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GROUND STATES OF THE QUASILINEAR PROBLEMS INVOLVING THE CRITICAL SOBOLEV EXPONENT AND POTENTIALS

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Abstract. In this paper we study a quasilinear elliptic problem which involves the critical Sobolev exponent and multiple Hardy-type terms. By the analytic technics and variational methods, the existence and nonexistence of the ground states for the problem is established under certain assumptions.

1. Introduction

In this paper, we study the following elliptic problem:

$$\begin{cases} -\Delta_p u - \sum_{i=1}^{k} \dfrac{\lambda_i u^{p-1}}{|x-a_i|^p} = u^{p^*-1}, \quad u>0, \\ x \in \mathbb{R}^N \setminus \{a_1, a_2, \cdots, a_k\}, \end{cases} \tag{1.1}$$

where $N \geqslant 3$, $k \geqslant 2$, $1 < p < N$, $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, $-\infty < \lambda_i < \bar{\lambda}$, $\bar{\lambda} := (\frac{N-p}{p})^p$ is the best Hardy constant, $a_i \in \mathbb{R}^N$ and $a_i \neq a_j$ for any $i,j = 1,2,\cdots,k$, $i \neq j$.

Problem (1.1) is related to the well-known Hardy inequality [4, 9, 12]:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x-a|^p}\,dx \leqslant \frac{1}{\bar{\lambda}} \int_{\mathbb{R}^N} |\nabla u|^p\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N),\ \ a \in \mathbb{R}^N. \tag{1.2}$$

In this paper, the space $D^{1,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla u|^p\,dx)^{1/p}$. The function $u \in D^{1,p}(\mathbb{R}^N)$ is said to be a solution of the problem (1.1) if $u > 0$ satisfies

$$\int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2} \nabla u \nabla v - \sum_{i=1}^{k} \frac{\lambda_i u^{p-1} v}{|x-a_i|^p} - u^{p^*-1} v \Big)\,dx = 0, \quad \forall v \in D^{1,p}(\mathbb{R}^N).$$

By the standard elliptic regularity argument, the solution $u \in C^{1,\alpha}(\mathbb{R}^N \setminus \{a_1, \cdots, a_k\})$.

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We mention that the elliptic problems involving the Hardy inequality were studied by many authors recently, see for example [1]-[11], [13]-[15], [18], [19] and the references therein. We also point out that in the recent papers [5]-[7], the semilinear problems involving the critical Sobolev exponent and multiple Hardy terms were studied and many important conclusions were obtained. Stimulated by these publications, in this paper we investigate the ground state of the problem (1.1), the solution which has the smallest energy and minimizes the corresponding Rayleigh quotient.

For all $\lambda \in (-\infty, \bar{\lambda})$ and $a \in \mathbb{R}^N$, by the Hardy and Sobolev inequalities we can define the following best Sobolev constant,

$$S(\lambda) := \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \Big(|\nabla u|^p - \lambda \frac{|u|^p}{|x-a|^p}\Big) dx}{\Big(\displaystyle\int_{\mathbb{R}^N} |u|^{p^*} dx\Big)^{\frac{p}{p^*}}}. \tag{1.3}$$

Note that $S(\lambda)$ is independent of the point a. Here we recall a recent result in [1], where the authors studied the limiting problem:

$$\begin{cases} -\Delta_p u - \lambda \dfrac{u^{p-1}}{|x-a|^p} = u^{p^*-1} & \text{in } \mathbb{R}^N\setminus\{a\}, \\ u \in D^{1,p}(\mathbb{R}^N), u > 0 & \text{in } \mathbb{R}^N\setminus\{a\}. \end{cases} \tag{1.4}$$

They proved that for $0 \leqslant \lambda < \bar{\lambda}$ and $1 < p < N$, (1.4) has the radially symmetric ground states

$$V^a_{p,\lambda,\varepsilon}(x) = \varepsilon^{\frac{p-N}{p}} U_{p,\lambda}\Big(\frac{x-a}{\varepsilon}\Big) = \varepsilon^{\frac{p-N}{p}} U_{p,\lambda}\Big(\frac{|x-a|}{\varepsilon}\Big), \ \forall\, \varepsilon > 0, \tag{1.5}$$

that satisfy

$$\int_{\mathbb{R}^N} \Big(|\nabla V^a_{p,\lambda,\varepsilon}(x)|^p - \lambda \frac{|V^a_{p,\lambda,\varepsilon}(x)|^p}{|x-a|^p}\Big) dx = \int_{\mathbb{R}^N} |V^a_{p,\lambda,\varepsilon}(x)|^{p^*} dx = S(\lambda)^{\frac{N}{p}}.$$

$U_{p,\lambda}(x) = U_{p,\lambda}(|x|)$ is the unique radial solution of (1.4) satisfying [1]:

$$U_{p,\lambda}(1) = \Big(\frac{N(\bar{\lambda} - \lambda)}{N-p}\Big)^{\frac{1}{p^*-p}},$$

$$\lim_{r\to 0} r^{a(\lambda)} U_{p,\lambda}(r) = C_1 > 0, \quad \lim_{r\to+\infty} r^{b(\lambda)} U_{p,\lambda}(r) = C_2 > 0,$$

$$\lim_{r\to 0} r^{a(\lambda)+1} |U'_{p,\lambda}(r)| = C_1 a(\lambda) \geqslant 0, \quad \lim_{r\to+\infty} r^{b(\lambda)+1} |U'_{p,\lambda}(r)| = C_2 b(\lambda) > 0,$$

where C_1 and C_2 are positive constants depending on λ, p and N and $a(\lambda)$ and $b(\lambda)$ are zeroes of the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \lambda, \ t \geqslant 0, \tag{1.6}$$

that satisfy

$$0 \leqslant a(\lambda) < \delta < b(\lambda), \qquad \delta := \frac{N-p}{p}. \tag{1.7}$$

Moreover, there exist positive constants $\mathscr{C}_1(\lambda)$ and $\mathscr{C}_2(\lambda)$ such that

$$0 < \mathscr{C}_1(\lambda) \leqslant U_{p,\lambda}(x)\Big(|x|^{\frac{a(\lambda)}{\delta}} + |x|^{\frac{b(\lambda)}{\delta}}\Big)^{\delta} \leqslant \mathscr{C}_2(\lambda). \tag{1.8}$$

The above results are useful for us to study the problem (1.1).

In this paper, the following assumption is needed:

$(\mathscr{H}_1)$: $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k < \bar{\lambda}$, $k \geqslant 2$ and there exists an $l \in \{0,1,2,\cdots,k-1\}$ such that $\lambda_l \leqslant 0 < \lambda_{l+1} \leqslant \lambda_{l+2} \leqslant \cdots \leqslant \lambda_k$ $(\lambda_l = 0 \ \ if \ \ l = 0)$; moreover,

$$\sum_{i=l+1}^{k} \lambda_i < \bar{\lambda} \ \text{ and } \ \sum_{i=1}^{k-1} \lambda_i < 0.$$

Under the assumption $\sum_{i=l+1}^{k} \lambda_i < \bar{\lambda}$ and by the Hardy inequality, we can define the following best constant $A = A(\lambda_1, \lambda_2, \cdots, \lambda_k)$,

$$A := \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N}\Big(|\nabla u|^p - \sum_{i=1}^{k}\frac{\lambda_i |u|^p}{|x-a_i|^p}\Big)dx}{\Big(\displaystyle\int_{\mathbb{R}^N}|u|^{p^*}dx\Big)^{\frac{p}{p^*}}}, \tag{1.9}$$

we shall prove that A is bounded later on, see Lemma 3.2 in this paper.

The main results of this paper can be concluded as the following theorems. We mention that if $p = 2$ and $0 \leqslant \lambda < \bar{\lambda}_2 := (\frac{N-2}{2})^2$, then

$$a(\lambda) = \sqrt{\bar{\lambda}_2} - \sqrt{\bar{\lambda}_2 - \lambda} \quad \text{and} \quad b(\lambda) = \sqrt{\bar{\lambda}_2} + \sqrt{\bar{\lambda}_2 - \lambda},$$

and our results are the same with those in [6]. The results are new in the case when $1 < p < N$ and $p \neq 2$. We can verify that the intervals in $(\mathscr{H}_1)$ and Theorem 1.1 for the parameter λ_k are not empty.

THEOREM 1.1. *Suppose* $(\mathscr{H}_1)$, $N > \max\{p^2, p+1\}$, $\lambda^* := (N-p^2)N^{p-1}p^{-p}$, δ, $b(\lambda_k)$, $\mathscr{C}_1(\lambda_k)$ *and* $\mathscr{C}_2(\lambda_k)$ *are defined as in* (1.6)-(1.8) *and* l *is defined as in* $(\mathscr{H}_1)$. *Assume that one of the following conditions is satisfied:*

$$(i) \quad \sum_{i=1}^{k-1}\frac{\lambda_i}{|a_i - a_k|^p} > 0, \quad 0 < \lambda_k < \lambda^*,$$

$$(ii) \quad \sum_{i=1}^{l}\frac{\big(\mathscr{C}_2(\lambda_k)\big)^p\lambda_i}{|a_i - a_k|^{p(b(\lambda_k)-\delta)}} + \sum_{j=l+1}^{k-1}\frac{\big(\mathscr{C}_1(\lambda_k)\big)^p\lambda_j}{|a_j - a_k|^{p(b(\lambda_k)-\delta)}} > 0, \quad \lambda^* \leqslant \lambda_k < \bar{\lambda}.$$

Then the infimum in (1.9) *is achieved and therefore the problem* (1.1) *has at least one ground state solution.*

THEOREM 1.2. *Assume that* $k \geqslant 2$ *and the following conditions are satisfied:*

$$\lambda_i > 0,\ i = 1,2,\cdots,k,\ \ and\ \ \sum_{i=1}^{k} \lambda_i < \bar{\lambda}.$$

Then the infimum in (1.9) *cannot be achieved and* (1.1) *has no ground state.*

This paper is organized as follows. In Section 2 we study a local Palais-Smale condition by the concentration compactness principle. In Sections 3 and 4 we prove Theorems 1.1 and 1.2 respectively. In this paper, $D^{-1,p}(\mathbb{R}^N)$ is the dual space of $D^{1,p}(\mathbb{R}^N)$. For $t > 0$, $O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leqslant C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ is a generic infinitesimal value. In the following argument, we employ C to denote the positive constants and omit dx in integrals for convenience if no confusion is caused.

2. Palais-Smale condition

We define the functional on the space $D^{1,p}(\mathbb{R}^N)$,

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N}\Big(|\nabla u|^p - \sum_{i=1}^{k}\frac{\lambda_i |u|^p}{|x-a_i|^p}\Big)dx \ - \ \frac{A}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}dx.$$

Note that if $u > 0$ is a critical point of J, then $v = A^{\frac{1}{p^*-p}}u$ is a solution to (1.1). The following lemma provides a local Palais-Smale condition for the functional J.

LEMMA 2.1. *The functional* J *satisfies the* $(PS)_c$ *condition for all* $c < c^*$, *where*

$$c^* = \frac{1}{N}A^{1-\frac{N}{p}}\left(\min\Big\{S(0),\ S(\lambda_1),\ S(\lambda_2),\cdots,S(\lambda_k),\ S\big(\sum_{i=1}^{k}\lambda_i\big)\Big\}\right)^{\frac{N}{p}}. \qquad (2.1)$$

Proof. Suppose that the sequence $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ satisfies $J(u_n) \to c < c^*$ and $J'(u_n) \to 0$ in $D^{-1,p}(\mathbb{R}^N)$. Then u_n is a bounded sequence in $D^{1,p}(\mathbb{R}^N)$. Up to a subsequence and for some $u \in D^{1,p}(\mathbb{R}^N)$ we have:

$$u_n \rightharpoonup u \quad \text{weakly in } H,$$
$$u_n \to u \quad \text{a. e. in } \mathbb{R}^N,$$
$$u_n \to u \quad \text{in } L^{\alpha}_{loc}(\mathbb{R}^N),\ \forall \alpha \in [1,p^*).$$

Then by the concentration compactness principle [16, 17] and up to a subsequence if necessary, there exist an at most countable set $\mathscr{J}$, $x_j \in \mathbb{R}^N \backslash \{a_1,a_2,\cdots,a_k\}$, real numbers $\mu_{x_j}, \nu_{x_j}, j \in \mathscr{J}$ and $\mu_{a_i}, \nu_{a_i}, \gamma_{a_i}, i = 1,2,\cdots,k$, such that the following convergences hold in the sense of measures:

$$|\nabla u_n|^p \rightharpoonup d\mu \geqslant |\nabla u|^p + \sum_{i=1}^{k} \mu_{a_i}\delta_{a_i} + \sum_{j\in\mathscr{J}} \mu_{x_j}\delta_{x_j}, \tag{2.2}$$

$$|u_n|^{p^*} \rightharpoonup d\nu = |u|^{p^*} + \sum_{i=1}^{k} \nu_{a_i}\delta_{a_i} + \sum_{j\in\mathscr{J}} \nu_{x_j}\delta_{x_j}, \tag{2.3}$$

$$\frac{\lambda_i\,|u_n|^p}{|x-a_i|^p} \rightharpoonup d\gamma_i = \frac{\lambda_i\,|u|^p}{|x-a_i|^p} + \gamma_{a_i}\delta_{a_i},\ \ i=1,2,\cdots,k, \tag{2.4}$$

where δ_x is the Dirac mass at x. By the Sobolev inequality we have

$$S(0)(\nu_{x_j})^{\frac{p}{p^*}} \leqslant \mu_{x_j},\ \ \forall\, j\in\mathscr{J}, \tag{2.5}$$

$$S(0)(\nu_{a_i})^{\frac{p}{p^*}} \leqslant \mu_{a_i},\ \ i=1,2,\cdots,k. \tag{2.6}$$

To study the concentration at infinity, we set:

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} |\nabla u_n|^p dx,$$
$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} |u_n|^{p^*} dx,$$
$$\gamma_\infty = \Big(\sum_{i=1}^{k}\lambda_i\Big)\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} \frac{|u_n|^p}{|x|^p}dx.$$

We need to verify the following claims.

Claim 1. The set $\mathscr{J}$ is finite and for any $j\in\mathscr{J}$, either $\nu_{x_j}=0$ or

$$\nu_{x_j} \geqslant \Big(\frac{S(0)}{A}\Big)^{N/p}.$$

In fact, choose $\varepsilon>0$ small such that $a_i \notin B_\varepsilon(x_j), i=1,2,\cdots,k$ and $B_\varepsilon(x_i)\cap B_\varepsilon(x_j)=\emptyset$ for $i\neq j,\ i,j\in\mathscr{J}$. Take the cut-off function $\phi_j\in C_0^\infty(B_\varepsilon(x_j))$ such that $0\leqslant\phi_j\leqslant 1$, $\phi_j=1$ in $B_{\varepsilon/2}(x_j)$ and $|\nabla\phi_j|\leqslant\frac{4}{\varepsilon}$. Then

$$\begin{aligned}\langle J'(u_n),u_n\phi_j\rangle = &\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_j dx + \int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_j dx\\ &-\int_{\mathbb{R}^N}\sum_{i=1}^{k}\frac{\lambda_i|u_n|^p}{|x-a_i|^p}\phi_j dx - A\int_{\mathbb{R}^N}|u_n|^{p^*}\phi_j dx.\end{aligned}$$

Note that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^p\phi_j dx = \int_{\mathbb{R}^N}\phi_j d\mu \geqslant \int_{\mathbb{R}^N}|\nabla u|^p\phi_j dx + \mu_{x_j}, \tag{2.7}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\phi_j dx = \int_{\mathbb{R}^N}\phi_j d\nu = \int_{\mathbb{R}^N}|u|^{p^*}\phi_j dx + \nu_{x_j}, \tag{2.8}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_j\Big|dx = 0, \tag{2.9}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x-a_i|^p}\phi_j\Big|dx \leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_{B_\varepsilon(x_j)}\frac{|u_n|^p}{(|x_j-a_i|-\varepsilon)^p}\phi_j\Big|dx = 0, \tag{2.10}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\phi_j dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\phi_j d\nu = \nu_{x_j}. \tag{2.11}$$

From (2.7)-(2.11) it follows that

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n),\, u_n\phi_j\rangle \geqslant \mu_{x_j} - A\,\nu_{x_j}. \tag{2.12}$$

By (2.5) and (2.12) we deduce that Claim 1 holds.

Claim 2. For $i = 1,2,\cdots,k$, either $\nu_{a_i} = 0$ or $\nu_{a_i} \geqslant \big(\frac{S(\lambda_i)}{A}\big)^{N/p}$.

In fact, for $\varepsilon > 0$ we can take $\psi_i(x) \in C_0^\infty(B_\varepsilon(a_i))$ such that $0 \leqslant \psi_i(x) \leqslant 1$, $\psi_i(x) = 1$ in $B_{\varepsilon/2}(a_i)$ and $|\nabla\psi_i| \leqslant \frac{4}{\varepsilon}$. From (1.3) it follows that

$$\int_{\mathbb{R}^N}\Big(|\nabla(u_n\psi_i)|^p - \lambda_i\frac{|u_n|^p\psi_i^p}{|x-a_i|^p}\Big)dx \geqslant S(\lambda_i)\Big(\int_{\mathbb{R}^N}|u_n\psi_i|^{p^*}dx\Big)^{\frac{p}{p^*}}. \tag{2.13}$$

Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^p\psi_i\,dx = \int_{\mathbb{R}^N}\psi_i d\mu \geqslant \int_{\mathbb{R}^N}|\nabla u|^p\psi_i\,dx + \mu_{a_i}, \tag{2.14}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\psi_i\,dx = \int_{\mathbb{R}^N}\psi_i\,d\nu = \int_{\mathbb{R}^N}|u|^{p^*}\psi_i\,dx + \nu_{a_i}, \tag{2.15}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{\lambda_i|u_n|^p}{|x-a_i|^p}\psi_i\,dx = \int_{\mathbb{R}^N}\psi_i d\gamma_i = \int_{\mathbb{R}^N}\frac{\lambda_i|u|^p\psi_i}{|x-a_i|^p}dx + \gamma_{a_i}, \tag{2.16}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{\lambda_j|u_n|^p}{|x-a_j|^p}\psi_i\,dx = 0, \quad i,j = 1,2,\cdots,k,\ i\neq j, \tag{2.17}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_i\,dx = 0. \tag{2.18}$$

From (2.14)-(2.18) it follows that

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n),\, u_n\psi_i\rangle \geqslant \mu_{a_i} - \gamma_{a_i} - A\,\nu_{a_i}. \tag{2.19}$$

By (2.13) we infer that

$$S(\lambda_i)\,(\nu_{a_i})^{\frac{p}{p^*}} \leqslant \mu_{a_i} - \gamma_{a_i}, \quad 1 \leqslant i \leqslant k.$$

Thus we have

$$S(\lambda_i)\,(\nu_{a_i})^{\frac{p}{p^*}} \leqslant \mu_{a_i} - \gamma_{a_i} \leqslant A\,\nu_{a_i},$$

which implies that Claim 2 holds.

Claim 3. We claim that either $\nu_\infty = 0$ or

$$\nu_\infty \geqslant \left(\frac{S(\sum_{i=1}^k \lambda_i)}{A}\right)^{N/p}.$$

In fact, for $R>0$ large we choose $\psi(x) \in C_0^\infty(\mathbb{R}^N \setminus B_R(0))$ such that $\psi(x)=1$ in $\mathbb{R}^N \setminus B_{2R}(0)$, $0 \leqslant \psi(x) \leqslant 1$ and $|\nabla\psi| \leqslant \frac{2}{R}$. By (1.3) we have

$$\int_{\mathbb{R}^N}\Big(|\nabla(u_n\psi)|^p - \big(\sum_{i=1}^k \lambda_i\big)\frac{|u_n|^p\psi^p}{|x|^p}\Big)dx \geqslant S\big(\sum_{i=1}^k \lambda_i\big)\Big(\int_{\mathbb{R}^N}|u_n\psi|^{p^*}dx\Big)^{\frac{p}{p^*}}. \tag{2.20}$$

By the elementary inequality

$$\big||X+Y|^p - |X|^p\big| \leqslant C\big(|X|^{p-1}|Y| + |Y|^p\big), \quad \forall X,Y \in \mathbb{R}^N,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^N}||\nabla(u_n\psi)|^p - \psi^p|\nabla u_n|^p|\,dx &= \int_{\mathbb{R}^N}||\psi\nabla u_n + u_n\nabla\psi|^p - \psi^p|\nabla u_n|^p|\,dx \\ &\leqslant \int_{\mathbb{R}^N}(|\psi\nabla u_n|^{p-1}|u_n\nabla\psi| + |u_n\nabla\psi|^p)\,dx. \end{aligned}\tag{2.21}$$

From the Hölder inequality it follows that

$$\int_{\mathbb{R}^N}|u_n|\,|\psi\nabla u_n|^{p-1}\,|\nabla\psi|\,dx \leqslant \Big(\int_{R<|x|<2R}|u_n|^p|\nabla\psi|^p dx\Big)^{\frac{1}{p}}\Big(\int_{R<|x|<2R}|\nabla u_n|^p dx\Big)^{\frac{p-1}{p}}.$$

Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|\,\psi^{p-1}\,|\nabla u_n|^{p-1}\,|\nabla\psi|dx &\leqslant C\Big(\int_{R<|x|<2R}|u|^p\,|\nabla\psi|^p dx\Big)^{\frac{1}{p}} \\ &\leqslant C\Big(\int_{R<|x|<2R}|u|^{p^*}dx\Big)^{\frac{p}{p^*}}\Big(\int_{R<|x|<2R}|\nabla\psi|^N dx\Big)^{\frac{p}{N}} \\ &\leqslant C\Big(\int_{R<|x|<2R}|u|^{p^*}dx\Big)^{\frac{p}{p^*}}.\end{aligned}$$

Furthermore,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|\,\psi^{p-1}\,|\nabla u_n|^{p-1}\,|\nabla\psi|\,dx \leqslant \lim_{R\to\infty}\int_{R<|x|<2R}|u|^{p^*}dx = 0.$$

Similarly,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p\,|\nabla\psi|^p dx = 0.$$

By (2.21) we have

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla(u_n\psi)|^p dx = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\psi^p|\nabla u_n|^p dx.$$

From (2.20) it follows that

$$\mu_\infty - \gamma_\infty \geqslant S\Big(\sum_{i=1}^{k}\lambda_i\Big)\ \ (\nu_\infty)^{\frac{p}{p^*}}\,. \tag{2.22}$$

On the other hand,

$$0 = \lim_{n\to\infty}\langle J'(u_n)\,,\,u_n\psi\rangle = \lim_{n\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^p\psi\,dx + \int_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi\,dx$$
$$-\int_{\mathbb{R}^N}\sum_{i=1}^{k}\frac{\lambda_i\,|u_n|^p\psi}{|x-a_i|^p}dx - A\int_{\mathbb{R}^N}\psi|u_n|^{p^*}dx\Big)\,. \tag{2.23}$$

Furthermore,

$$\left|\frac{|u_n|^p\psi}{|x-a_i|^p} - \frac{|u_n|^p\psi}{|x|^p}\right| = \frac{|u_n|^p\psi}{|x|^p}\ \frac{\big|\,|x|^p - |x-a_i|^p\,\big|}{|x-a_i|^p} \leqslant C\,\frac{|u_n|^p\psi}{|x|^{p+1}},$$

where C is a constant independent of R. Thus by the Hölder inequality we have

$$\int_{\mathbb{R}^N}\frac{|u_n|^p\psi}{|x|^{p+1}}\,dx \leqslant \Big(\int_{|x|>R}|u_n|^{p^*}dx\Big)^{\frac{p}{p^*}}\ \Big(\int_{|x|>R}|x|^{-\frac{N(p+1)}{p}}dx\Big)^{\frac{p}{N}}\ = O(R^{-1}).$$

Consequently,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\sum_{i=1}^{k}\frac{\lambda_i|u_n|^p\psi}{|x-a_i|^p}\,dx = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\sum_{i=1}^{k}\lambda_i\frac{|u_n|^p\psi}{|x|^p}\,dx = \gamma_\infty\,. \tag{2.24}$$

From (2.2)-(2.4) and (2.23)-(2.24) we obtain

$$\mu_\infty - \gamma_\infty \leqslant A\,\nu_\infty\,. \tag{2.25}$$

By (2.22) and (2.25) we deduce that Claim 3 holds.

Now we come to the conclusion of Lemma 2.1. From Claims 1-3 it follows that

$$\begin{aligned} c &= J(u_n) - \frac{1}{p}\langle J'(u_n)\,,\,u_n\rangle + o(1)\\ &= \Big(\frac{1}{p}-\frac{1}{p^*}\Big)A\int_{\mathbb{R}^N}|u_n|^{p^*}dx + o(1)\\ &= \frac{1}{N}A\Big(\int_{\mathbb{R}^N}|u|^{p^*}dx + \nu_\infty + \sum_{i=1}^{k}\nu_{a_i} + \sum_{j\in\mathscr{J}}\nu_{x_j}\Big). \end{aligned}$$

Thus from (2.1) and Claims 1-3, it follows that

$$\nu_\infty = 0;\qquad \nu_{a_i} = 0,\ \ i = 1,2,\cdots,k;\qquad \nu_{x_j} = 0,\ \forall\, j\in\mathscr{J}\,.$$

Up to a subsequence, $u_n \to u$ strongly in $D^{1,p}(\mathbb{R}^N)$. Thus the proof of Lemma 2.1 is complete.

3. Proof of Theorem 1.1

In the following arguments, we always set

$$\hat{\lambda} := \sum_{i=1}^{k} \lambda_i, \quad \lambda_i \in \mathbb{R}, \tag{3.1}$$

$$\alpha_\lambda := \mathscr{C}_1(\lambda)\Big(\int_{\mathbb{R}^N} |U_{p,\lambda}(x)|^{p^*}\Big)^{-\frac{1}{p^*}}, \tag{3.2}$$

$$\bar{\alpha}_\lambda := \mathscr{C}_2(\lambda)\Big(\int_{\mathbb{R}^N} |U_{p,\lambda}(x)|^{p^*}\Big)^{-\frac{1}{p^*}}, \tag{3.3}$$

$$\beta := b(\lambda) - \delta, \ \ \beta_i := b(\lambda_i) - \delta, \quad \lambda_i \in [0, \bar{\lambda}), \ \ i = 1, 2, \cdots, k, \tag{3.4}$$

$$\gamma_\lambda := \int_{\mathbb{R}^N} \frac{dx}{|x|^p\,|x - e_1|^{p\,b(\lambda)}}, \quad e_1 := (1, 0, \cdots, 0) \in \mathbb{R}^N, \tag{3.5}$$

$$z_\mu^\lambda(x) := \mu^{-\delta}\, U_{p,\lambda}\big(|x|\,\mu^{-1}\big)\left(\int_{\mathbb{R}^N} |U_{p,\lambda}(x)|^{p^*} dx\right)^{-\frac{1}{p^*}}, \ \ \mu \in (0, +\infty), \tag{3.6}$$

where $\mathscr{C}_1(\lambda)$, $\mathscr{C}_2(\lambda)$, $b(\lambda)$ and $b(\lambda_i)$ are defined as in (1.6)-(1.8) and $U_{p,\lambda}$ is the minimizers of $S(\lambda)$ in (1.5).

LEMMA 3.1. *Suppose $\xi \in \mathbb{R}^N \backslash \{0\}$ and $N > \max\{p^2, p+1\}$. Then as $\mu \to 0$ we have*

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx = \frac{\mu^p}{|\xi|^p} \int_{\mathbb{R}^N} |z_1^\lambda|^p dx + o(\mu^p) \quad \text{if} \ \ 0 < \lambda < \lambda^*,$$

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx \geqslant (\alpha_\lambda)^p\, \frac{\mu^p |ln\,\mu|}{|\xi|^p} + o(\mu^p |ln\,\mu|) \quad \text{if} \ \ \lambda = \lambda^*,$$

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx \leqslant (\bar{\alpha}_\lambda)^p\, \frac{\mu^p |ln\,\mu|}{|\xi|^p} + o(\mu^p |ln\,\mu|) \quad \text{if} \ \ \lambda = \lambda^*,$$

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx \geqslant (\alpha_\lambda)^p \gamma_\lambda\, |\xi|^{-p\beta} \mu^{p\beta} + o(\mu^{p\beta}) \ \ \text{if} \ \lambda^* < \lambda < \bar{\lambda},$$

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx \leqslant (\bar{\alpha}_\lambda)^p \gamma_\lambda\, |\xi|^{-p\beta} \mu^{p\beta} + o(\mu^{p\beta}) \quad \text{if} \ \lambda^* < \lambda < \bar{\lambda},$$

where $\lambda^* = \frac{(N-p^2)N^{p-1}}{p^p}$ is defined as in Theorem 1.1.

Proof. Here we need to investigate the properties of $a(\lambda)$ and $b(\lambda)$ for $\lambda \in [0, \bar{\lambda})$. We can verify that the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \lambda, \quad t \in [0, +\infty),$$

has the only minimal point $\delta = \frac{N-p}{p}$ and is increasing on the interval $(\delta, +\infty)$. Since $\frac{N}{p}$, $b(\lambda) \in (\delta, +\infty)$, then for $N > p^2$ and $\lambda > 0$ we have that:

$$\beta > 1 \iff \frac{N}{p} < b(\lambda) \iff f\big(\frac{N}{p}\big) < f\big(b(\lambda)\big) = 0 \iff \lambda < \lambda^*,$$
$$\beta = 1 \iff \frac{N}{p} = b(\lambda) \iff f\big(\frac{N}{p}\big) = f\big(b(\lambda)\big) = 0 \iff \lambda = \lambda^*,$$
$$\beta < 1 \iff \frac{N}{p} > b(\lambda) \iff f\big(\frac{N}{p}\big) > f\big(b(\lambda)\big) = 0 \iff \lambda > \lambda^*.$$

Note the fact that $0 < a(\lambda) < \delta < b(\lambda)$. Then

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p} dx = \mu^p \int_{\mathbb{R}^N} \frac{|z_1^\lambda|^p}{|\mu x+\xi|^p} dx$$
$$= \mu^p \int_{|x|<\frac{\xi}{2\mu}} \frac{|z_1^\lambda|^p}{|\mu x+\xi|^p} dx + \mu^p \int_{|x|\geqslant\frac{\xi}{2\mu}} \frac{|z_1^\lambda|^p}{|\mu x+\xi|^p} dx. \tag{3.7}$$

For the first part, from (1.8) we have

$$\mu^p \left| \int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^p \Big(\frac{1}{|\mu x+\xi|^p} - \frac{1}{|\xi|^p} \Big) dx \right|$$
$$\leqslant C\mu^p \int_{|x|<\frac{|\xi|}{2\mu}} \Big(|x|^{\frac{a(\lambda)}{\delta}} + |x|^{\frac{b(\lambda)}{\delta}} \Big)^{-(N-p)} \left| \frac{1}{|\mu x+\xi|^p} - \frac{1}{|\xi|^p} \right| dx.$$

If $|x| < |\xi|/(2\mu)$, then there exists some constant $C(\xi) > 0$ depending only on ξ such that $||\mu x+\xi|^{-p} - |\xi|^{-p}| \leqslant C(\xi)$. Consequently,

$$\left| \mu^p \int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^p \Big(\frac{1}{|\mu x+\xi|^p} - \frac{1}{|\xi|^p} \Big) dx \right|$$
$$\leqslant C\mu^p \int_0^{\frac{|\xi|}{2\mu}} \frac{r^{N-1}\, dr}{\big(r^{\frac{a(\lambda)}{\delta}} + r^{\frac{b(\lambda)}{\delta}}\big)^{N-p}} = O(\mu^{p(b(\lambda)-\delta)}) \to 0 \text{ as } \mu \to 0. \tag{3.8}$$

On the other hand, from (1.8) it follows that

$$\mu^p \int_{|x|\geqslant\frac{\xi}{2\mu}} \frac{|z_1^\lambda|^p}{|\mu x+\xi|^p} dx$$
$$\leqslant C\mu^{p(b(\lambda)-\delta)} \int_{|x-\xi|\geqslant\frac{|\xi|}{2}} \frac{\big(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\big)^{-(N-p)}}{|x|^p\, |x-\xi|^{p\,a(\lambda)}} dx$$
$$= C\mu^{p(b(\lambda)-\delta)} \Big(\int_{|x-\xi|\geqslant\frac{|\xi|}{2},\, |x|<2|\xi|} + \int_{|x-\xi|\geqslant\frac{|\xi|}{2},\, |x|\geqslant2|\xi|} \Big)$$
$$\leqslant C\mu^{p(b(\lambda)-\delta)} \Big(\int_0^{2|\xi|} r^{N-p-1}\, dr + \int_{2|\xi|}^{+\infty} \frac{dr}{r^{1+p(b(\lambda)-\delta)}} \Big)$$
$$= O(\mu^{p(b(\lambda)-\delta)}) \to 0 \text{ as } \mu \to 0. \tag{3.9}$$

(i). If $N > p^2,\ 0 < \lambda < \lambda^*$, then $b(\lambda) > \frac{N}{p},\ p(b(\lambda) - \delta) > p$ and $z_1^\lambda \in L^p(\mathbb{R}^N)$. From (3.6)-(3.8) it follows that

$$\mu^p \int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^p \frac{1}{|\mu x + \xi|^p} dx = \frac{\mu^p}{|\xi|^p} \int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^p dx + o(\mu^p)$$
$$= \frac{\mu^p}{|\xi|^p} \int_{\mathbb{R}^N} |z_1^\lambda(x)|^p dx + o(\mu^p).$$

Consequently,

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p} dx = \frac{\mu^p}{|\xi|^p} \int_{\mathbb{R}^N} |z_1^\lambda(x)|^p dx + o(\mu^p).$$

(ii). If $N > p^2$ and $\lambda = \lambda^*$, then $b(\lambda) = \frac{N}{p}$. From (1.8) it follows that

$$\mu^p \int_{|x|<\frac{|\xi|}{2\mu}} |z_1^\lambda(x)|^p dx = \mu^p \int_0^{\frac{|\xi|}{2\mu}} |z_1^\lambda(r)|^p\, r^{N-1} dr$$
$$= \mu^p \int_1^{\frac{|\xi|}{2\mu}} |z_1^\lambda(r)|^p\, r^{N-1} dr + O(\mu^p) \geqslant (\alpha_{\lambda,N})^p \mu^p |ln\,\mu| + O(\mu^p)$$
$$= (\alpha_{\lambda,N})^p \mu^p |ln\,\mu| + o(\mu^p |ln \mu|). \tag{3.10}$$

From (3.7)-(3.10) it follows that

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p} dx \geqslant \frac{(\alpha_{\lambda,N})^p \mu^p |ln\,\mu|}{|\xi|^p} + o(\mu^p |\ln \mu|).$$

On the other hand, by (1.8) and following the similar argument we obtain that

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p} dx \leqslant \frac{(\bar{\alpha}_{\lambda,N})^p \mu^p |ln\,\mu|}{|\xi|^p} + o(\mu^p |\ln \mu|)$$

for $b(\lambda) = \frac{N}{p}$ and $\mu \to 0$.

(iii). If $N > \max\{p^2, p+1\}$ and $\lambda > \lambda^*$, then $b(\lambda) < \frac{N}{p}$. From (1.8) it follows

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p} dx \geqslant (\alpha_{\lambda,N})^p \int_{\mathbb{R}^N} \frac{\mu^{p-N} dx}{|x+\xi|^p \Big(|\frac{x}{\mu}|^{\frac{a(\lambda)}{\delta}} + |\frac{x}{\mu}|^{\frac{b(\lambda)}{\delta}} \Big)^{N-p}}$$
$$= (\alpha_{\lambda,N})^p \mu^{p\beta} \int_{\mathbb{R}^N} \frac{\big(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\big)^{-(N-p)}}{|x|^p\, |x-\xi|^{p\,a(\lambda)}} dx$$
$$= (\alpha_{\lambda,N})^p \mu^{p\beta} \int_{\mathbb{R}^N} \left(\frac{\big(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\big)^{p-N}}{|x|^p |x-\xi|^{pa(\lambda)}} - \frac{1}{|x|^p |x-\xi|^{pb(\lambda)}} \right) dx$$
$$+\ (\alpha_{\lambda,N})^p\, \mu^{p\beta} \int_{\mathbb{R}^N} \frac{dx}{|x|^p\, |x-\xi|^{p\,b(\lambda)}}. \tag{3.11}$$

For $a,b \geqslant 0$ and $\tau > 1$, the following elementary inequality holds:

$$0 \leqslant (a+b)^{\tau} - a^{\tau} \leqslant C\,(a^{\tau-1}b + b^{\tau}),$$

where $C = C(\tau) > 0$ is some constant. From the assumption $N > p+1$ it follows that

$$\begin{aligned}
&\left|\frac{\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p|x-\xi|^{pa(\lambda)}} - \frac{1}{|x|^p|x-\xi|^{pb(\lambda)}}\right| \\
&= \frac{\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)}} \times \\
&\qquad \left(\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{N-p} - \left(|x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{N-p}\right) \\
&\leqslant C\,\frac{\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)}} \times \\
&\qquad \left(|x-\xi|^{\frac{(b(\lambda)-a(\lambda))(N-p-1)}{\delta}}\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + \mu^{p(b(\lambda)-a(\lambda))}\right) \\
&= C\Bigg(\frac{\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}}\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)-\frac{(b(\lambda)-a(\lambda))(N-p-1)}{\delta}}} \\
&\qquad + \frac{\mu^{p(b(\lambda)-a(\lambda))}\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)}}\Bigg).
\end{aligned} \tag{3.12}$$

If $\mu \to 0$, there follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}}\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)-\frac{(b(\lambda)-a(\lambda))(N-p-1)}{\delta}}}\,dx \\
&\leqslant C\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} \int_{|x-\xi|>\frac{|\xi|}{2},\,|x|<2|\xi|} \frac{dx}{|x|^p} \\
&\qquad + C\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} \int_{|x-\xi|>\frac{|\xi|}{2},\,|x|>2|\xi|} \frac{dx}{|x|^{p+pb(\lambda)+\frac{p}{N-p}(b(\lambda)-a(\lambda))}} \\
&\qquad + C\,\mu^{N-pb(\lambda)} \int_0^{\frac{|\xi|}{2\mu}} \frac{r^{N-1}\,dr}{r^{p\,b(\lambda)-\frac{(b(\lambda)-a(\lambda))(N-p-1)}{\delta}}\left(1+r^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{N-p}} \\
&= O\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right) = o(1) \quad \text{as } \mu \to 0.
\end{aligned}$$

By the same arguments we also have

$$\int_{\mathbb{R}^N} \frac{\mu^{p(b(\lambda)-a(\lambda))}\left(\mu^{\frac{b(\lambda)-a(\lambda)}{\delta}} + |x-\xi|^{\frac{b(\lambda)-a(\lambda)}{\delta}}\right)^{-(N-p)}}{|x|^p\,|x-\xi|^{p\,b(\lambda)}}\,dx = o(1) \quad \text{as } \mu \to 0.$$

From (3.11) and (3.12) we deduce that

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p}\,dx \geqslant (\alpha_{\lambda,N})^p \mu^{p\beta} \int_{\mathbb{R}^N} \frac{dx}{|x|^p|x-\xi|^{p\,b(\lambda)}} + o(\mu^{p\beta}).$$

If $b(\lambda) < \frac{N}{p}$ and $\mu \to 0$, from (1.8) and by the similar argument we have

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda|^p}{|x+\xi|^p}dx \leqslant (\bar{\alpha}_{\lambda,N})^p \mu^{p\beta} \int_{\mathbb{R}^N} \frac{dx}{|x|^p|x-\xi|^{pb(\lambda)}} + o(\mu^{p\beta}).$$

On the other hand, the function

$$\varphi(\xi) = \int_{\mathbb{R}^N} \frac{1}{|x|^p\,|x-\xi|^{pb(\lambda)}}dx$$

is invariant by rotation. Furthermore,

$$\varphi(\eta\xi) = \eta^{p(\delta-b(\lambda))}\varphi(\xi) = \eta^{-p\beta}\varphi(\xi), \quad \forall \eta > 0,$$

$$\varphi(\xi) = \varphi\big(|\xi|\,(\xi/|\xi|)\big) = |\xi|^{-p\beta}\,\varphi(\xi/|\xi|) = |\xi|^{-p\beta}\,\varphi(e_1).$$

Thus the proof of the lemma is complete.

LEMMA 3.2. *Suppose* $(\mathscr{H}_1)$, $N > \max\{p^2, p+1\}$ *and* $j \in \{1,2,\cdots,k\}$. *Assume that one of the following conditions is satisfied:*

(i) $0 < \lambda_j < \lambda^*$ *and* $\displaystyle\sum_{i\neq j, i=1}^{k} \frac{\lambda_i}{|a_i-a_j|^p} > 0,$

(ii) $\lambda^* \leqslant \lambda_j < \bar{\lambda}$ *and* $\displaystyle\Lambda_j := \sum_{i=1}^{l} \frac{\big(\mathscr{C}_2(\lambda_j)\big)^p\lambda_i}{|a_i-a_j|^{p\beta_j}} + \sum_{i\neq j, i=l+1}^{k} \frac{\big(\mathscr{C}_1(\lambda_j)\big)^p\lambda_i}{|a_i-a_j|^{p\beta_j}} > 0.$

Then $A < S(\lambda_j)$.

Proof. If $0 < \lambda_j < \bar{\lambda}$ and $\mu \to 0$, from Lemma 3.1 it follows that

$$\begin{aligned}
A &\leqslant \int_{\mathbb{R}^N}\Big(|\nabla z_\mu^{\lambda_j}(x-a_j)|^p - \sum_{i=1}^{k}\lambda_i\frac{|z_\mu^{\lambda_j}(x-a_j)|^p}{|x-a_i|^p}\Big)dx \\
&= \int_{\mathbb{R}^N}\Big(|\nabla z_\mu^{\lambda_j}(x)|^p - \lambda_j\frac{|z_\mu^{\lambda_j}(x)|^p}{|x|^p}\Big)dx - \sum_{i\neq j, i=1}^{k}\lambda_i\int_{\mathbb{R}^N}\frac{|z_\mu^{\lambda_j}(x)|^p}{|x+a_j-a_i|^p}dx \\
&= S(\lambda_j) - \sum_{i\neq j, i=1}^{k}\lambda_i\int_{\mathbb{R}^N}\frac{|z_\mu^{\lambda_j}(x)|^p}{|x+a_j-a_i|^p}dx + o(1) \\
&\leqslant S(\lambda_j) - \begin{cases} C\mu^p\Big(\displaystyle\sum_{i\neq j, i=1}^{k}\frac{\lambda_i}{|a_i-a_j|^p} + o(1)\Big), \ 0 < \lambda_j < \lambda^*, \\ C\mu^p\,|ln\mu|\,\big(\Lambda_j + o(1)\big), \ \lambda_j = \lambda^*, \\ C\mu^{p\beta_j}\,\big(\Lambda_j + o(1)\big), \ \lambda^* < \lambda_j < \bar{\lambda} \end{cases}.
\end{aligned}$$

Under the assumption either (i) or (ii), we have that $A < S(\lambda_j)$.

Proof of Theorem 1.1. Let $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ be a minimizing sequence for A. By the homogeneity of the quotient we can assume that $\int_{\mathbb{R}^N} |u_n|^{p^*} = 1$. From the Ekeland's variational principle we can assume that the sequence has the Palais-Smale property:

$$o(\|v\|) = \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla v\, dx - \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{\lambda_i |u_n|^{p-2} u_n v}{|x-a_i|^p} dx - A \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n v\, dx.$$

Hence $J'(u_n) \to 0$ in $D^{-1,p}(\mathbb{R}^N)$ and $J(u_n) \to \frac{1}{N}A$. Note that $S(\lambda)$ is strictly decreasing with respect to λ. By $(\mathscr{H}_1)$ and Lemma 3.2 we have

$$A < S(\lambda_k) \leqslant S(\lambda_{k-1}) \leqslant \cdots \leqslant S(\lambda_1), \quad S(\lambda_k) < S(\hat{\lambda}), \quad S(\lambda_k) < S(0).$$

Consequently,

$$A < \min\Big\{S(0),\ S(\lambda_1),\ S(\lambda_2),\ \cdots,\ S(\lambda_k),\ S(\hat{\lambda})\Big\},$$
$$\frac{1}{N}A < \frac{1}{N}A^{1-\frac{N}{p}}\Big(\min\{S(0),\ S(\lambda_1),\ S(\lambda_2),\ \cdots,\ S(\lambda_k),\ S(\hat{\lambda})\}\Big)^{\frac{N}{p}}.$$

By Lemma 2.1 we conclude that $\{u_n\}$ has a subsequence converging strongly to some $u_0 \in D^{1,p}(\mathbb{R}^N)$. Moreover, $J(u_0) = \frac{1}{N}A$. Thus u_0 achieves the infimum in (1.9). From the fact that $J(u_0) = J(|u_0|)$ it follows that $|u_0|$ is also a minimizer in (1.9) and therefore $v_0 = A^{\frac{1}{p^*-p}}|u_0|$ is a nonnegative solution of (1.1). By the maximum principle [20], we have $v_0 > 0$ in $\mathbb{R}^N \setminus \{a_1, a_2, \cdots, a_k\}$. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first establish several lemmas.

LEMMA 4.1. *Let $z_\mu^\lambda(x)$ be the function defined as in* (3.6). *Then for any $\xi \in \mathbb{R}^N$, $\lambda \in (0, \bar{\lambda})$ and $\mu \to +\infty$ we have*

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx = \int_{\mathbb{R}^N} \frac{|z_1^\lambda(x)|^p}{|x|^p} dx + o(1) = \int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x|^p} dx + o(1). \tag{4.1}$$

Proof. Setting $x = \mu y$, we have

$$\int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(x)|^p}{|x+\xi|^p} dx = \mu^{p-N} \int_{\mathbb{R}^N} \frac{|z_1^\lambda(\frac{x}{\mu})|^p}{|x+\xi|^p} dx = \int_{\mathbb{R}^N} \frac{|z_1^\lambda(y)|^p}{|y+\frac{\xi}{\mu}|^p} dy. \tag{4.2}$$

Note that $|\frac{\xi}{\mu}| \to 0$ as $\mu \to \infty$. By the continuity property of convolution and density arguments we have

$$\int_{\mathbb{R}^N} \frac{|z_1^\lambda(y)|^p}{|y+\frac{\xi}{\mu}|^p} dy = \int_{\mathbb{R}^N} \frac{|z_1^\lambda(y)|^p}{|y|^p} dy + o(1) = \int_{\mathbb{R}^N} \frac{|z_\mu^\lambda(y)|^p}{|y|^p} dy + o(1). \tag{4.3}$$

From (4.2) and (4.3) we can reach the desired conclusion (4.1).

LEMMA 4.2. *Assume that* $a_i \in \mathbb{R}^N$, $\lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k$, *and* $\hat{\lambda} \in (0, \bar{\lambda})$ *is defined as in* (3.1). *Then* $A \leqslant S(\hat{\lambda})$.

Proof. By Lemma 4.1, as μ large enough we have

$$A \leqslant \left(\int_{\mathbb{R}^N} \left(|\nabla z_\mu^{\hat{\lambda}}|^p - \sum_{i=1}^k \frac{\lambda_i |z_\mu^{\hat{\lambda}}|^p}{|x-a_i|^p} \right) dx \right) \left(\int_{\mathbb{R}^N} |z_\mu^{\hat{\lambda}}|^{p^*} dx \right)^{-\frac{p}{p^*}}$$
$$= \left(\int_{\mathbb{R}^N} \left(|\nabla z_\mu^{\hat{\lambda}}|^p - \hat{\lambda} \frac{|z_\mu^{\hat{\lambda}}|^p}{|x|^p} \right) dx \right) \left(\int_{\mathbb{R}^N} |z_\mu^{\hat{\lambda}}|^{p^*} dx \right)^{-\frac{p}{p^*}} + o(1) = S(\hat{\lambda}) + o(1).$$

Taking $\mu \to +\infty$ we can conclude the lemma.

LEMMA 4.3. *Assume that* $\lambda_i > 0$, $i = 1, 2, \cdots, k$ *and* $\hat{\lambda} < \bar{\lambda}$. *Then*

$$A = S(\hat{\lambda}). \tag{4.4}$$

Proof. For any $a \in \mathbb{R}^N, u \in D^{1,p}(\mathbb{R}^N)$ and $u \geqslant 0$ a.e. in $\mathbb{R}^N$, we have [21]:

$$\int_{\mathbb{R}^N} u^{p^*} dx = \int_{\mathbb{R}^N} |u^*(x)|^{p^*} dx, \tag{4.5}$$
$$\int_{\mathbb{R}^N} \frac{u^p}{|x-a|^p} dx \leqslant \int_{\mathbb{R}^N} |u^*(x)|^p \left(\left(\frac{1}{|x-a|} \right)^* \right)^p dx,$$

where $u^* = \inf\{t > 0 : |\{y \in \mathbb{R}^N, u(y) > t\}| \leqslant \omega_N |x|^N\}$ is the Schwarz symmetrization of u and $|\cdot|$ is the Lebesgue measure of $\mathbb{R}^N$, ω_N is the volume of the unit ball in $\mathbb{R}^N$. Direct calculation shows that

$$\left(\frac{1}{|x-a|} \right)^* = \frac{1}{|x|}.$$

Consequently,

$$\int_{\mathbb{R}^N} \frac{u^p}{|x-a|^p} dx \leqslant \int_{\mathbb{R}^N} \frac{|u^*(x)|^p}{|x|^p} dx. \tag{4.6}$$

From the Pólya-Szegö inequality it follows that

$$\int_{\mathbb{R}^N} |\nabla u^*|^p dx \leqslant \int_{\mathbb{R}^N} |\nabla u|^p dx. \tag{4.7}$$

Thus for all $u \in D^{1,p}(\mathbb{R}^N)$ and $u \geqslant 0$ a.e. in $\mathbb{R}^N$ we have that

$$\left(\int_{\mathbb{R}^N} \left(|\nabla u|^p - \sum_{i=1}^k \frac{\lambda_i u^p}{|x-a_i|^p} \right) dx \right) \left(\int_{\mathbb{R}^N} \sum_{i=1}^k u^{p^*} dx \right)^{-\frac{p}{p^*}}$$
$$\geqslant \left(\int_{\mathbb{R}^N} \left(|\nabla u^*|^p - \hat{\lambda} \frac{|u^*|^p}{|x|^p} \right) dx \right) \left(\int_{\mathbb{R}^N} |u^*|^{p^*} dx \right)^{-\frac{p}{p^*}} \geqslant S(\hat{\lambda}). \tag{4.8}$$

Note that the Rayleigh quotient above remains unchanged when replacing u with $|u|$. Thus we have

$$A = \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}, u \geqslant 0} \frac{\displaystyle\int_{\mathbb{R}^N} \Big(|\nabla u|^p - \sum_{i=1}^{k} \frac{\lambda_i u^p}{|x-a_i|^p}\Big) dx}{\Big(\displaystyle\int_{\mathbb{R}^N} u^{p^*} dx\Big)^{p/p^*}}.$$

From (4.8) it follows that $A \geqslant S(\hat{\lambda})$, which together with Lemma 4.2 implies that (4.4) holds.

Proof of Theorem 1.2. We argue by contradiction. Assume that the infimum in (1.9) is attained by some $u_0 \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}$. Since $|u_0|$ is also a minimizer in (1.9), we may assume that $u_0 \geqslant 0$ a.e. in $\mathbb{R}^N$ and therefore the Schwarz symmetrization u_0^* can be employed. From (4.5)-(4.8) we obtain

$$\begin{aligned} A = &\Big(\int_{\mathbb{R}^N} \Big(|\nabla u_0|^p - \sum_{i=1}^{k} \frac{\lambda_i u_0^p}{|x-a_i|^p}\Big) dx\Big) \Big(\int_{\mathbb{R}^N} u_0^{p^*} dx\Big)^{-\frac{p}{p^*}} \\ &\geqslant \Big(\int_{\mathbb{R}^N} \Big(|\nabla u_0^*|^p - \hat{\lambda} \frac{|u_0^*|^p}{|x|^p}\Big) dx\Big) \Big(\int_{\mathbb{R}^N} |u_0^*|^{p^*} dx\Big)^{-\frac{p}{p^*}} \geqslant S(\hat{\lambda}). \quad (4.9) \end{aligned}$$

From Lemma 4.3 we deduce that all inequalities in (4.9) are indeed equalities. In particular we have:

$$\int_{\mathbb{R}^N} \Big(|\nabla u_0^*|^p - \hat{\lambda} \frac{|u_0^*|^p}{|x|^p}\Big) dx = S(\hat{\lambda}) \Big(\int_{\mathbb{R}^N} |u_0^*|^{p^*} dx\Big)^{\frac{p}{p^*}}, \quad (4.10)$$

$$\int_{\mathbb{R}^N} \Big(|\nabla u_0|^p - \sum_{i=1}^{k} \frac{\lambda_i u_0^p}{|x-a_i|^p}\Big) dx = \int_{\mathbb{R}^N} \Big(|\nabla u_0^*|^p - \hat{\lambda} \frac{|u_0^*|^p}{|x|^p}\Big) dx. \quad (4.11)$$

Thus from (4.6), (4.7) and (4.11) it follows that

$$0 \leqslant \int_{\mathbb{R}^N} |\nabla u_0|^p dx - \int_{\mathbb{R}^N} |\nabla u_0^*|^p dx = \sum_{i=1}^{k} \frac{\lambda_i u_0^p}{|x-a_i|^p} - \hat{\lambda} \frac{|u_0^*|^p}{|x|^p} \leqslant 0.$$

Therefore

$$\int_{\mathbb{R}^N} |\nabla u_0|^p dx = \int_{\mathbb{R}^N} |\nabla u_0^*|^p dx. \quad (4.12)$$

By (4.10) we infer that u_0^* is a minimizer of $S(\hat{\lambda})$ and solves the equation (1.4) with $a = 0$ and $\lambda = \hat{\lambda}$. From [1] it follows that u_0^* must belongs to the family of $\mu^{-\delta} U_{p,\lambda}\big(\frac{|x|}{\mu}\big)$ for some $\mu > 0$ and $\lambda = \hat{\lambda}$, where $U_{p,\lambda}$ is defined as in (1.5). From the fact that $u^*(|x|)$ is strictly decreasing we have

$$\big|\{x \in \mathbb{R}^N \,|\, \nabla u_0^*(x) = 0\}\big| = 0. \quad (4.13)$$

By (4.12) and (4.13) we conclude that there exists some point $x_0 \in \mathbb{R}^N$ such that $u_0 = u_0^*(\cdot - x_0)$ [3]. Thus u_0 is spherically symmetric with respect to x_0. Since u_0 is a minimizer in (1.9), $v_0 = A^{\frac{1}{p^*-p}} u_0$ is a solution of the problem (1.1). Consequently, $\sum_{i=1}^k \frac{\lambda_i}{|x-a_i|^p}$ must be spherically symmetric with respect to x_0, which gives a contradiction. Therefore the infimum in (1.9) cannot be achieved. The proof of Theorem 1.2 is complete. □

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