

ON A CONJECTURE FOR THREE-DIMENSIONAL COMPETITIVE LOTKA-VOLTERRA SYSTEMS WITH A HETEROCLINIC CYCLE

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Abstract. For three-dimensional competitive Lotka-Volterra systems, Zeeman (1993) identified 33 stable equivalence classes. In this paper we show that: in the case of a heteroclinic cycle on the boundary of the carrying simplex of three-dimensional competitive Lotka-Volterra systems (class 27 in Zeeman’s classification), the conditions (a) there is a pair of purely imaginary eigenvalues at an interior equilibrium, (b) the first focal value vanishes, (c') the second focal value vanishes, and (c) the heteroclinic cycle is neutrally stable do *not* imply (d) the third focal value vanishes. In particular, the conditions (a), (b), (c'), and (c) do not imply that the interior equilibrium is a center. This proves a conjecture by Gyllenberg and Yan (2009).

1. Introduction

In this paper we consider the three-dimensional competitive Lotka-Volterra (LV) system

$$\frac{dx_i}{dt} = x_i \left(r_i - \sum_{j=1}^3 a_{ij}x_j \right), \quad i = 1, 2, 3. \quad (1)$$

Recall that the assumption of competitiveness means that $r_i > 0$, $a_{ij} > 0$, $i, j = 1, 2, 3$. For such systems quite a lot is known. M.W. Hirsch [6] has shown that all nontrivial orbits approach a “carrying simplex” Σ , a Lipschitz two-dimensional manifold-with-corner homeomorphic to the standard simplex in \mathbb{R}_+^3 . This then leads to a Poincaré-Bendixson theorem for three-dimensional systems (see H.L. Smith [13]). Thus, the long-term behavior of system (1) is determined by the dynamics on Σ , and the nonzero forward limit sets in \mathbb{R}_+^3 all lie on Σ . Based on this result of Hirsch, M.L. Zeeman [16] defined a combinatorial equivalence relation on the set of all three-dimensional competitive LV systems and identified 33 stable equivalence classes. Of these, classes 1-25 and classes 32-33 exhibit convergence to an equilibrium for all orbits while limit cycles are possible for the remaining six classes, i.e., in classes 26 to 31 (see [14, 16]).

In the case of a heteroclinic cycle on the boundary of the carrying simplex Σ of a three-dimensional competitive LV system, we have (see Hofbauer and Sigmund [8]):

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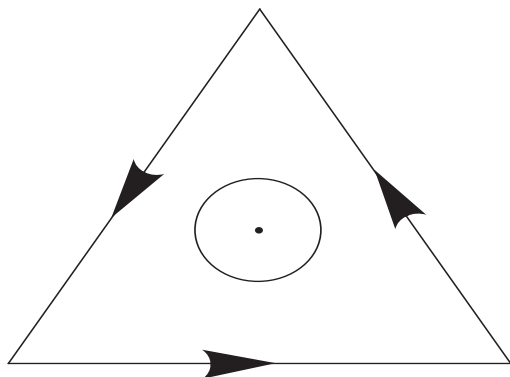


Figure 1: The phase portrait on Σ of class 27 with interior fixed point [16].

(i) the heteroclinic cycle is repelling if the characteristic number of the heteroclinic cycle is positive, i.e.,

$$P := \lambda_{12}\lambda_{23}\lambda_{31} + \lambda_{21}\lambda_{13}\lambda_{32} > 0, \text{ where } \lambda_{ij} = r_j - \frac{a_{ij}r_i}{a_{ii}};$$

(ii) the heteroclinic cycle is attracting if $P < 0$;

(iii) the stability of the heteroclinic cycle is undecided if $P = 0$.

Hofbauer and So [10] conjectured that, in the case of a heteroclinic cycle on the boundary of the carrying simplex, the following three conditions are equivalent to having a center:

- (a) there is a pair of purely imaginary eigenvalues at a positive equilibrium;
- (b) the first focal value vanishes;
- (c) $P = 0$.

Furthermore, Hofbauer and So [10] conjectured that condition (c) might be replaced by the condition:

- (c') the second focal value vanishes.

In the recent paper [3], we refuted both conjectures by proving the following theorems:

THEOREM 1.1. *In the case of a heteroclinic cycle on the boundary of the carrying simplex of a three-dimensional competitive LV system, the conditions:*

- (a) *there is a pair of purely imaginary eigenvalues at a positive equilibrium;*
- (b) *the first focal value vanishes;*
- (c') *the second focal value vanishes;*

do not imply,

(c) $P = 0$.

In particular, the conditions (a), (b) and (c') do not imply that the interior equilibrium is a center.

THEOREM 1.2. *In the case of a heteroclinic cycle on the boundary of the carrying simplex of a three-dimensional competitive LV system, the conditions:*

(a) *there is a pair of purely imaginary eigenvalues at a positive equilibrium;*

(b) *the first focal value vanishes;*

(c) $P = 0$;

do not imply the condition,

(c') *the second focal value vanishes.*

In particular, the conditions (a), (b) and (c) do not imply that the interior equilibrium is a center.

In [3] we conjectured that in the case of a heteroclinic cycle on the boundary of the carrying simplex of a three-dimensional competitive LV system, the conditions (a), (b), (c), and (c') do not imply the condition:

(d) the third focal value vanishes.

In particular, the conditions (a), (b), (c), and (c') do not imply that the interior equilibrium is a center.

In this paper we present an example of a three-dimensional competitive LV system with a heteroclinic cycle for class 27 in Zeeman's classification, which proves this conjecture.

We denote by \mathbb{R}_+^3 and $\text{Int}\mathbb{R}_+^3$ the closed and open positive cone, respectively. The restriction of system (1) to the i th coordinate axis is the logistic equation $\dot{x}_i = x_i(r_i - a_{ii}x_i)$, which has a fixed point R_i at the carrying capacity r_i/a_{ii} . Note that we are abusing notation here, allowing R_i to denote a point in \mathbb{R}_+ or in \mathbb{R}_+^3 as dictated by the context.

It is easy to see that the origin is a repelling fixed point of system (1), and that the basin of repulsion of 0 in \mathbb{R}_+^3 is bounded. The boundary of that basin is called the carrying simplex of system (1), and is denoted by Σ . We refer to Hirsch [6] and Zeeman [16] for more details.

A vector x is called positive if $x \in \mathbb{R}_+^3$, strictly positive if $x \in \text{Int}\mathbb{R}_+^3$. Two points $u, v \in \mathbb{R}^3$ are related if either $u - v$ or $v - u$ is strictly positive. A set S is called balanced if no two distinct points of S are related. The unit simplex, Δ , in \mathbb{R}_+^3 has the standard meaning of $\Pi \cap \mathbb{R}_+^3$, where Π denotes the plane with equation $\sum_{i=1}^3 x_i = 1$.

The following theorem of Hirsch [6] shows that Σ is topologically and geometrically simple and that all the nonzero equilibria and other ω -limit sets of system (1) lie on Σ . In particular, the equilibria R_i and any nontrivial periodic orbit lie on Σ .

THEOREM 1.3. (Hirsch) *Given system (1), every trajectory in $\mathbb{R}_+^3 \setminus \{0\}$ is asymptotic to one in Σ , and Σ is a balanced Lipschitz submanifold, homeomorphic to the unit simplex in \mathbb{R}_+^3 by radial projection.*

By Theorem 1.1 the long term dynamics on \mathbb{R}_+^3 is completely determined by the dynamics on Σ . Moreover, Σ can be viewed (via radial projection) as the unit simplex Δ , which can then be removed from the ambient \mathbb{R}_+^3 and pictured as an equilateral triangle.

If there is no equilibrium in the interior of the carrying simplex Σ , then the dynamics of system (1) is trivial: every orbit converges to the boundary [8]. Therefore we are interested only in the case where system (1) has a positive equilibrium in the interior of Σ . Without loss of generality, we can assume, as indicated by the context, that $E = (1, 1, 1)$ is a positive equilibrium of system (1) and E has no zero eigenvalues.

2. Example for Class 27 with a heteroclinic cycle

In this section, we present an example of a three-dimensional competitive LV system with a heteroclinic cycle for class 27 in Zeeman’s classification, which show that: the conditions (a) there is a pair of purely imaginary eigenvalues at an interior equilibrium, (b) the first focal value vanishes, (c’) the second focal value vanishes, and (c) the heteroclinic cycle is neutrally stable do *not* imply (d) the third focal value vanishes. In particular, the conditions (a), (b), (c’), and (c) do not imply that the interior equilibrium is a center.

EXAMPLE. Consider the three-dimensional competitive LV system

$$\dot{x}_i = x_i[A(x - E)]_i, \quad i = 1, 2, 3, \tag{2}$$

where

$$A = (a_{ij}) = \begin{pmatrix} -1 & -\frac{3}{2} & \lambda \\ n & -1 & \mu \\ -1 & -3 & -2 \end{pmatrix}$$

with three negative parameters μ , λ and n . By elementary linear algebra, a necessary condition that A has a negative real eigenvalue and a pair of purely imaginary eigenvalues is

$$\det(A) = (M_{23} + M_{13} + M_{12}) \cdot \text{tr}A,$$

where $\text{tr}(A) = \sum_{i=1}^3 a_{ii}$, $M_{23} = a_{22}a_{33} - a_{23}a_{32}$, $M_{13} = a_{11}a_{33} - a_{13}a_{31}$ and $M_{12} = a_{22}a_{11} - a_{12}a_{21}$. A simple calculation yields that

$$\mu = \mu(n, \lambda) := -\frac{12}{7} - \frac{2}{7}n + \frac{2}{7}n\lambda - \frac{2}{7}\lambda. \tag{3}$$

Let $y_i = x_i - 1$, $i = 1, 2, 3$, and set $z = Ty$, where $y = (y_1, y_2, y_3)^T$, $z = (z_1, z_2, z_3)^T$, and

$$T = \begin{pmatrix} n & 3 & -\frac{12}{7} - \frac{2}{7}n + \frac{2}{7}n\lambda - \frac{2}{7}\lambda \\ -1 & -3 & \frac{2}{7} \\ \frac{6}{7} - \frac{6}{7}n + \frac{6}{7}n\lambda - \frac{6}{7}\lambda & 3 - 3\lambda & \frac{18}{7} + \frac{3}{7}n - \frac{3}{7}n\lambda - \frac{18}{7}\lambda \end{pmatrix}.$$

The system (2) is transformed into a new one whose linear part is in the block diagonal form

$$\text{linear part} = \begin{pmatrix} \frac{2+5n}{2(-1+n)} & \frac{-8n\lambda+4n^2\lambda+3n^2+22n+4\lambda+24}{14(-1+n)} & 0 \\ -\frac{1+6n}{2(-1+n)} & -\frac{2+5n}{2(-1+n)} & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

This can be reduced to the two-dimensional case by computing the center manifold

$$z_3 = F(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2) + f_3(z_1, z_2) + f_4(z_1, z_2) + f_5(z_1, z_2) + f_6(z_1, z_2) + h.o.t.,$$

where $f_i(z_1, z_2) = \sum_{j=0}^i c_{ij}z_1^{i-j}z_2^j$, $i = 1, 2, 3, 4, 5, 6$, and h.o.t. denotes the terms with order greater than or equal to seven. Solving for the c_{ij} 's and substituting by appealing to the method in [11] one obtains the following rather complicated and lengthy expressions for the first focal value $LV_1(n, \lambda)$ and the second focal value $LV_2(n, \lambda)$:

$$LV_1(n, \lambda) = \frac{f_1(n, \lambda)}{f_2(n, \lambda)} \quad \text{and} \quad LV_2(n, \lambda) = \frac{g_1(n, \lambda)}{g_2(n, \lambda)},$$

where

$$\begin{aligned} f_1(n, \lambda) = & 16(-3144 + 12724n + 384n^5\lambda^4 - 1104\lambda \\ & - 1873n^3 - 13095n^4 + 41444n\lambda - 22701n^4\lambda^2 \\ & - 28578n^4\lambda + 14105n^3\lambda^2 - 4843n^3\lambda - 2106n^5 \\ & + 11490n\lambda^3 - 19886n^2\lambda^3 - 2816n^4\lambda^4 + 18654n^3\lambda^3 \\ & - 13072n^4\lambda^3 + 5856n^3\lambda^4 + 7494n^2 - 2248n^2\lambda \\ & - 27153n^2\lambda^2 + 35413n\lambda^2 + 2226\lambda^2 - 4512n^2\lambda^4 \\ & + 800n\lambda^4 - 4671n^5\lambda + 1734\lambda^3 \\ & + 288\lambda^4 - 1890n^5\lambda^2 + 1080n^5\lambda^3), \end{aligned}$$

$$\begin{aligned} f_2(n, \lambda) = & 62208n^5\lambda^3 + 139968n^5\lambda^2 + 104976n^5\lambda \\ & + 26244n^5 + 51840n^4\lambda^3 + 2674944n^4\lambda^2 + 3924936n^4\lambda \\ & + 1460916n^4 + 17280n^3\lambda^3 + 1744416n^3\lambda^2 + 33593292n^3\lambda \\ & + 24221025n^3 + 2880n^2\lambda^3 + 432864n^2\lambda^2 + 16147350n^2\lambda \\ & + 103760514n^2 + 240n\lambda^3 + 47916n\lambda^2 + 2655288n\lambda \\ & + 32595372n + 8\lambda^3 + 146520\lambda + 1992\lambda^2 + 2661336, \end{aligned}$$

$g_1(n, \lambda)$ and $g_2(n, \lambda)$ are given in the Appendix.

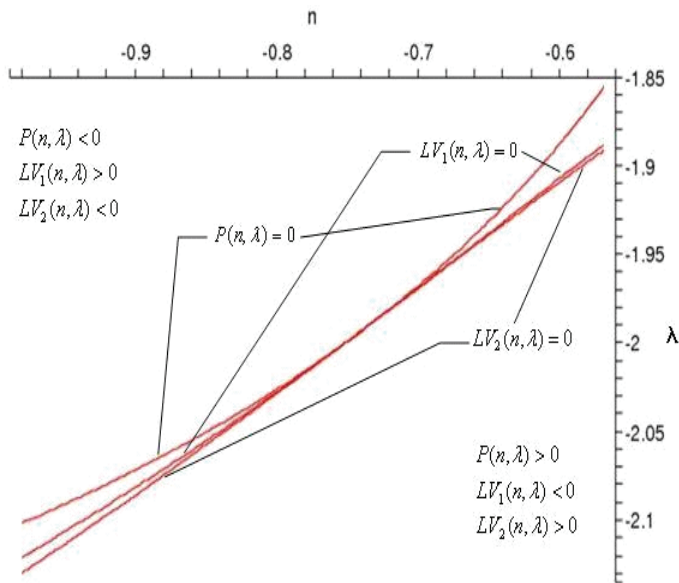


Figure 2: Three curves $P(n, \lambda) = 0$, $LV_1(n, \lambda) = 0$ and $LV_2(n, \lambda) = 0$ intersect at the point $(n_0, \lambda_0) = (-0.75, -2)$.

We computed $LV_1(n, \lambda)$ and $LV_2(n, \lambda)$ as rational functions by using the computer algebraic system Maple. The expression of $P(n, \lambda)$ (i.e., the characteristic number of the heteroclinic cycle) is given as follows:

$$\begin{aligned}
 P(n, \lambda) = & -(-n(5/2 - \lambda) - 19/7 + 5n/7 + 2n\lambda/7 - 2\lambda/7) \\
 & (15/7 - 15n/7 - 6n\lambda/7 + 6\lambda/7)(-2\lambda - 5/2) \\
 & -(11/7 - 15n/14 - 3n\lambda/7 + 10\lambda/7) \\
 & (-7/2 - \lambda)(17/7 + 11n/7 - 4n\lambda/7 + 4\lambda/7).
 \end{aligned}$$

A straightforward calculation yields that $LV_1(n, \lambda)$ and $LV_2(n, \lambda)$ have a unique root $(n_0, \lambda_0) = (-0.75, -2) \in U_\varepsilon$, the ε -neighborhood ($\varepsilon = 10^{-5}$) of the point $(-0.75, -2)$ (see Figure 2, above).

Now, we choose $n = n_0$ and $\lambda = \lambda_0$ and adjust $\mu = \mu(n_0, \lambda_0) = -\frac{12}{7} - \frac{2}{7}n_0 + \frac{2}{7}n_0\lambda_0 - \frac{2}{7}\lambda_0$ which keeps the linear part of system (2) in a center-focus form. Noting that $P(n_0, \lambda_0) = 0$, $LV_1(n_0, \lambda_0) = 0$, $LV_2(n_0, \lambda_0) = 0$, and

$$LV_3(n_0, \lambda_0) = \frac{794314633104083832799}{23397286781621005120311720} > 0.$$

Moreover, $\det(A) < 0$ for $(n, \lambda) \in U_\varepsilon$, and $\mu = \mu(n, \lambda) < 0$ for $(n, \lambda) \in U_\varepsilon$. It follows that for $(n, \lambda) \in U_\varepsilon$ and $\mu = \mu(n, \lambda)$ system (2) is a competitive system that satisfies the condition of the eigenvalues; that is, for $(n, \lambda) \in U_\varepsilon$ and $\mu = \mu(n, \lambda)$, the equilibrium E of system (2) has a negative real eigenvalue and a pair of purely imaginary

eigenvalues. Since for $(n, \lambda) \in U_\varepsilon$ and $\mu = \mu(n, \lambda)$, $R_{12} = 1$, $R_{13} = -1$, $R_{21} = -1$, $R_{23} = 1$, $R_{31} = 1$, $R_{32} = -1$ (where $R_{ij} = \text{sgn}(\alpha_{ij})$ with $\alpha_{ij} = r_i a_{ji} / a_{ii} - r_j$), then system (2) with $(n, \lambda) \in U_\varepsilon$ and $\mu = \mu(n, \lambda)$ belongs to class 27 in Zeeman's classification. By the above discussion, we have the following theorem.

THEOREM 2.1. *In the case of a heteroclinic cycle on the boundary of the carrying simplex of three-dimensional competitive Lotka-Volterra systems, the conditions (a), (b), (c), and (c') do not imply (d). In particular, the conditions (a), (b), (c), and (c') do not imply that the interior equilibrium is a center.*

3. Appendix: The explicit expressions for the polynomials g_1 and g_2

The polynomials g_1 and g_2 are as follows:

$$\begin{aligned}
 g_1(n, \lambda) = & -8/5(2088342451079156603879424n \\
 & +427114949657639107166208\lambda \\
 & +4670582623656054574841856n\lambda \\
 & -1538210013412371568888104n^{11} \\
 & +1274230043769757154445n^{14} \\
 & +197057486090050498215n^{15} \\
 & -15997073309887942326474n^{13} \\
 & -233034784794613776384\lambda^{12}n^{12} \\
 & -105410494156350013440\lambda^{12}n^{10} \\
 & -77226061024198656n^{18}\lambda^{12} \\
 & -2710795117817856\lambda^{12}n^2 \\
 & +16705452734969230282401792n^2\lambda^2 \\
 & +146400042392810422272\lambda^{12}n^{13} \\
 & +15405050456699830272\lambda^{12}n^{15} \\
 & +5240143632797422683070464n\lambda^2 \\
 & -678070962302976\lambda^{12}n \\
 & +224377242601700130816\lambda^{12}n^{11} \\
 & -5890011808314477728\lambda^{12}n^{14} \\
 & +463918244908423501873152\lambda^2 \\
 & +8512664285691445150162944n^2 \\
 & +16648923307993026867302400n^2\lambda \\
 & +7153684592984064n^{19}\lambda^{12} \\
 & +320779856598872059330560\lambda^3 \\
 & -7312552507028932308n^{17}\lambda \\
 & +155610350896070218383360\lambda^4 \\
 & +19184119846008319852736000n^5\lambda^5 \\
 & -3766775971986479138527296n^{10} \\
 & +4062511633508931729858560n^4\lambda^5 \\
 & -270473093729703360272124n^{12}
 \end{aligned}$$

$$\begin{aligned}
& -32824208078224516885623552n^4\lambda^2 \\
& -39694020384815528966876160n^4\lambda \\
& +3776787691565137248043008n\lambda^3 \\
& +10671745266078595923747840n^2\lambda^3 \\
& -16014481250525298206368768n^3\lambda^3 \\
& -9790123482514249173765120n^3\lambda^4 \\
& +4426009920690000887790592n^2\lambda^4 \\
& +6304619005550403497649664n^4\lambda^4 \\
& -5379325052307743488684032n^4\lambda^3 \\
& +1851615146510956089280512n\lambda^4 \\
& -389947093560426312n^{18}\lambda \\
& +52372966280262646001664\lambda^5 \\
& -14906741272305985952113152n^3\lambda^2 \\
& -2126832778576442404657152n^3\lambda \\
& -1463711477450423063296\lambda^{10}n^6 \\
& -7280855155102051719354744\lambda n^{11} \\
& -2140030569697957440n^{18}\lambda^2 \\
& -28584881673599664n^{19}\lambda^2 \\
& -6822578717469709599744\lambda^{10}n^9 \\
& +17272475206627128768768\lambda^{10}n^8 \\
& +2164993618645099854n^{16} \\
& +13247695239309239459074560n^7 \\
& +7305815309568016158627840n^3 \\
& -10192027944306729462008832n^4 \\
& -19497875396163754137039360n^5 \\
& -41328988235481483592229160\lambda^2n^{10} \\
& -82609347063714652160\lambda^{11}n^6 \\
& +817279289585882615808\lambda^{11}n^9 \\
& -579263688986018536234287\lambda^2n^{13} \\
& +634739536944241825792\lambda^{11}n^8 \\
& +475446724480991232\lambda^{11}n^{17} \\
& -1209049207964953067979\lambda^2n^{15} \\
& +40147983705226825728\lambda^{11}n^5 \\
& -60769495195593240121821\lambda^2n^{14} \\
& +10700074450944\lambda^{13}n \\
& -3149874278562004992\lambda^{10}n^{17} \\
& +54065662579241421242174464n^5\lambda^3 \\
& +1589766230079047497984\lambda^{10}n^5 \\
& +12336386926007810048\lambda^{11}n^4 \\
& +5450333544781762560\lambda^9 \\
& -9573112961705705472\lambda^{11}n^{16} \\
& -336983608230143014912\lambda^{11}n^7 \\
& -6888503874960479232n^{18}\lambda^3
\end{aligned}$$

$$\begin{aligned}
 & -218244176536864768\lambda^{11}n^3 \\
 & -738263503753445376n^{18}\lambda^9 \\
 & -109551306989294208^{19}\lambda^3 \\
 & +24454357411037346396242688n^5\lambda^2 \\
 & -22699989063322528384184832n^5\lambda \\
 & -267047751327324098281968n^8\lambda^6 \\
 & +11973077950118294576564352n^8\lambda \\
 & -65804976623962487174271936n^9\lambda^3 \\
 & -57232485545950314366840864n^9\lambda^2 \\
 & -21551155324910948726788032n^9\lambda \\
 & -2126364105605186837701152n^9\lambda^6 \\
 & -13639623120638502150957120n^9\lambda^5 \\
 & -39661773236816626596932904n^9\lambda^4 \\
 & -2524234430043243085824\lambda^{11}n^{12} \\
 & +201182805493183130112\lambda^{10}n^4 \\
 & +1013422558920205271040\lambda^{11}n^{13} \\
 & -675637716179551212n^{17} \\
 & -13847170127706453061486242\lambda^2n^{11} \\
 & -424598443175182336\lambda^{11}n^2 \\
 & -3257093682992061628416\lambda^{11}n^{10} \\
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 & -298462637965672710144\lambda^{11}n^{14} \\
 & +3921236138643255558144\lambda^{11}n^{11} \\
 & -277307775629141760n^{19}\lambda^4 \\
 & -53243718011723776\lambda^{11}n \\
 & -20060737114147246080n^{18}\lambda^5 \\
 & +75718954694007300328627968n^6\lambda^2 \\
 & +32254191532249430717417472n^6\lambda \\
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 & -34580538332282880n^{19}\lambda^9 \\
 & -3102349426244381120980068\lambda^3n^{12} \\
 & -18843856037824905216n^{18}\lambda^6 \\
 & -67001453043661501440\lambda^{10}n^3 \\
 & -8998512301351281675520\lambda^{10}n^7 \\
 & -2360850696743878656\lambda^{10}n^{16} \\
 & -7090789488324829925376\lambda^{10}n^{10} \\
 & +132822647387994415104\lambda^{10}n^{12} \\
 & +52453364433543168\lambda^{10} \\
 & -586594440973541376n^{19}\lambda^6 \\
 & -44588319436513927135157148\lambda^3n^{10} \\
 & -483884000420990976n^{19}\lambda^5
 \end{aligned}$$

$$\begin{aligned}
& -11808791323167398912\lambda^{10}n^2 \\
& -983002939072838389009332\lambda^3n^{13} \\
& -234965563628545621951296\lambda^8n^6 \\
& -46662479796891524931712\lambda^8n^9 \\
& +265570502332051994720640\lambda^8n^5 \\
& -49715108634764132352\lambda^8n^{17} \\
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& -2246022539013098668032\lambda^{10}n^{13} \\
& -71010444885388886016\lambda^{10}n^{15} \\
& -9394034853816778724280\lambda^3n^{15} \\
& +896299322675025936384\lambda^{10}n^{14} \\
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& -173509193058435630822528\lambda^3n^{14} \\
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& -274410599379586552497152\lambda^8n^7 \\
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& -12877808522936090247415764\lambda^3n^{11} \\
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& +124691364362733249161216\lambda^6n \\
& -5393737054062614470642416n^8\lambda^5 \\
& -19195644145099164052150368n^8\lambda^4 \\
& -28653759458087117328979968n^8\lambda^3 \\
& -1849167059210218091244800n^7\lambda^5 \\
& +8239840794819093032480448n^7\lambda^4 \\
& -13111602723988644023259552n^8\lambda^2 \\
& +6251090412376750935735680n^5\lambda^6
\end{aligned}$$

$$\begin{aligned}
 &+9364708184440920755301504n^8 \\
 &-1571941329743427043123008n^9 \\
 &+253974919321928717298624n^6\lambda^6 \\
 &-2163143737480841773057312n^7\lambda^6 \\
 &+89464410663754627464192\lambda^8n^{11} \\
 &+1269160552731333896192\lambda^8n \\
 &-699912929134279736928\lambda^4n^{16} \\
 &-4178382214002282601852928n^3\lambda^5 \\
 &-110862225328038727680\lambda^7n^{17} \\
 &-4093268366384529408n^{18}\lambda^8 \\
 &+599684183205503382757376n\lambda^5 \\
 &+1071105137021814246606848n^2\lambda^5 \\
 &+273966990632726649176448\lambda^7n^{12} \\
 &-446292618085348190208\lambda^7n^{16} \\
 &-951790428934819686855936\lambda^7n^{10} \\
 &-1108455277068990771965928\lambda^4n^{12} \\
 &+445872912755271043503744\lambda^7n^{11} \\
 &-20342520205197080020992\lambda^7n^{14} \\
 &-2109771675414528\lambda^{11} \\
 &+1909798371928293101568\lambda^7n^{15} \\
 &-27724806581868329187446604\lambda^4n^{10} \\
 &-114991262416185939802368\lambda^7n^{13} \\
 &-918751239960999602358600\lambda^4n^{13} \\
 &-235011236476883790344940\lambda^4n^{14} \\
 &-18091749646572268378896\lambda^4n^{15} \\
 &-235500612854611968n^{19}\lambda^8 \\
 &+315043872768\lambda^{13} \\
 &+1642117223499020709888\lambda^7 \\
 &+134597677997837033472\lambda^8 \\
 &-228604266762262145866752n^3\lambda^7 \\
 &-781833743146215924539200n^6\lambda^7 \\
 &+1552141707064877750317056n^5\lambda^7 \\
 &-856264400195929304743616n^7\lambda^7 \\
 &+206989785669569542383872n^4\lambda^7 \\
 &+430652287816590546490816n^8\lambda^7 \\
 &+2345551581811936564224\lambda^7n^2 \\
 &+16273625146153409175552\lambda^7n \\
 &+22600934805036783201600n^9\lambda^7 \\
 &-5236574417228843915521824\lambda^4n^{11} \\
 &-3856278596334155429260032n^6 \\
 &-216073745881558191360\lambda^5n^{17} \\
 &+831322655490048n^3\lambda^{13} \\
 &-198601225374007296n^{18}\lambda^{10}
 \end{aligned}$$

$$\begin{aligned}
& -2757493432629854208n^{12}\lambda^{13} \\
& +1254685223878656n^4\lambda^{13} \\
& +840546338960572416n^{15}\lambda^{13} \\
& +67369174384508928n^{17}\lambda^{13} \\
& -282950099452035072n^{16}\lambda^{13} \\
& +2731590061865828352n^{13}\lambda^{13} \\
& -10373057306689536n^{18}\lambda^{13} \\
& -1800232512799113216n^{14}\lambda^{13} \\
& +55204764572516352n^7\lambda^{13} \\
& -24463679351881728n^6\lambda^{13} \\
& -8047306230398976n^5\lambda^{13} \\
& -407510779538178048n^9\lambda^{13} \\
& -27383868828457955668736\lambda^9n^6 \\
& -183293168473945912320\lambda^6n^{17} \\
& +146069826362474496n^8\lambda^{13} \\
& +1547537387608866816n^{11}\lambda^{13} \\
& -100229659824488448n^{10}\lambda^{13} \\
& +744989726343168n^{19}\lambda^{13} \\
& +48671830141934149399170816n^7\lambda \\
& +54457739414859266414332992n^7\lambda^2 \\
& +33233974161645920162685312n^7\lambda^3 \\
& -3416334229011752780805120\lambda^6n^{10} \\
& +617090980846969685746224\lambda^6n^{12} \\
& -1083097653235041957120\lambda^6n^{16} \\
& -271652229572557581687744\lambda^6n^{13} \\
& -57164695438848166903552\lambda^9n^9 \\
& -74161227864783113748864\lambda^6n^{14} \\
& -3122401811494244792832\lambda^6n^{15} \\
& -313308499136256n^{19} - 31900184616960\lambda^{12} \\
& +122376713153411683226624\lambda^9n^8 \\
& -16672968165994463232\lambda^9n^{17} \\
& -1201925897661364358400\lambda^5n^{16} \\
& +140960653836288n^2\lambda^{13} \\
& +27384594828931085713408\lambda^9n^5 \\
& +453409625357876636998560\lambda^5n^{12} \\
& +11686969745073917829120\lambda^6 \\
& -176977947218387681664\lambda^4n^{17} \\
& +2925907507351612272384\lambda^9n^4 \\
& -31726121091414696n^{18} \\
& -2032135927386300338176\lambda^9n^3 \\
& -11273828057155986796734192\lambda^5n^{10} \\
& -68773482094759357939968\lambda^9n^7
\end{aligned}$$

$$\begin{aligned}
 & -549934237499147022708144\lambda^5n^{13} \\
 & +51829793077040971776\lambda^9n^{16} \\
 & -98086201926263954496\lambda^3n^{17} \\
 & +20664520594498353549312\lambda^9n^{12} \\
 & -187271544790056960n^{18}\lambda^{11} \\
 & -183545376872515998912\lambda^3n^{16} \\
 & -171753955121106411967296\lambda^5n^{14} \\
 & -14779427861048802536448\lambda^5n^{15} \\
 & -9798830175985207590912\lambda^9n^{10} \\
 & -185229010613455613952\lambda^9n^2 \\
 & +35754120053784576n^{19}\lambda^{10} \\
 & +152986082226959680009200\lambda^5n^{11} \\
 & +25798736647028736n^{19}\lambda^{11} \\
 & -35080134418967016456\lambda^2n^{17} \\
 & -18836840500949092737024\lambda^9n^{13} \\
 & +6887268491138165340\lambda^2n^{16} \\
 & -703473579973214797824\lambda^9n^{15} \\
 & -3148097554050635795164227\lambda^2n^{12} \\
 & +5688365804607297208320\lambda^9n^{14} \\
 & +5749502195598197772288\lambda^9n^{11} \\
 & +970983640823690430653184\lambda^6n^{11} \\
 & -4448076913218096n^{19}\lambda \\
 & -1482673931073388498404924\lambda n^{12} \\
 & +14259922406973689598\lambda n^{16} \\
 & -19641232520640541673221104\lambda n^{10} \\
 & -164410778699166878843706\lambda n^{13} \\
 & -5229811423386783991530\lambda n^{14} \\
 & +672689980659322746198\lambda n^{15} \\
 & +49117158160779304960\lambda^9n \\
 & -3674629887337271296\lambda^{12}n^6 \\
 & -7040967940493492224\lambda^{12}n^9 \\
 & +27892689303662923776\lambda^{12}n^8 \\
 & +426978410300964864\lambda^{12}n^{17} \\
 & +35726505368834048\lambda^{12}n^3 \\
 & +331811829988282368\lambda^{12}n^4 \\
 & +374411105650126848\lambda^{12}n^5 \\
 & -4379953231555497984\lambda^{12}n^7 \\
 & -2729511369990733824\lambda^{12}n^{16} \\
 & +188881204051397271748608),
 \end{aligned}$$

$$\begin{aligned}
g_2(n, \lambda) = & (1 + 216n^3 + 108n^2 + 18n) \\
& (33481450248803564195119104n \\
& + 1310374346696324212654080\lambda \\
& + 20707418191521713105977344n\lambda \\
& + 4651051622273062517952n^{11} \\
& + 151735990238788257n^{14} + 1866168852828012n^{15} \\
& + 7203134977023237336n^{13} + 98820113301504\lambda^{12}n^{12} \\
& - 108066439692288\lambda^{12}n^{10} + 308772864\lambda^{12}n^2 \\
& + 35907455264313301275598848n^2\lambda^2 \\
& + 540910093860864\lambda^{12}n^{13} \\
& - 641959232274432\lambda^{12}n^{15} \\
& + 5457301158576126304862208n\lambda^2 \\
& + 10027008\lambda^{12}n - 208042343792640\lambda^{12}n^{11} \\
& - 53496602689536\lambda^{12}n^{14} \\
& + 308046429533841802149888\lambda^2 \\
& + 168489949226583816530546688n^2 \\
& + 116996399604971716771086336n^2\lambda \\
& + 38065183602967199711232\lambda^3 \\
& + 2759080637042253840384\lambda^4 \\
& + 1057894608476486814695424n^5\lambda^5 \\
& + 68222242316239464504240n^{10} \\
& + 735429484001822266589184n^4\lambda^5 \\
& + 221689912616960537736n^{12} \\
& + 207848159523841688771447808n^4\lambda^2 \\
& + 502740552895989531698663424n^4\lambda \\
& + 791132396812543178440704n\lambda^3 \\
& + 6208992958864372771184640n^2\lambda^3 \\
& + 23318793420166557179854848n^3\lambda^3 \\
& + 3018167933034134156967936n^3\lambda^4 \\
& + 653365039544105138380800n^2\lambda^4 \\
& + 7099748121004876119767040n^4\lambda^4 \\
& + 47749455967494855161548800n^4\lambda^3 \\
& + 68291295345529234587648n\lambda^4 \\
& + 124626430386710249472\lambda^5 \\
& + 113383557930184423599255552n^3\lambda^2 \\
& + 310929499128914171770650624n^3\lambda \\
& + 156927379015249920\lambda^{10}n^6 \\
& + 43038184772640588898752\lambda n^{11} \\
& - 1096043450768031744\lambda^{10}n^9 \\
& - 22240862324195328\lambda^{10}n^8
\end{aligned}$$

$$\begin{aligned}
 &+10167463313316n^{16} + 27002330030227417756591104n^7 \\
 &+344019230519242880203112448n^3 \\
 &+368565166629588224556804096n^4 \\
 &+235859075759569313044998144n^5 \\
 &+1774446149678038028295264\lambda^2n^{10} \\
 &+1152396565512192\lambda^{11}n^6 \\
 &-6984516798775296\lambda^{11}n^9 \\
 &+456769101651370329024\lambda^2n^{13} \\
 &+3745591320379392\lambda^{11}n^8 \\
 &+182072182600262592\lambda^2n^{15} \\
 &+245071183429632\lambda^{11}n^5 \\
 &+12075991724416092336\lambda^2n^{14} \\
 &+70529504763367593368764416n^5\lambda^3 \\
 &+42572293810421760\lambda^{10}n^5 \\
 &+34413934600192\lambda^{11}n^4 \\
 &+7310521073664\lambda^9 + 2888816545234944\lambda^{11}n^{16} \\
 &+3278570358767616\lambda^{11}n^7 + 3234834907136\lambda^{11}n^3 \\
 &+285622920759661535933245440n^5\lambda^2 \\
 &+492996633623550855370091520n^5\lambda \\
 &+256304276841891762239616n^8\lambda^6 \\
 &+27768362491685934313673472n^8\lambda \\
 &+16025912681414041529187840n^9\lambda^3 \\
 &+12294115340025380849872704n^9\lambda^2 \\
 &+4674626476171220319676992n^9\lambda \\
 &-105514945032869639156736n^9\lambda^6 \\
 &+2402651837278454398621056n^9\lambda^5 \\
 &+10226198270251254981853440n^9\lambda^4 \\
 &+60441790878056448\lambda^{11}n^{12} + 7163877318140928\lambda^{10}n^4 \\
 &+7436935895187456\lambda^{11}n^{13} \\
 &+170176312891925970345072\lambda^2n^{11} \\
 &+197060026368\lambda^{11}n^2 \\
 &-25024973692207104\lambda^{11}n^{10} \\
 &+38889058121809920\lambda^{11}n^{15} \\
 &-84152632725209088\lambda^{11}n^{14} - 1953960661352448\lambda^{11}n^{11} \\
 &+7066025984\lambda^{11}n + 274735757629604805305619456n^6\lambda^2 \\
 &+294651493554680142905364480n^6\lambda \\
 &+83104692732191736718543872n^6\lambda^3 \\
 &+28694999662797310426080\lambda^3n^{12} + 779894448091136\lambda^{10}n^3 \\
 &+296051830893232128\lambda^{10}n^7 + 11916368249094144\lambda^{10}n^{16} \\
 &-700649092830855168\lambda^{10}n^{10} + 1140156597304295424\lambda^{10}n^{12} \\
 &+38137184256\lambda^{10} + 3085845741750551315536224\lambda^3n^{10} \\
 &+53823317909504\lambda^{10}n^2 + 1411657947722924870976\lambda^3n^{13}
 \end{aligned}$$

$$\begin{aligned}
&+338751715174527368448\lambda^8 n^6 + 837577480288222494720\lambda^8 n^9 \\
&+194022430937566930944\lambda^8 n^5 - 561774448721287148544\lambda^8 n^8 \\
&-4660212647344472064\lambda^{10} n^{13} + 344667466003120128\lambda^{10} n^{15} \\
&+727228301634743040\lambda^3 n^{15} + 2064996540877897728\lambda^{10} n^{14} \\
&+52734027152685957120\lambda^8 n^4 + 42783112385311996800\lambda^3 n^{14} \\
&+10161992470225316832018432n^5\lambda^4 + 8273974668517490688\lambda^8 n^3 \\
&-11021483784877839360\lambda^8 n^7 + 50272178550865920\lambda^8 n^{16} \\
&+2720132479669764096\lambda^{10} n^{11} + 1345414977212525936640\lambda^8 n^{12} \\
&+372043476458248837451328\lambda^3 n^{11} + 1464304978492101107712\lambda^8 n^{10} \\
&+2152352595968\lambda^{10} n + 552598105499368685568\lambda^8 n^{13} \\
&+770455463956992000\lambda^8 n^2 + 3008676101503057920\lambda^8 n^{15} \\
&+63960713188909056000\lambda^8 n^{14} + 1176506162373311115300864n^6\lambda^5 \\
&+14558172787357772189589504n^6\lambda^4 + 49890421953374828507136\lambda^6 n^4 \\
&12801887181708745236480\lambda^6 n^3 + 1754888546281857122304\lambda^6 n^2 \\
&+124733047102205165568\lambda^6 n + 712665109365397062488064n^8\lambda^5 \\
&+17224518897795487547688768n^8\lambda^4 + 49734335050181632749504384n^8\lambda^3 \\
&+2158816883647241883184128n^7\lambda^5 + 12851218850176328105410560n^7\lambda^4 \\
&+55600963862385272698385856n^8\lambda^2 + 93218256517273087592448n^5\lambda^6 \\
&+5207048865459685496392704n^8 + 707805303974119540372608n^9 \\
&+61645235223882312640512n^6\lambda^6 + 88934206785168882203136n^7\lambda^6 \\
&-2531538082833936777216\lambda^8 n^{11} + 39781830708854784\lambda^8 n \\
&+15906431494609920\lambda^4 n^{16} + 248461722712035719921664n^3\lambda^5 \\
&+3656312309699450929152n\lambda^5 + 42692970370785570127872n^2\lambda^5 \\
&+10989107725253705220096\lambda^7 n^{12} + 60326614261039104\lambda^7 n^{16} \\
&-25794699418475547807744\lambda^7 n^{10} + 47094428654217919316160\lambda^4 n^{12} \\
&+20763139733719812366336\lambda^7 n^{11} + 130840278026411114496\lambda^7 n^{14} \\
&+113606656\lambda^{11} + 4543177459385106432\lambda^7 n^{15} \\
&+2973096509016980702125728\lambda^4 n^{10} + 1776379977548443975680\lambda^7 n^{13} \\
&+2798250213221984973312\lambda^4 n^{13} + 99365962081849758720\lambda^4 n^{14} \\
&+1935740708471623680\lambda^4 n^{15} + 69571136029261824\lambda^7 \\
&+881621602762752\lambda^8 + 411434839913383673856n^3\lambda^7 \\
&+5414569147320142537728n^6\lambda^7 + 5567807415732563570688n^5\lambda^7 \\
&-2449591855130324616192n^7\lambda^7 + 2084522960972753989632n^4\lambda^7 \\
&+7754698340191482356736n^8\lambda^7 + 45929098048749944832\lambda^7 n^2 \\
&+2757674039379296256\lambda^7 n + 23393637822145424283648n^9\lambda^7 \\
&+483867696452109907142016\lambda^4 n^{11} + 97172160398288251269154560n^6 \\
&+10331928483762364416\lambda^9 n^6 + 111512684213400500773060608n^7\lambda \\
&+158848766033921031850550784n^7\lambda^2 + 86021475391459403453675520n^7\lambda^3 \\
&+251466478820522150072832\lambda^6 n^{10} + 31419299665286507907072\lambda^6 n^{12}
\end{aligned}$$

$$\begin{aligned}
 &+52785787478409216\lambda^6 n^{16} + 3212518828335938666496\lambda^6 n^{13} \\
 &-31430236213118730240\lambda^9 n^9 + 174025580819806937088\lambda^6 n^{14} \\
 &+4791450647345725440\lambda^6 n^{15} + 147456\lambda^{12} \\
 &-19756285885018718208\lambda^9 n^8 + 33933720521834496\lambda^5 n^{16} \\
 &+3843897155133170688\lambda^9 n^5 + 49067507623608212207616\lambda^5 n^{12} \\
 &+3631780623181971456\lambda^6 + 802961681530982400\lambda^9 n^4 \\
 &+103503159505944576\lambda^9 n^3 + 1468328904880796361844224\lambda^5 n^{10} \\
 &+9378077886336061440\lambda^9 n^7 + 3684223319688637286400\lambda^5 n^{13} \\
 &+29790920622735360\lambda^9 n^{16} - 141490856019422085120\lambda^9 n^{12} \\
 &+5302143831536640\lambda^3 n^{16} + 158001275242572521472\lambda^5 n^{14} \\
 &+3604899225683238912\lambda^5 n^{15} + 63322468550877118464\lambda^9 n^{10} \\
 &+8219580582952960\lambda^9 n^2 + 370959000070163459286528\lambda^5 n^{11} \\
 &+64326445975467196416\lambda^9 n^{13} + 1192982362095744\lambda^2 n^{16} \\
 &+1322293918134435840\lambda^9 n^{15} + 10878211682252694350676\lambda^2 n^{12} \\
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 &+150691532229253206374400\lambda^6 n^{11} + 2349964504783516591152\lambda n^{12} \\
 &+162679413013056\lambda n^{16} + 540150166466770664730240\lambda n^{10} \\
 &+86084967963180624228\lambda n^{13} + 2018430200359283328\lambda n^{14} \\
 &+27343363272642000\lambda n^{15} + 370913390542848\lambda^9 n \\
 &+3264261783552\lambda^{12} n^6 + 8173092077568\lambda^{12} n^9 \\
 &+26430114889728\lambda^{12} n^8 + 5668798464\lambda^{12} n^3 \\
 &+68627349504\lambda^{12} n^4 + 570389299200\lambda^{12} n^5 \\
 &+12342322003968\lambda^{12} n^7 + 320979616137216\lambda^{12} n^{16} \\
 &+2236602623405684208500736).
 \end{aligned}$$

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