is complete. □

Proof of Theorem 1.2. Choose $\{\delta_i\}_{i=1}^{\infty}$, $\{\varepsilon_i\}_{i=1}^{\infty}$, $\{\tau_i\}_{i=1}^{\infty}$ satisfying $0<\delta_i,\ \varepsilon_i<1$ and $\tau_i>1$ with $\delta_i,\ \varepsilon_i,\ \tau_i\to 1$ as $i\to\infty$. Putting $(\delta,\varepsilon,\tau)=(\delta_i,\varepsilon_i,\tau_i)$ in Lemma 2.3, we get $\widetilde{T_i}<T$ such that the corresponding (i)-(iv) of Lemma 2.3 hold for all $\widetilde{T_i}\leqslant t<T$.

(i) Let $m<0$ and $q<0$. From (i) of Lemma 2.2 if follows that for such sequences $\{\delta_i\}_{i=1}^{\infty}$ and $\{\tau_i\}_{i=1}^{\infty}$, there exists $\{t_i\}_{i=1}^{\infty}: t_i<T$ with $t_i\to T$ as $i\to\infty$ such that

$$-\frac{1}{m}\ln[-\delta_i mG(t)] \leqslant u(x_0,t) \leqslant -\frac{1}{m}\ln[-\tau_i mG(t)], \ t_i\leqslant t<T. \tag{2.19}$$

Denote $T_i^*=\max\{t_i,\widetilde{T_i}\}$, then (2.19) and ($i$) of Lemma 2.3 assert that for $T_i^*\leqslant t<T$,

$$\begin{aligned} F'(t) &\geqslant \delta_i^{p/(-m)}[-mG(t)]^{p/(-m)}\\ &\geqslant \delta_i^{-\frac{p\theta}{q\sigma}}(\delta_i/\tau_i)^{-\frac{p^2}{m(p-m)}}(\varepsilon_i\sigma/\theta)^{\frac{p}{p-m}}[-qF(t)]^{-\frac{p\theta}{q\sigma}}, \end{aligned} \tag{2.20}$$

$$F'(t) \leqslant \tau_i^{-\frac{p\theta}{q\sigma}}(\tau_i/\delta_i)^{-\frac{p^2}{m(p-m)}}(\sigma/(\varepsilon_i\theta))^{\frac{p}{p-m}}[-qF(t)]^{-\frac{p\theta}{q\sigma}}. \tag{2.21}$$

Notice that $1+\frac{p\theta}{q\sigma}=\frac{pn-qm}{q(p-m)}=\frac{1}{q\sigma}<0$ if $pn>qm$. Integrating (2.20) and (2.21) from t to T and using of $\lim_{t\to T}F(t)=\infty$, we obtain that, for $T_i^*\leqslant t<T$,

$$C_i^{-1}\sigma(\sigma/\theta)^{-\frac{p}{p-m}} \leqslant (T-t)[-qF(t)]^{-\frac{1}{q\sigma}} \leqslant c_i^{-1}\sigma(\sigma/\theta)^{-\frac{p}{p-m}}, \tag{2.22}$$

where

$$c_i=\delta_i^{-\frac{p\theta}{q\sigma}}(\delta_i/\tau_i)^{-\frac{p^2}{m(p-m)}}\varepsilon_i^{\frac{p}{p-m}},\ \ C_i=\tau_i^{-\frac{p\theta}{q\sigma}}(\tau_i/\delta_i)^{-\frac{p^2}{m(p-m)}}\varepsilon_i^{-\frac{p}{p-m}}.$$

Since $c_i\to 1$ and $C_i\to 1$ on the account of $\delta_i,\ \varepsilon_i,\ \tau_i\to 1$ and $T_i^*\to T$ as $i\to\infty$, by letting $i\to\infty$ in (2.22) we find

$$[-qF(t)]^{1/(-q)} \sim \sigma^{\sigma}(\theta/\sigma)^{p/(pn-qm)}(T-t)^{-\sigma}. \tag{2.23}$$

Similar as above, it can be inferred that

$$[-mG(t)]^{1/(-m)} \sim \theta^{\theta}(\sigma/\theta)^{n/(pn-qm)}(T-t)^{-\theta}. \tag{2.24}$$

From (i) of Lemma 2.2, (2.23) and (2.24), we know that

$$(T-t)^{\theta}e^{u(x,t)} \sim \theta^{\theta}(\sigma/\theta)^{n/(pn-qm)},\ \ (T-t)^{\sigma}e^{v(x,t)} \sim \sigma^{\sigma}(\theta/\sigma)^{p/(pn-qm)} \tag{2.25}$$

uniformly on any compact subset of Ω. That is, uniformly on any compact subset of Ω holds

$$\lim_{t\to T}(T-t)^{\theta}e^{u(x,t)}=\theta^{\theta}(\sigma/\theta)^{n/(pn-qm)},\ \ \lim_{t\to T}(T-t)^{\sigma}e^{v(x,t)}=\sigma^{\sigma}(\theta/\sigma)^{p/(pn-qm)}. \tag{2.26}$$

(ii) Let $m=0$ and $q<0$. Analogous as the proof of case (i), it follows from (ii) of Lemma 2.2 and (iii) of Lemma 2.3 that for $T_i^*\leqslant t<T$

$$\begin{aligned} G'(t) &\geqslant \delta_i^{n/(-q)}[-qF(t)]^{n/(-q)} \\ &\geqslant \delta_i^{n/(-q)}[\varepsilon_i\delta_i\tau_i^{n/q}p^{-1}(n-q)]^{\frac{n}{n-q}}\exp\{\tfrac{np}{n-q}G(t)\} \\ &= \delta_i^{\frac{n(n-2q)}{-q(n-q)}}\tau_i^{\frac{n^2}{q(n-q)}}[\varepsilon_i p^{-1}(n-q)]^{\frac{n}{n-q}}\exp\{\tfrac{np}{n-q}G(t)\}, \\ G'(t) &\leqslant \tau_i^{\frac{n(n-2q)}{-q(n-q)}}\delta_i^{\frac{n^2}{q(n-q)}}[(\varepsilon_i p)^{-1}(n-q)]^{\frac{n}{n-q}}\exp\{\tfrac{np}{n-q}G(t)\}. \end{aligned} \tag{2.27}$$

And hence, for $T_i^*\leqslant t\leqslant T$

$$\begin{aligned} \exp\{-\tfrac{np}{n-q}G(t)\}G'(t) &\geqslant \delta_i^{\frac{n(n-2q)}{-q(n-q)}}\tau_i^{\frac{n^2}{q(n-q)}}[\varepsilon_i p^{-1}(n-q)]^{\frac{n}{n-q}}, \\ \exp\{-\tfrac{np}{n-q}G(t)\}G'(t) &\leqslant \tau_i^{\frac{n(n-2q)}{-q(n-q)}}\delta_i^{\frac{n^2}{q(n-q)}}[(\varepsilon_i p)^{-1}(n-q)]^{\frac{n}{n-q}}. \end{aligned} \tag{2.28}$$

Define $A=-\ln(np)-\frac{q}{n-q}\ln(n-q)$. Integrating (2.28) from t to T and using $\lim_{t\to T}G(t)=\infty$, we deduce that for $t\in[T_i^*,T)$,

$$\widehat{c_i}+|\ln(T-t)| \leqslant \frac{np}{n-q}G(t) \leqslant \widehat{C_i}+|\ln(T-t)|, \tag{2.29}$$

where

$$\widehat{c}_i = A + \tfrac{n(n-2q)}{q(n-q)} \ln \tau_i + \tfrac{n}{n-q} \ln[p\varepsilon_i \delta_i^{n/(-q)}],$$
$$\widehat{C}_i = A + \tfrac{n(n-2q)}{q(n-q)} \ln \delta_i + \tfrac{n}{n-q} \ln[p\varepsilon_i^{-1} \delta_i^{n/(-q)}].$$

By joining (2.29) and (*ii*) of Lemma 2.3, it follows that for $T_i^* \leqslant t < T$,

$$c_i + |\ln(T-t)| \leqslant -\frac{n}{q} \ln(-qF(t)) \leqslant C_i + |\ln(T-t)|, \tag{2.30}$$

where

$$c_i = \widehat{c}_i - \tfrac{n}{n-q} \ln[\delta_i^{-1} \varepsilon_i^{-1} \tau_i^{-\frac{n}{q}} \tfrac{p}{n-q}], \;\; C_i = \widehat{C}_i - \tfrac{n}{n-q} \ln[\varepsilon_i \tau_i^{-1} \delta_i^{-\frac{n}{q}} \tfrac{p}{n-q}].$$

Consequently, (2.29) and (2.30) guarantee that for $T_i^* \leqslant t < T$,

$$\begin{aligned} \frac{\widehat{c}_i + |\ln(T-t)|}{|\ln(T-t)|} &\leqslant \frac{npG(t)}{(n-q)|\ln(T-t)|} \leqslant \frac{\widehat{C}_i + |\ln(T-t)|}{|\ln(T-t)|}, \\ \frac{c_i + |\ln(T-t)|}{|\ln(T-t)|} &\leqslant \frac{n\ln[-qF(t)]}{-q|\ln(T-t)|} \leqslant \frac{C_i + |\ln(T-t)|}{|\ln(T-t)|}. \end{aligned} \tag{2.31}$$

Note that $\widehat{c}_i$, $\widehat{C}_i \to A + \frac{n}{n-q} \ln p$, and c_i, $C_i \to -\ln(np) + \ln(n-q)$ because of δ_i, ε_i, $\tau_i \to 1$ as $i \to \infty$. By letting $i \to \infty$ in (2.31), we get

$$\lim_{t\to T} \ln[-qF(t)]|\ln(T-t)|^{-1} = -\frac{q}{n}, \;\; \lim_{t\to T} G(t)|\ln(T-t)|^{-1} = \frac{n-q}{np}. \tag{2.32}$$

Therefore, it can be deduces from (*ii*) of Lemma 2.2 and (2.32) that uniformly on any compact subset of Ω,

$$u(x,t) \sim G(t) \sim \frac{n-q}{np}|\ln(T-t)|, \;\; v(x,t) \sim -\frac{1}{q} \ln[-qF(t)] \sim \frac{1}{n}|\ln(T-t)|. \tag{2.33}$$

And therefore, uniformly on any compact subset of Ω,

$$\lim_{t\to T} |\ln(T-t)|^{-1} u(x,t) = (n-q)/(np), \;\; \lim_{t\to T} |\ln(T-t)|^{-1} v(x,t) = 1/n. \tag{2.34}$$

Finally, we can verify the cases (*iii*) and (*iv*) by similar means of cases (*i*) and (*ii*). So, we complete the proof of Theorem 1.2. □

3. Proofs of Theorems 1.3-1.7

In this section, we pay attention to system (1.1) with m, $q > 0$. By the results of [4] and [20], applying standard methods we find from the assumptions (H2) and (H3) that the following results are true:

(R1) $u(x,t) > 0$ and $v(x,t) > 0$ in $B(0;R) \times (0,T)$, where T is the maximal existence time of the solution (u,v) to problem (1.1).

(R2) $u(x,t) = u(r,t)$, $v(x,t) = v(r,t)$ and $u_r(r,t) \leqslant 0$, $v_r(r,t) \leqslant 0$ in $(0,R) \times (0,T)$.

(R3) $u_t \geqslant 0$ and $v_t \geqslant 0$ for $(x,t) \in B(0;R) \times (0,T)$.

To prove Theorems 1.3-1.6, we begin with an elementary lemma, which will play an important role in the following proof.

LEMMA 3.1. *Let assumptions (H1)-(H3) be satisfied. Suppose that (u,v) is a classical solution of problem (1.1) which blows up in finite time T, then for some $t_1 < T$, there exists a positive constant $\varepsilon \leqslant 1$ such that*

$$u_t \geqslant \varepsilon \exp\{mu(x,t)+nv(0,t)\},\ v_t \geqslant \varepsilon \exp\{pu(0,t)+qv(x,t)\},\ x \in B(0;R),\ t_1 \leqslant t < T. \tag{3.1}$$

In addition,

$$u_t(0,t) \leqslant \exp\{mu(0,t)+nv(0,t)\},\ v_t(0,t) \leqslant \exp\{pu(0,t)+qv(0,t)\},\ 0 < t < T. \tag{3.2}$$

Proof. By the result (R2) listed above

$$u(0,t) = \max_{x\in\overline{B}(0;R)} u(x,t),\ \ v(0,t) = \max_{x\in\overline{B}(0;R)} v(x,t),\ \ 0 < t < T. \tag{3.3}$$

Hence, $\Delta u(0,t) \leqslant 0$, $\Delta v(0,t) \leqslant 0$ for any $0 < t < T$. And thus

$$u_t(0,t) \leqslant \exp\{mu(0,t)+nv(0,t)\},\ v_t(0,t) \leqslant \exp\{pu(0,t)+qv(0,t)\},\ 0 < t < T. \tag{3.4}$$

which is just the assertion (3.2).

Next, we infer the assertion (3.1). Since (u,v) blows up in finite time T, and $u_t,\ v_t \geqslant 0$ for all $(x,t) \in B(0;R)\times(0,T)$, it can be deduced that for any $t_0 : 0 < t_0 < T$, $u_t(x,t_0) \not\equiv 0$ or $v_t(x,t_0) \not\equiv 0$ in $B(0;R)$. Otherwise, (u,v) can not blow up in finite time. Denote $\varphi = u_t,\ \psi = v_t$, then

$$\begin{array}{ll}
\varphi_t = \Delta\varphi + \exp\{mu(x,t)+nv(0,t)\}(m\varphi(x,t)+n\psi(0,t)), & x \in B(0;R),\ t_0 \leqslant t < T,\\
\psi_t = \Delta\psi + \exp\{pu(0,t)+qv(x,t)\}(p\varphi(0,t)+q\psi(x,t)), & x \in B(0;R),\ t_0 \leqslant t < T,\\
\varphi(x,t) = \psi(x,t) = 0, & |x| = R,\ t_0 \leqslant t < T,\\
\varphi(x,t_0) \geqslant, \not\equiv 0,\ \psi(x,t_0) \geqslant, \not\equiv 0, & x \in \overline{B}(x;R).
\end{array} \tag{3.5}$$

The maximal principle shows that

$$\begin{array}{lll}
\varphi(x,t) > 0, & \psi(x,t) > 0, & \forall (x,t) \in B(0;R)\times(t_0,T),\\
\frac{\partial\varphi}{\partial\eta} < 0, & \frac{\partial\psi}{\partial\eta} < 0, & \forall (x,t) \in \partial B(0;R)\times(t_0,T),
\end{array} \tag{3.6}$$

where η is the unit outward normal. By the standard method it follows that for any $t_1 : t_0 < t_1 < T$, there exists $0 < \varepsilon \leqslant 1$ such that

$$\varphi(x,t_1) \geqslant \varepsilon \exp\{mu(x,t_1)+nv(0,t_1)\},\ \psi(x,t_1) \geqslant \varepsilon \exp\{pu(0,t_1)+qv(x,t_1)\},\ x \in \overline{B}(0;R), \tag{3.7}$$

i.e., for $x \in \overline{B}(0;R)$

$$\begin{array}{l}
\Delta u(x,t_1) + \exp\{mu(x,t_1)+nv(0,t_1) \geqslant \varepsilon \exp\{mu(x,t_1)+nv(0,t_1)\},\\
\Delta v(x,t_1) + \exp\{pu(0,t_1)+qv(x,t_1) \geqslant \varepsilon \exp\{pu(0,t_1)+qv(x,t_1).
\end{array} \tag{3.8}$$

Since $w(x,t) = u_t(x,t) - \varepsilon\exp\{mu(x,t)+nv(0,t)\}$, $z_t = v_t - \varepsilon\exp\{pu(0,t)+qv(x,t)\}$. By using the ideas of [5, 21], we are sure that $w(x,t) \geqslant 0$, $z(x,t) \geqslant 0$. Indeed,

$$\begin{aligned}
w_t - \Delta w &= (u_t - \Delta u)_t - \varepsilon m\exp\{mu(x,t)+nv(0,t)\}(u_t - \Delta u)\\
&\quad -\varepsilon n\exp\{mu(x,t)+nv(0,t)\}v_t(0,t)\\
&\quad +\varepsilon m^2\exp\{mu(x,t)+nv(0,t)\}|\nabla u|^2\\
&\geqslant (u_t - \Delta u)_t - \varepsilon m\exp\{mu(x,t)+nv(0,t)\}(u_t - \Delta u)\\
&\quad -\varepsilon n\exp\{mu(x,t)+nv(0,t)\}v_t(0,t)\\
&\geqslant m\exp\{mu(x,t)+nv(0,t)\}w, \qquad x \in B(0;R),\ t_1 < t < T,\\
z_t - \Delta z &\geqslant q\exp\{pu(0,t)+qv(x,t)\}z, \qquad x \in B(0;R),\ t_1 < t < T,\\
w(x,t_1) &= \Delta u(x,t_1) + \exp\{mu(x,t_1)+nv(0,t_1)\}\\
&\quad -\varepsilon\exp\{mu(x,t_1)+nv(0,t_1)\} \geqslant 0, \qquad x \in B(0;R),\\
z(x,t_1) &= \Delta v(x,t_1) + \exp\{pu(0,t_1)+qv(x,t_1)\}\\
&\quad -\varepsilon\exp\{pu(0,t_1)+qv(x,t_1)\} \geqslant 0, \qquad x \in B(0;R),\\
w(x,t) &= z(x,t) = 0,\ |x| = R, \qquad t_1 < t < T.
\end{aligned} \tag{3.9}$$

The maximum principle implies that $w,\ z \geqslant 0$. Therefore,

$$u_t \geqslant \varepsilon\exp\{mu(x,t)+nv(0,t)\},\ v_t \geqslant \varepsilon\exp\{pu(0,t)+qv(x,t)\},\ x \in B(0;R),\ t_1 \leqslant t < T, \tag{3.10}$$

which means the assertion (3.1) is true. The proof of Lemma 3.1 is complete. □

3.1. Proofs of Theorems 1.3, 1.4 and 1.5.

In this subsection, we prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Assume on the contrary that u blows up in finite time T and v is bounded in $B(0;R)\times(0,T)$. By (3.1) and (3.2) in Lemma 3.1, we have

$$\begin{aligned}
\varepsilon\exp\{mu(0,t)+nv(0,t)\} \leqslant u_t(0,t) \leqslant \exp\{mu(0,t)+nv(0,t)\},\quad t \in [t_1,T),\\
\varepsilon\exp\{pu(0,t)+qv(0,t)\} \leqslant v_t(0,t) \leqslant \exp\{pu(0,t)+qv(0,t)\},\quad t \in [t_1,T).
\end{aligned} \tag{3.11}$$

As v is non-negative and bounded in $B(0,R)\times(0,T)$, we claim that $v(0,t) \geqslant c > 0$, where c is a constant. Indeed, let w be the solution of the heat equation $w_t = \Delta w$ with null Dirichlet boundary condition and $w(x,0) = v_0(x)$, then the comparison principle asserts that $v \geqslant w$ in $B(0,R)\times(0,T)$. Since $0 \in B(0;R)$ and $v_0(x) \geqslant 0, \not\equiv 0$, for any fixed $t_1 \in (0,T)$, there exists some constant $c = c(t_1,T) > 0$ such that $w(0,t) \geqslant c$ for all $t_1 \leqslant t \leqslant T$, and so does v. Without loss of generality, we assume that $t_1 = 0$, thus $v(0,t) \geqslant c > 0$ for all $t \in [0,T)$. Therefore, there exist positive constants $C_1 \geqslant C_2$ and $C_3 \geqslant C_4$ such that for all $t \in [t_1,T)$,

$$C_2 e^{mu(0,t)} \leqslant u_t(0,t) \leqslant C_1 e^{mu(0,t)},\ \ C_4 e^{pu(0,t)} \leqslant v_t(0,t) \leqslant C_5 e^{pu(0,t)}. \tag{3.12}$$

Due to $m > 0$ and $\lim_{t\to T} u(0,t) = \infty$, integrating the first equation of (3.12) yields

$$mC_2(T-t) \leqslant e^{-mu(0,t)} \leqslant mC_1(T-t),\ \ t_1 \leqslant t < T. \tag{3.13}$$

Consequently, for $t \in [t_1,T)$

$$C_4[C_1 m(T-t)]^{-\frac{p}{m}} \leqslant v_t(0,t) \leqslant C_3[C_2 m(T-t)]^{-\frac{p}{m}}. \tag{3.14}$$

As $p \geqslant m > 0$, it can be deduced that $\lim_{t\to T} v(0,t) = \infty$. This is a contradiction. Therefore, Theorem 1.3 is completed. □

Proof of Theorem 1.4. By (3.1) and (3.2) in Lemma 3.1, we have

$$\frac{\varepsilon \exp\{(m-p)u(0,t)\}}{\exp\{(q-n)v(0,t)\}} \leqslant \frac{du(0,t)}{dv(0,t)} \leqslant \frac{\exp\{(m-p)u(0,t)\}}{\varepsilon \exp\{(q-n)v(0,t)\}}. \tag{3.15}$$

In view of the right-hand side of (3.15)

$$\varepsilon \exp\{(p-m)u(0,t)\}du(0,t) \leqslant \exp\{(n-q)v(0,t)\}dv(0,t), \tag{3.16}$$

when $p \geqslant m$, suppose on the contrary that $n < q$. By integrating (3.16), we see that

$$\begin{array}{ll} u(0,s)|_{t_1}^{t} \leqslant \frac{1}{\varepsilon(n-q)} \exp\{(n-q)v(0,s)\}|_{t_1}^{t} \leqslant \frac{1}{\varepsilon(q-n)} \exp\{(n-q)v(0,t_1)\}, & \text{if } p = m, \\ \frac{1}{p-m} \exp\{(p-m)u(0,s)\}|_{t_1}^{t} \leqslant \frac{1}{\varepsilon(q-n)} \exp\{(n-q)v(0,t_1)\}, & \text{if } p > m. \end{array} \tag{3.17}$$

Since $\lim_{t\to T} u(0,t) = \infty$, taking $t \to T$ in the above leads to a contradiction. Consequently, $n \geqslant q$. When $n \geqslant q$, by using of analogous argument, we can show $p \geqslant m$. Similarly, we may conclude (b) of Theorem 1.4. □

Proof of Theorems 1.5. Assume on the contrary that (u,v) blows up at another point $x^* \neq 0$. Furthermore, we may consider without loss of generality that u blows up at the point x^* as $t \to T$, i.e. $\limsup_{t\to T} u(x^*,t) = \infty$. Set $r^* = |x^*|$, then $r^* > 0$. Since $u(x,t) = u(r,t)$ is non-increasing in r, $\limsup_{t\to T} u(r,t) = \infty$ for any $r \in [0,r^*]$ with $r = |x|$.

Let a be fixed number satisfying $a = r^*/3$ and

$$B_a^+(0;R) = B(0;R) \cap \{x \in \mathbb{R}^N | x_1 > a\} = \{x \in B(0;R) | x_1 > a\}.$$

Define

$$J(x,t) = u_{x_1}(x,t) + c(x_1)\exp\{m_0 u(x,t)\}, \ \ (x,t) \in \overline{B}_a^+(0;R) \times [0,T), \tag{3.18}$$

where $0 < m_0 < m$, and $c(x_1) = \varepsilon(x_1 - a)^2$ with $\varepsilon > 0$ is a small constant to be determined.

A straight computation yields

$$\begin{aligned} J_t - \Delta J &= (u_t - \Delta u)_{x_1} + m_0 c(x_1)\exp[m_0 u(x,t)](u_t - \Delta u) - c''(x)\exp[m_0 u(x,t)] \\ &\quad -2m_0 \exp[m_0 u(x,t)]c'(x_1)u_{x_1} - m_0^2 c(x_1)\exp[m_0 u(x,t)]|\nabla u|^2 \\ &\leqslant \{m\exp[mu(x,t) + nv(0,t)] - 4\varepsilon m_0(x_1 - a)\exp[m_0 u(x,t)]\}J \\ &\quad -c(x_1)\exp[m_0 u(x,t)]\{(m - m_0)\exp[mu(x,t) + nv(0,t)]\} \\ &\quad -c(x_1)\exp[m_0 u(x,t)]\{-4\varepsilon m_0(x_1 - a)\exp[m_0 u(x,t)] + 2(x_1 - a)^{-2}\} \\ &\leqslant bJ - c(x_1)\exp[m_0 u(x,t)]\{(m - m_0)\exp[mu(x,t) + nv(0,t)]\} \\ &\quad -c(x_1)\exp[m_0 u(x,t)]\{-4\varepsilon m_0 R\exp[m_0 u(x,t)] + 2R^{-2}\}, \end{aligned} \tag{3.19}$$

where $b \equiv m\exp[mu(x,t)+nv(0,t)] - 4\varepsilon m_0(x_1 - a)\exp[m_0 u(x,t)]$. Remember that $v(r,t) > 0$ in $[0,R) \times [0,T)$ and $v(0,t) = \max_{0\leqslant r\leqslant R} v(r,t)$ for $t \in [0,T)$, then $v(0,t) > c > 0$ for some constant c. By $m_0 < m$, there exists ε_1 so small that for $0 < \varepsilon \leqslant \varepsilon_1$

$$(m - m_0)\exp[mu(x,t) + nv(0,t)] - 4\varepsilon m_0 R\exp[m_0 u(x,t)] + 2R^{-2} \geqslant 0, \tag{3.20}$$

in $B_a^+(0;R)\times(0,T)$. Consequently, from (3.19) and (3.20) follows

$$J_t-\Delta J-bJ\leqslant 0,\quad (x,t)\in B_a^+(0;R)\times(0,T). \tag{3.21}$$

Moreover, since $u_{0r}\leqslant,\not\equiv 0$ (otherwise, $u_0\equiv 0$, which contradicts the assumption on u_0), by standard methods one can deduce that $u_r<0$ provided that $r\neq 0$ and $t>0$. And thus, $u_{x_1}(x,t)<0$ for $(x,t)\in\overline{B}_a^+(0;R)\times(0,T)$. Replacing $[0,T)$ by $[t^*,T)$ for some $t^*\in(0,T)$ in following discussion, we may assume that $u_{x_1}(x,t)<0$ holds on $\overline{B}_a^+(0;R)\times[0,T)$. Hence, there exists ε_2 so small that for $0<\varepsilon\leqslant\varepsilon_2$,

$$J(x,t)\leqslant u_{x_1}(x,t)+\varepsilon(R-a)^2\leqslant 0,\quad (x,t)\in\partial B_a^+(0;R)\times(0,T). \tag{3.22}$$

On the other hand,

$$\begin{aligned} J(x,0)&=u_{x_1}(x,0)+c(x_1)\exp\{m_0u_0(x)\}\\ &\leqslant u_{x_1}(x,0)+\varepsilon R^2\max\nolimits_{x\in\overline{B}(0;R)}\exp\{m_0u_0(x)\}\\ &\leqslant 0,\qquad \forall x\in B_a^+(0;R). \end{aligned} \tag{3.23}$$

Provided that $0<\varepsilon\leqslant\varepsilon_3$ for some sufficiently small ε_3.

Set $\varepsilon=\min\{1,\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, then (3.21) and (3.23) hold. Application of the maximum principle to (3.21)-(3.23) ensures that

$$J(x,t)\leqslant 0,\quad (x,t)\in B_a^+(0;R)\times(0,T). \tag{3.24}$$

Namely

$$-\exp\{-m_0u(x,t)\}u_{x_1}\geqslant c(x_1),\quad (x,t)\in B_a^+(0;R)\times(0,T). \tag{3.25}$$

Taking $y=(2a,0,0,\cdots,0)$ and $z=(r^*,0,0,\cdots,0)$, then $y,\ z\in B_a^+(0;R)$. Integrating (3.25) yields that

$$0<\int_y^z c(x_1)dx_1\leqslant\frac{1}{m_0}\exp\{-m_0u(z,t)\},\quad 0<t<T. \tag{3.26}$$

The fact that $\limsup_{t\to T}u(z,t)=\infty$ and $m_0>0$ leads to a contradiction. Therefore, u blows up at a single point $x=0$, and so does the solution (u,v) of system (1.1). Consequently, we conclude the Theorem 1.5. □

3.2. Proof of Theorem 1.6.

In this subsection, we prove Theorem 1.6 by introducing a lemma first, which shows the relationship between $u(0,t)$ and $v(0,t)$.

LEMMA 3.2. *Under the conditions of Theorem 1.4, for any given $0<\delta<1$, there exists $t_1\leqslant T_0<T$ such that the following statements hold for all $t\in[T_0,T)$.*

(*i*) *If $p>m$ and $n>q$, or $p<m$ and $n<q$, then*

$$\delta\varepsilon\sigma\exp[(n-q)v(0,t)]\leqslant\theta\exp[(p-m)u(0,t)],\ \delta\varepsilon\theta\exp[(p-m)u(0,t)]\leqslant\sigma\exp[(n-q)v(0,t)].$$

(*ii*) *If* $p > m$ *and* $n = q$, *then*

$$\delta\varepsilon(p-m)v(0,t) \leqslant \exp[(p-m)u(0,t)], \ \ \delta\varepsilon\exp[(p-m)u(0,t)] \leqslant (p-m)v(0,t).$$

(*iii*) *If* $p = m$ *and* $n > q$, *then*

$$\delta\varepsilon\exp[(n-q)v(0,t)] \leqslant (n-q)u(0,t), \ \ \delta\varepsilon(n-q)u(0,t) \leqslant \exp[(n-q)v(0,t)].$$

(*iv*) *If* $p = m$ *and* $n = q$, *then* $\delta\varepsilon v(0,t) \leqslant u(0,t)$, $\delta\varepsilon u(0,t) \leqslant v(0,t)$.

Proof. (*i*) (a) Let $p > m$ and $n > q$. One can deduce from the right-hand side of (3.15) that

$$\frac{1}{p-m}\exp[(p-m)u(0,s)]\big|_{t_1}^{t} \leqslant \frac{1}{\varepsilon(n-q)}\exp[(n-q)v(0,s)]\big|_{t_1}^{t} \leqslant \frac{\exp[(n-q)v(0,t)]}{\varepsilon(n-q)}. \tag{3.27}$$

Notice that $\lim_{t\to T} u(0,t) = \infty$ and $p > m$, for given $0 < \delta < 1$, there exists $t_2 : t_1 \leqslant t_2 < T$ such that $\exp[(p-m)u(0,t_1)] \leqslant (1-\delta)\exp[(p-m)u(0,t)]$ for $t_2 \leqslant t < T$, thus (3.27) ensures $\frac{\delta}{p-m}\exp[(p-m)u(0,t)] \leqslant \frac{1}{\varepsilon(n-q)}\exp[(n-q)v(0,t)]$ for $t \in [t_2, T)$, i.e.,

$$\delta\varepsilon(n-q)\exp[(p-m)u(0,t)] \leqslant (p-m)\exp[(n-q)v(0,t)], \quad t \in [t_2, T). \tag{3.28}$$

On the other hand, application of similar analysis to left-hand side of (3.15) derives that for given $0 < \delta < 1$, there exists $t_2^* : t_1 \leqslant t_2^* < T$ such that

$$\delta\varepsilon(p-m)\exp[(n-q)v(0,t)] \leqslant (n-q)\exp[(p-m)u(0,t)], \quad t \in [t_2^*, T). \tag{3.29}$$

Define $T_0 = \max\{t_2, t_2^*\}$, then we come to the conclusion (*i*) from (3.28), (3.29) and the definition of θ and σ. Analogously, we can demonstrate other cases. □

Proof of Theorem 1.6. (*i*) We need only prove the case (a) $p > m$ and $n > q$, since the case (b) $p < m$ and $n < q$ can be treated similarly. Combining the first inequality of (3.2) and (*i*) of Lemma 3.2, we see that

$$u_t(0,t) \leqslant [\theta/(\delta\varepsilon\sigma)]^{n/(n-q)}\exp[(m+n\sigma/\theta)u(0,t)]. \tag{3.30}$$

Since $-m - n\sigma/\theta = -m - n\frac{p-m}{n-q} = -\frac{pn-qm}{n-q} = -1/\theta < 0$ and $\lim_{t\to T} u(0,t) = \infty$, by integrating (3.30) we find that there exists constant $c_1 > 0$ such that

$$u(0,t) \geqslant \ln[c_1(T-t)^{-\theta}], \quad T_0 \leqslant t < T. \tag{3.31}$$

Applying the above argument to the first inequality of (3.1) and using (*i*) of Lemma 3.2 show that there exists a constant $C_1 > 0$ such that

$$u(0,t) \leqslant \ln[C_1(T-t)^{-\theta}], \quad T_0 \leqslant t < T. \tag{3.32}$$

Consequently, it follows from (3.31) (3.32) and (*i*) of Lemma 3.2 that there exist positive constants $0 < c_2 \leqslant C_2$ such that

$$\ln[c_2(T-t)^{-\sigma}] \leqslant v(0,t) \leqslant \ln[C_2(T-t)^{-\sigma}]. \tag{3.33}$$

Let $c=\min\{c_1,c_2\}$ and $C=\max\{C_1,C_2\}$, the (3.31)-(3.33) imply the desired conclusion (i) of Theorem 1.6.

(ii) Let $p>m$ and $n=q$. By joining the second inequality of (3.2) and (ii) of Lemma 3.2, we have

$$v_t(0,t)\leqslant[(p-m)/(\delta\varepsilon)]^{p/(p-m)}[v(0,t)]^{p/(p-m)}\exp[qv(0,t)],\ \ T_0\leqslant t<T. \tag{3.34}$$

Observe that $\lim_{t\to T}v(0,t)=\infty$, it follows (3.34) that

$$\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds\leqslant[(p-m)/(\delta\varepsilon)]^{p/(p-m)}(T-t),\ \ T_0\leqslant t<T. \tag{3.35}$$

Similarly, in view of the second inequality of (3.1) and (ii) of Lemma 3.2 we have

$$\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds\geqslant\varepsilon[\delta\varepsilon(p-m)]^{p/(p-m)}(T-t),\ \ T_0\leqslant t<T. \tag{3.36}$$

Since

$$\begin{aligned}&\lim_{t\to T}\frac{\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds}{e^{-qv(0,t)}(v(0,t))^{-p/(p-m)}}\\ \Leftrightarrow&\lim_{v(0,t)\to\infty}\frac{\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds}{e^{-qv(0,t)}(v(0,t))^{-p/(p-m)}}\\ =&\lim_{v\to\infty}\frac{-e^{-qv}v^{-p/(p-m)}}{-qe^{-qv}v^{-p/(p-m)}-\frac{p}{p-m}e^{-qv}v^{-1-p/(p-m)}}\\ =&\lim_{v\to\infty}\frac{-1}{-q-\frac{p}{p-m}v^{-1}}=\frac{1}{q},\end{aligned} \tag{3.37}$$

i.e.,

$$q\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds\sim e^{-qv(0,t)}(v(0,t))^{-p/(p-m)}. \tag{3.38}$$

Therefore, by (3.38) there exists $T_1<T$ such that for all $t\in[T_1,T)$

$$\frac{\exp[-qv(0,t)]}{2[v(0,t)]^{p/(p-m)}}\leqslant q\int_{v(0,t)}^{\infty}e^{-qs}s^{-p/(p-m)}ds\leqslant\frac{2\exp[-qv(0,t)]}{[v(0,t)]^{p/(p-m)}}. \tag{3.39}$$

If $T^*=\max\{T_0,T_1\}$, then it can be deduce from (3.35), (3.36) and (3.39) that there exist come positive constants $c\leqslant C$ such that

$$c(T-t)^{-1}\leqslant\exp[qv(0,t)][v(0,t)]^{p/(p-m)}\leqslant C(T-t)^{-1},\quad T^*\leqslant t<T. \tag{3.40}$$

In addition, on account of the first inequalities of (3.1) and (3.2), and n=q

$$\varepsilon\exp[qv(0,t)]\leqslant\exp[-mu(0,t)]u_t(0,t)\leqslant\exp[qv(0,t)],\quad T^*\leqslant t<T. \tag{3.41}$$

Hence, (ii) of Lemma 3.2, (3.40) and (3.41) ensure that exist some constants $c_1\leqslant C_1$ such that

$$c_1(T-t)^{-1}\leqslant\exp[(p-m)u(0,t)]u_t(0,t)\leqslant C_1(T-t)^{-1},\quad T^*\leqslant t<T. \tag{3.42}$$

Due to $p > m$, for some constant $c \leqslant C$, integrating (3.42) in $[T^*, t)$ yields

$$c|\ln(t-t)| \leqslant \exp[(p-m)u(0,t)] \leqslant C|\ln(t-t)|, \quad T^* \leqslant t < T. \tag{3.43}$$

Therefore, we come to the conclusion (*ii*) of Theorem 1.6 from (3.40) (3.43) and $n = q$.

(*iii*) If $p = m$ and $n > q$, the conclusion (*iii*) of Theorem 1.6 follows the similar way as case (*ii*).

(*iv*) When $p = m$ and $n = q$, it follows from (3.1), (3.2) and (*iv*) of Lemma 3.2 that

$$\varepsilon \exp[(m + n\delta\varepsilon)u(0,t)] \leqslant u_t(0,t) \leqslant \exp[(m + n/(\delta\varepsilon))u(0,t)], \quad T_0 \leqslant t < T. \tag{3.44}$$

Recall $\lim_{t\to T} u(0,t) = \infty$, integrating the above yields that for $T_0 \leqslant t < T$

$$\ln[(T-t)(m\delta\varepsilon + n)/(\delta\varepsilon)]^{-\frac{\delta\varepsilon}{m\delta\varepsilon+n}} \leqslant u(0,t) \leqslant \ln[\varepsilon(m + n\delta\varepsilon)(T-t)]^{-\frac{1}{m+n\delta\varepsilon}}. \tag{3.45}$$

Consequently, for some positive constants $c \leqslant C$, there exists $T_0 \leqslant T^* < T$ such that

$$c|\ln(T-t)| \leqslant u(0,t) \leqslant C|\ln(T-t)|. \tag{3.46}$$

Moreover, from (3.46) and (*iv*) of Lemma 2.3 that there exist positive constants $c \leqslant C$ such that

$$c|\ln(T-t)| \leqslant v(0,t) \leqslant C|\ln(T-t)|. \tag{3.47}$$

Therefore, (3.46) and (3.37) imply the conclusion (*iv*) of Theorem 1.6. □

3.3. Proof of Theorem 1.7

Similar as in subsection 3.2, we will apply the ideas of [5] to proceed our discussion for Theorem 1.7. We only need to verify the estimate of u, since the estimate of v can be obtained analogously. Set $J(r,t) = u_r(r,t) + c(r)\exp[m_0 u(r,t)]$, $(r,t) \in [0,R] \times (0,T)$, where $0 < m_0 < m$ and $c(r) = \varepsilon r^{1+\delta}$ with any constant $\delta > 0$ and small ε to be defined. Direct computation for J shows that in $(0,R) \times (0,T)$,

$$\begin{aligned}
J_t &- \tfrac{N-1}{r}J_r - J_{rr} \\
&= (u_t - \tfrac{N-1}{r}u_r - u_{rr})_r + m_0 c(r)\exp[m_0 u(r,t)](u_t - \tfrac{N-1}{r}u_r - u_{rr}) \\
&\quad -(N-1)r^{-1}c'(r)\exp[m_0 u(r,t)] - c''(r)\exp[m_0 u(r,t)] \\
&\quad -2m_0 c'(r)\exp[m_0 u(r,t)]u_r - m_0^2 c(r)\exp[m_0 u(r,t)](u_r)^2 \\
&\leqslant \{m\exp[mu(r,t)+nv(0,t)] - 2\varepsilon m_0(1+\delta)r^\delta \exp[m_0 u(r,t)]\}u_r - c''(r)\exp[m_0 u(r,t)] \\
&\quad +m_0 c(r)\exp[(m+m_0)u(r,t) + nv(0,t)] - (N-1)r^{-1}c'(r)\exp[m_0 u(r,t)] \\
&\leqslant bJ - c(r)\exp[m_0 u(r,t)]\{(m-m_0)\exp[mu(r,t)+nv(0,t)] + \delta(N-1+\delta)R^{-2}\} \\
&\quad -c(r)\exp[m_0 u(r,t)]\{-2\varepsilon m_0(1+\delta)R^\delta \exp[m_0 u(x,t)]\},
\end{aligned} \tag{3.48}$$

where $b \equiv m\exp[mu(r,t)+nv(0,t)] - 2\varepsilon m_0(1+\delta)r^\delta \exp[m_0 u(r,t)]$. Note that $v(r,t) > 0$ in $[0,R) \times [0,T)$ and $v(0,t) = \max_{0\leqslant r\leqslant R} v(r,t)$ for $t \in [0,T)$, then $v(0,t) > c_1 > 0$ for some constant c_1. Consequently, by $0<m_0<m$, we know that there exists $\varepsilon_1>0$ small enough such that for $0<\varepsilon\leqslant\varepsilon_1$

$$(m-m_0)\exp[mu(r,t)+nv(0,t)]+\delta(N-1+\delta)R^{-2}-2\varepsilon m_0(1+\delta)R^{\delta}\exp[m_0u(r,t)]\geqslant 0 \tag{3.49}$$

in $(0,R)\times(0,T)$. Thus from (3.48) and (3.49), we get

$$J_t-\frac{N-1}{r}J_r-J_{rr}-bJ\leqslant 0,\quad (r,t)\in(0,R)\times(0,T). \tag{3.50}$$

In addition, as $u(r,t)>0$ for $(r,t)\in(0,R)\times(0,T)$ and $u(R,t)=0$ for all $t\in(0,T)$, the strong maximum principle for parabolic equations guarantees that $u_r(R,t)<0$ for $t\in(0,T)$. Hence, there exists $\varepsilon_2>0$ small enough such that for $0<\varepsilon\leqslant\varepsilon_2$

$$J(0,t)=u_r(0,t)=0,\ \ J(R,t)=u_r(R,t)+\varepsilon R^{1+\delta}\leqslant 0,\quad t\in(0,T). \tag{3.51}$$

$$J(r,0)=u_0'(r)+\varepsilon r^{1+\delta}\exp[m_0u_0(r)]\leqslant -cr+\varepsilon rR^{\delta}\exp[m_0u_0(0)]\leqslant 0,\quad r\in(0,R), \tag{3.52}$$

provided that $\varepsilon\leqslant\varepsilon_3=cR^{-\delta}\exp[-m_0u_0(0)]$. Therefore, choose $\varepsilon=\min\{1,\varepsilon_1,\varepsilon_2,\varepsilon_3\}$, then (3.50)-(3.52) hold. Application of the maximum principle to (3.50)-(3.52) asserts that $J(r,t)\leqslant 0,\ \ (r,t)\in(0,R)\times(0,T)$. That is

$$-\exp[-m_0u(r,t)]u_r\geqslant\varepsilon r^{1+\delta},\quad (r,t)\in(0,R)\times(0,T). \tag{3.53}$$

Integrating this inequality we obtain that

$$u(r,t)\leqslant\ln\Big(\frac{\varepsilon m_0}{2+\delta}r^{2+\delta}+e^{-m_0u(0,t)}\Big)^{-1/m_0}\leqslant\ln\{[\varepsilon m_0/(2+\delta)]^{-1/m_0}r^{-\frac{2+\delta}{m_0}}\}. \tag{3.54}$$

Since $\delta>0$ is arbitrary and $0<m_0<m$, it is obvious that $(2+\delta)/m_0>2/m$, and $(2+\delta)/m_0>2/m$ can be made arbitrarily to $2/m$. Consequently, (3.54) implies the assertion of u. So, we conclude Theorem 1.7. □

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