Differential
Equations
& Applications
Volume 2, Number 1 (2010), 39–51

NEW RESULTS ON THE ASYMPTOTIC BEHAVIOR OF A THIRD–ORDER NONLINEAR DIFFERENTIAL EQUATION

MATHEW OMEIKE

(*Communicated by D. Blackmore*)

Abstract. Sufficient conditions are established for the asymptotic behavior of a third-order nonlinear differential equation. Our results improve on Qian's [C. Qian, Asymptotic behavior of a third-order nonlinear differential equation, J. Math. Anal. Appl., 284 (2003), 191–205]

1. Introduction

We consider the third-order nonlinear ordinary differential equation

$$x''' + \psi(x,x')x'' + f(x,x') = p(t), \tag{1.1}$$

or its equivalent system

$$x' = y, \;\; y' = z, \;\; z' = -\psi(x,y)z - f(x,y) + p(t), \tag{1.2}$$

where

$$\psi,\; f,\; \psi_x,\; f_x \in C(\mathbb{R}\times\mathbb{R},\mathbb{R}) \;\text{ and }\; p \in C([0,\infty),\mathbb{R}). \tag{1.3}$$

It is assumed that solutions of (1.1) exist and are unique. Stability and boundedness are very important problems in the theory and applications of differential equations, and an effective method for studying the stability and boundedness of nonlinear differential equations is the second method of Lyapunov (see [1-11]).

Recently, Qian [6] discussed the boundedness and asymptotic behavior of solutions of Eq.(1.1) and the following results were proved.

THEOREM A. (Qian [6]) *Assume that*:

(1) $\int_0^x f(u,0)du > 0$ *for* $x \neq 0$,

(2) $\lim_{|x|\to\infty} \sup \int_0^x f(u,0)du = \infty$,

(3) $\int_0^y f(0,v)dv \geqslant 0$,

(4) $\int_0^\infty |p(t)|dt < \infty$,

Mathematics subject classification (2010): 34B15, 34C11.
Keywords and phrases: nonlinear differential equation, asymptotic behavior, Lyapunov's method.

Paper DEA-02-04

and there is a positive number B *such that*:

(5) $\psi(x,y) \geqslant B$,

(6) $B[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v)vdv]y \geqslant y\int_0^y f_x(x,v)dv$,

(7) $4B\int_0^x f(u,0)du\left\{\int_0^y [f(x,v)-f(x,0)]dv + B\int_0^y [\psi(x,v)-B]vdv\right\} \geqslant y^2 f^2(x,0)$.

Then, for any solution $x(t)$ *of Eq.* (1.1), *there are constants* c_1, c_2 *and* c_3 *such that*

$$|x(t)| < c_1, \quad |x'(t)| < c_2 \text{ and } |x''(t)| < c_3, \quad t \geqslant 0.$$

THEOREM B. (Qian [6]) *Assume that*:

(1) $xf(x,0) > 0$ *for* $x \neq 0$,

(2) $\lim_{|x|\to\infty} \sup \int_0^x f(u,0)du = \infty$,

(3) $\int_0^y f(0,v)dv \geqslant 0$,

(4) $\int_0^\infty |p(t)|dt < \infty$,

and there is a positive number B *such that*:

(5) $\psi(x,y) \geqslant B$,

(6) $B[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v)vdv]y \geqslant y\int_0^y f_x(x,v)dv$,

(7) $4B\int_0^x f(u,0)du\left\{\int_0^y [f(x,v)-f(x,0)]dv + B\int_0^y [\psi(x,v)-B]vdv\right\} \geqslant y^2 f^2(x,0)$,

(8) $B[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v)vdv]y + \psi(x,y) \geqslant y\int_0^y f_x(x,v)dv + B$ *for* $y \neq 0$.

Then, every solution $x(t)$ *of Eq.* (1.1) *satisfies*

$$\lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} x'(t) = 0 \text{ and } \lim_{t\to\infty} x''(t) = 0.$$

Theoretically, these are interesting results since (1.1) is a rather general third-order nonlinear differential equation. For example, many third-order differential equations that have been discussed in [8] are special cases of Eq.(1.1), and some known results can be obtained using these theorems. However, it is not easy to apply Theorems A and B to these special cases to obtain new or better results since Theorems A and B have some hypotheses which are not necessary for the stability and boundedness of many nonlinear equations.

Our aim in this paper is to further study the stability and boundedness of Eq.(1.1). In the next section, we will state our results and also obtain sufficient conditions for every solution of Eq.(1.1) to be bounded by using Liapunov's direct method. In Section 3, we will establish criteria for every solutions of Eq. (1.1) to converge to zero by employing the method introduced by Yoshizawa [11]. Finally, in Section 4, we will discuss the asymptotic behavior of solutions of some interesting special cases of Eq. (1.1).

In the following discussion, we always assume (1.3) holds.

2. Main results

Our main results in this section are the following theorems.

THEOREM 1. *Let* $a>0,\ b>0,\ c>0,\ \delta_0>0$ *be constants such that* $ab>c$. *Assume that:*

(i) $\frac{f(x,0)}{x}\geqslant\delta_0>0$ *for* $x\neq 0,\ f(0,0)=0$,

(ii) $f'(x,0)\leqslant c$,

(iii) $f_y(x,\theta y)\geqslant b$ *for* $0\leqslant\theta\leqslant 1$,

(iv) $\psi(x,y)>a$,

(v) $a\left[f(x,y)-f(x,0)-\int_0^y\psi_x(x,v)vdv\right]y\geqslant y\int_0^y f_x(x,v)dv$,

(vi) $\int_0^\infty |p(t)|dt<\infty$.

Then for any solution $x(t)$ *of Eq.* (1.1), *there exists a constant* D, *depending on* a,b *and* c, *such that:*

$$|x(t)|\leqslant D,\ \ |x'(t)|\leqslant D\ \ \text{and}\ \ |x''(t)|\leqslant D, \tag{2.1}$$

for any arbitrary large t.

THEOREM 2. *Let* $a>0,\ b>0,\ c>0,\ \delta_0>0$ *be constants such that* $ab>c$. *Assume that:*

(i) $\frac{f(x,0)}{x}\geqslant\delta_0>0$ *for* $x\neq 0,\ f(0,0)=0$,

(ii) $f'(x,0)\leqslant c$,

(iii) $f_y(x,\theta y)\geqslant b$ *for* $0\leqslant\theta\leqslant 1$,

(iv) $\psi(x,y)>a$,

(v) $a\left[f(x,y)-f(x,0)-\int_0^y\psi_x(x,v)vdv\right]y\geqslant y\int_0^y f_x(x,v)dv$,

(vi) $\int_0^\infty |p(t)|dt<\infty$.

Then, every solution $x(t)$ *of Eq.* (1.1) *satisfies*

$$\lim_{t\to\infty}x(t)=0,\ \ \lim_{t\to\infty}x'(t)=0\ \ \text{and}\ \ \lim_{t\to\infty}x''(t)=0. \tag{2.2}$$

Proof of Theorem 1. Clearly, Eq. (1.1) is equivalent to the system (1.2), and it suffices to show that every solution of (1.2) is bounded. To this end, consider the function

$$V(t,x,y,z)=e^{-P(t)}U(x,y,z), \tag{2.3}$$

where

$$U(x,y,z)=\int_0^x f(u,0)du+\int_0^y\psi(x,v)vdv$$
$$+a^{-1}\int_0^y f(x,v)dv+\frac{1}{2}a^{-1}z^2+yz+2a^{-1}$$

and

$$P(t) = \int_0^t |p(s)|ds.$$

We claim that V is a positive function. To show this, it suffices to show that U is positive.

Now, rewrite U above thus:

$$\begin{aligned} U(x,y,z) = {} & \frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 \\ & + \int_0^y [\psi(x,v)-a]vdv + a^{-1}\int_0^y [f_v(x,\theta v)-b]vdv \\ & + \int_0^x \left[1-\frac{a^{-1}}{b}f'(u,0)\right] f(u,0)du + 2a^{-1} \end{aligned}$$

where

$$f_v(x,\theta v) = v^{-1}\{f(x,v)-f(x,0)\}, \quad v \neq 0.$$

By using hypotheses (i)-(iv) of Theorem 1, we obtain:

$$U(x,y,z) \geqslant \frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2 + \frac{\delta_1}{2}x^2. \tag{2.4}$$

Combining (2.3) and (2.4), we have that

$$V(t,x,y,z) \geqslant \frac{1}{2}e^{-P(t)}\left\{a^{-1}(ay+z)^2 + \frac{a^{-1}}{b}(f(x,0)+by)^2 + \delta_1 x^2\right\}.$$

It follows that there exists a constant $K > 0$ small enough that

$$V(t,x,y,z) \geqslant K(x^2+y^2+z^2). \tag{2.5}$$

Hence $V(t,x,y,z)$ is a positive function.

Next, we show that the derivative of $V(t,x,y,z)$ with respect to t along the solution path of (1.2) satisfies

$$V'_{(1.2)} \equiv \frac{d}{dt}V(t,x(t),y(t),z(t))|_{(1.2)} \leqslant -D_1 \tag{2.6}$$

provided that $x^2+y^2+z^2 \geqslant D_2$, where D_1, D_2 are some positive constants.

$$\begin{aligned} V'_{(1.2)} = e^{-P(t)}\Bigg\{ & -|p(t)|\left[\frac{a^{-1}}{2}(ay+z)^2 + \frac{a^{-1}}{2b}(f(x,0)+by)^2\right. \\ & + \int_0^y (\psi(x,v)-a)vdv + a^{-1}\int_0^y (f_v(x,\theta v)-b)vdv \\ & \left. + \int_0^x \left(1-\frac{a^{-1}}{b}f'(u,0)\right) f(u,0)du + 2a^{-1}\right] \\ & + a^{-1}y\int_0^y f_x(x,v)dv - (a^{-1}\psi(x,y)-1)z^2 \\ & - \left[f(x,y)-f(x,0)-\int_0^y \psi_x(x,v)vdv\right]y \\ & + a^{-1}(z+ay)p(t)\Bigg\}. \end{aligned}$$

Clearly, if $|z+ay|<2$, then $(z+ay)p(t)\leqslant 2|p(t)|$; if $|z+ay|\geqslant 2$, then $(z+ay)p(t)\leqslant \frac{1}{2}(z+ay)^2|p(t)|$. Hence for any t,x and y

$$(z+ay)p(t)\leqslant\left(2+\frac{1}{2}(z+ay)^2\right)|p(t)|$$

and so

$$\begin{aligned}V'_{(1.2)}\leqslant e^{-P(t)}\Big\{&-|p(t)|\Big[\frac{a^{-1}}{2b}(f(x,0)+by)^2\\&+\int_0^y(\psi(x,v)-a)v\,dv+a^{-1}\int_0^y[f_v(x,\theta v)-b]\,v\,dv\\&+\int_0^x\left(1-\frac{a^{-1}}{b}f'(u,0)\right)f(u,0)\,du\Big]\\&-(a^{-1}\psi(x,y)-1)z^2+a^{-1}y\int_0^y f_x(x,v)\,dv\\&-\left[f(x,y)-f(x,0)-\int_0^y\psi_x(x,v)v\,dv\right]y\Big\}.\end{aligned}\tag{2.7}$$

Then, by noting (i)-(v), we can find an $\eta>0$, small enough so that

$$V'_{(1.2)}\leqslant -e^{-P(t)}\left\{\eta z^2+\frac{a^{-1}}{2b}(f(x,0)+by)^2+\frac{1}{2}\delta_1x^2\right\}|p(t)|,$$

where $\delta_1=\delta_0\left(1-\dfrac{c}{ab}\right)$.

Thus, there exists a constant $D_3>0$ such that

$$V'_{(1.2)}\leqslant -D_3(x^2+y^2+z^2).$$

Hence $V'_{(1.2)}\leqslant -D_4$, provided $x^2+y^2+z^2\geqslant D_4D_3^{-1}$. This completes the verification of (2.7).

Now, we are ready to show that every solution of (1.2) satisfies (2.1).

Following [2], let $(x(t),y(t),z(t))$ be any solution of (1.2). Then there is evidently a $t_0\geqslant 0$ such that

$$x^2(t_0)+y^2(t_0)+z^2(t_0)<D_2,$$

where D_2 is the constant defined earlier; for otherwise, that is if

$$x^2(t)+y^2(t)+z^2(t)\geqslant D_2,\ \ t\geqslant 0,$$

then, by (2.6),

$$V'_{(1.2)}(t)\leqslant -D_1<0,\ \ t\geqslant 0,$$

and this in turn implies that $V(t)\to-\infty$ as $t\to\infty$, which contradicts (2.5). Hence to prove (2.1) it will suffice to show that if

$$x^2(t)+y^2(t)+z^2(t)<D_5\ \text{ for }\ t=T,\tag{2.8}$$

where $D_5 \geqslant D_2$ is a finite constant, then there is a constant $D_6 > 0$ depending on a,b,c and D_5, such that

$$x^2(t)+y^2(t)+z^2(t) \leqslant D_6 \text{ for } t \geqslant T. \tag{2.9}$$

Our proof of (2.9) is based essentially on an extension of an argument in the proof of [10; Lemma 1]. For any given constant $d > 0$ let $S(d)$ denote the surface: $x^2+y^2+z^2 = d$. Because V is continuous in t,x,y,z and tends to $+\infty$ as $x^2+y^2+z^2 \to \infty$, there is evidently a constant $D_7 > 0$ depending on D_5 as well as on a,b,c such that

$$\min_{(x,y,z)\in S(D_7)} V(t,x,y,z) > \max_{(x,y,z)\in S(D_5)} V(t,x,y,z). \tag{2.10}$$

It is easy to see from (2.8) and (2.10) that

$$x^2(t)+y^2(t)+z^2(t) < D_7,\ \ t \geqslant T. \tag{2.11}$$

For suppose on the contrary that there is a $t > T$ such that

$$x^2(t)+y^2(t)+z^2(t) \geqslant D_7.$$

Then, by (2.8) and by the continuity of the quantities $x(t),y(t),z(t)$ in the argument displayed, there exist $t_1,t_2,T < t_1 < t_2$ such that

$$x^2(t_1)+y^2(t_1)+z^2(t_1) = D_5 \tag{2.12a}$$

$$x^2(t_2)+y^2(t_2)+z^2(t_2) = D_7 \tag{2.12b}$$

and such that

$$D_5 \leqslant x^2(t)+y^2(t)+z^2(t) \leqslant D_7,\ \ t_1 \leqslant t \leqslant t_2. \tag{2.13}$$

But writing $V(t) \equiv V(t,x(t),y(t),z(t))$, since $D_5 \geqslant D_2$, (2.13) obviously implies [in view of (2.7)] that

$$V(t_2) < V(t_1)$$

and this contradicts the conclusion [from (2.10) and (2.12)]:

$$V(t_2) > V(t_1).$$

Hence (2.11) holds. This completes the proof of (2.1), and the theorem now follows.

REMARK 1. Clearly, Theorem 1 is an improvement and extension of Theorem A. In particular, from Theorem 1 we see that (3) and (7) assumed in Theorem A are not necessary, and (1) and (2) can be replaced by (*i*) of Theorem 1 for the boundedness of solutions of Eq. (1.1).

EXAMPLE 1. Consider Eq. (1.1) with

$$\psi(x,y) = \ln(1+x^2)+1, \quad f(x,y) = \frac{x}{1+x^2}(1+y^2)+y+\frac{1}{3}y^3$$

and $p(t) = \dfrac{\sin t}{1+t^2}$. It is easy to check that the hypotheses (i)-(iv) in Theorem 1 are satisfied. Since $\psi(x,y) \geqslant 1$ and

$$\begin{aligned}
&\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)v\,dv\right] y \\
&\quad = \left[\frac{x}{1+x^2}y^2 + y + \frac{1}{3}y^3 - \frac{x}{1+x^2}y^2\right] y \\
&\quad = y^2 + \frac{1}{3}y^4 \geqslant \frac{1-x^2}{(1+x^2)^2}\left[y^2 + \frac{1}{3}y^4\right] = y\int_0^y f_x(x,v)\,dv,
\end{aligned}$$

we see that (v) of Theorem 1 hold also. Hence all the hypotheses in Theorem 1 are satisfied, and so for every solution $x(t)$ of Eq. (1.1) there is a constant $D > 0$ such that

$$|x(t)| < D, \quad |x'(t)| < D \text{ and } |x''(t)| < D \text{ for } t \geqslant 0.$$

3. Asymptotic behavior of solutions

In this section, we use the method introduced by Yoshizawa [11] to study the asymptotic behavior of solutions of Eq. (1.1). The following lemma extracted from [11] will be needed in the proof of Theorem 2.

LEMMA 1. *Let Q be an open set in $\mathbb{R}^n$ and $I = [0,\infty)$. Consider the differential system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x}) + \mathbf{G}(t,\mathbf{x}), \tag{3.1}$$

where $\mathbf{H}$ is continuous on Q, $\mathbf{G}$ is continuous on $I \times Q$ and for any continuous and bounded function $\mathbf{x}(t)$ on $t_0 \leqslant t < \infty$,

$$\int_{t_0}^{\infty} \|\mathbf{G}(s,\mathbf{x}(s))\|\,ds < \infty.$$

Assume that all the solutions of (3.1) are bounded, and that there exists a nonnegative continuous function $V(t,\mathbf{x})$ which satisfies locally a Lipschitz condition with respect to $\mathbf{x}$ in Q such that $V'(t,\mathbf{x}) \leqslant -W(\mathbf{x})$, where $W(\mathbf{x})$ is positive definite with respect to a closed set Ω in Q. Then all the solutions of (3.1) *approach the largest semi-invariant set contained in Ω of the equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}(\mathbf{x}) \tag{3.2}$$

on Q.

Proof of Theorem 2. Consider the system (1.2) and let V be defined by (2.3). Then by noting

$$\frac{a^{-1}}{2}(ay+z)^2+\frac{a^{-1}}{2b}(f(x,0)+by)^2+\int_0^y(\psi(x,v)-a)vdv \\ +a^{-1}\int_0^y[f_v(x,\theta v)-b]vdv+\int_0^x\left(1-\frac{a^{-1}}{b}f'(u,0)\right)f(u,0)du\geqslant 0,$$

it follows from (2.7) that

$$\frac{dV}{dt}|_{(1.2)}\leqslant -e^{-P(\infty)}\left\{y\left[f(x,y)-f(x,0)-\int_0^y\psi_x(x,v)vdv\right.\right. \\ \left.\left.-a^{-1}\int_0^y f_x(x,v)dv\right]+[a^{-1}\psi(x,y)-1]z^2\right\}. \tag{3.3}$$

Set

$$W(x,y,z)=e^{-P(\infty)}\left\{y\left[f(x,y)-f(x,0)-\int_0^y\psi_x(x,v)vdv\right.\right. \\ \left.\left.-a^{-1}\int_0^y f_x(x,v)dv\right]+[a^{-1}\psi(x,y)-1]z^2\right\}.$$

Conditions (iv) and (v) yield $W(x,y,z)\geqslant 0$. Now, consider the set

$$\Omega=\{(x,y,z):W(x,y,z)=0\}.$$

Since the function W is continuous, the set Ω is closed and W is positive definite with respect to Ω. Next, consider the system

$$x'=y,\quad y'=z\quad z'=-f(x,y)-\psi(x,y)z. \tag{3.4}$$

The asymptotic behavior of solutions of (3.4) has been discussed in [5,6,7]. With the same hypotheses we have here, it has been shown in the proof of the main theorem in [7] that $(0,0,0)$ is the largest semi-invariant set of (3.4) contained in Ω. In addition, since all the hypotheses of Theorem 1 are satisfied, we know that every solution of (1.2) is bounded. Now, let

$$\mathbf{x}=(x,y,z)^T,\ \ H(\mathbf{x})=(y,z,-f(x,y)-\psi(x,y)z)^T$$

and

$$G(t,\mathbf{x})=(0,0,p(t))^T.$$

We see that (1.2) is in the form (3.1), and from the above discussion, all the hypotheses in Lemma 1 are satisfied. Hence, by Lemma 1, every solution of (1.2) tends to the largest semi-invariant set contained in Ω of (3.4) on Ω, that is, $(0,0,0)$. The proof is complete. □

REMARK 2. Clearly, Theorem 2 is an improvement and extension of Theorem B. In particular, from Theorem 2 we see that (3), (7) and (8) assumed in Theorem B are not necessary, and (1) can be replaced by (i) of Theorem 2 for the asymptotic stability of the trivial solutions of Eq. (1.1).

EXAMPLE 2. Consider Eq. (1.1) with

$$\psi(x,y) = 2\arctan x + y^2 + 2\pi, \quad f(x,y) = \frac{x}{1+x^2}(1+y^2) + 2y + y^3$$

and $\quad p(t) = \dfrac{\sin t}{1+t^2}$. It is easy to check that the hypotheses (i)-(iv) of Theorem 2 are satisfied. Observe that $\psi(x,y) > 2$,

$$\begin{aligned} &2\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)v\,dv\right] y \\ &\quad = 2\left[4 + 2\frac{x-1}{1+x^2}y + 2y^2\right] y^2 \\ &\quad \geqslant [4 - 3|y| + 2y^2]y^2 \geqslant \left[1 + y^2 + \left(\frac{3}{2} - |y|\right)^2\right] y^2 \\ &\quad \geqslant [1+y^2]y^2 \geqslant \frac{1-x^2}{(1+x^2)^2}\left[1 + \frac{1}{3}y^2\right] y^2 = y\int_0^y f_x(x,v)\,dv. \end{aligned} \tag{3.5}$$

We see that (v) of Theorem 2 hold also. Hence all the hypotheses in Theorem 2 are satisfied, and so every solution $x(t)$ of Eq. (1.1) satisfies (2.2).

4. Some special cases

In this section, we discuss asymptotic behavior of solutions of some special cases of (1.1). When $p(t) = 0$, these special cases have been discussed by several authors (see, for example, [5,6,7,8] and references cited therein). First, let us discuss the equation

$$x''' + \psi(x,x')x'' + \phi(x') + g(x) = p(t), \tag{4.1}$$

where

$$\psi, \psi_x \in C(\mathbb{R}\times\mathbb{R},\mathbb{R}), \quad \phi, g' \in C(\mathbb{R},\mathbb{R}) \text{ and } p \in C([0,\infty),\mathbb{R}). \tag{4.2}$$

Clearly, Eq.(4.1) is a special case of Eq.(1.1) with $f(x,y) = g(x) + \phi(y)$. Hence the following result is an immediate consequence of Theorem 2.

COROLLARY 1. *Assume that:*

(1) $\dfrac{g(x)}{x} \geqslant \delta_0 > 0$ *for* $x \neq 0$,

(2) $g'(x) \leqslant c$,

(3) $\dfrac{\phi(y)}{y} \geqslant b > 0$ *for* $y \neq 0$,

(4) $\psi(x,y) > a > 0$,

(5) $\int_0^\infty |p(t)|dt < \infty$,

(6) $a[\phi(y) - \int_0^y \psi_x(x,v)vdv]y \geqslant g'(x)y^2$.

Then every solution of Eq. (4.1) *satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0 \text{ and } \lim_{t\to\infty} x''(t) = 0. \tag{4.3}$$

Ezeilo [3] discussed the asymptotic behavior of the following equation:

$$x''' + \psi(x,x')x'' + \phi(x') + g(x) = p(t,x,x',x''). \tag{4.4}$$

When $p(t,x,x',x'') \equiv p(t)$, his result reduces to the following theorem.

THEOREM C. [3] *Assume that there are positive numbers* a,b,c *and* δ *such that*:

(i) $g(x)x > 0$ *for* $x \neq 0$,

(ii) $g'(x) \leqslant c$,

(iii) $\lim_{|x|\to\infty} \int_0^x g(x)dx = \infty$,

(iv) $\frac{\phi(y)}{y} \geqslant b > 0$ for $y \neq 0$,

(v) $\psi(x,y) \geqslant a > 0(ab - c > 0)$,

(vi) $y\psi_x(x,y) \leqslant \delta(0 < \delta < \frac{(ab-c)}{a})$,

(vii) $\sup_{t\geqslant 0} |p(t)| < \infty$,

$(viii)$ $\int_0^\infty |p(t)|dt < \infty$.

Then every solution $x(t)$ *of Eq.* (4.1) *satisfies* (4.3).

Now, we show that Corollary 1 is an improvement of the above result. Suppose all the hypotheses in Theorem C hold. It suffices to show that all the hypotheses in Corollary 1 hold also. Clearly, (i)-(v) and $(viii)$ imply (1)-(5). By noting (iv) and (vi), we see that $y\phi(y) \geqslant by^2$, $(\int_0^y \psi_x(x,v)vdv)y \leqslant \delta y^2$ and so

$$a\left[\phi(y) - \int_0^y \psi_x(x,v)vdv\right] y \geqslant a\phi(y)y \geqslant aby^2 > cy^2 \geqslant g'(x)y^2$$

and so (6) holds. Therefore, all the hypotheses of Corollary 1 hold.
When $g(x) = cx$ and $\phi(y) = by$, where b and c are positive constants, Eq.(4.1) reduces to

$$x''' + \psi(x,x')x'' + bx' + cx = p(t). \tag{4.5}$$

Then since $a > c/b$, we have the following result immediately from Corollary 1.

COROLLARY 2. *Assume that:*

(1) $\psi(x,y) > c/b$,

(2) $y\int_0^y \psi_x(x,v)vdv \leqslant 0$,

(3) $\int_0^\infty |p(t)|dt < \infty$.

Then every solution $x(t)$ *of Eq.* (4.5) *satisfies* (4.3).

EXAMPLE 3. Consider the equation

$$x''' + [(\cos x)x' + (x')^2 + 2]x'' + (x')^3 + x' + \frac{x}{1+x^2} = \frac{1}{1+t^2}. \tag{4.6}$$

Equation (4.6) is in the form of (4.1) with

$$\psi(x,y) = (\cos x)x' + (x')^2 + 2, \;\; g(x) = \frac{x}{1+x^2}, \;\; \phi(y) = y^3 + y$$

and $p(t) = \dfrac{1}{1+t^2}$. Take $a = 1$. Observe that

$$\begin{aligned}\left[\phi(y) - \int_0^y \psi_x(x,v)v\,dv\right] y &= \left[y^3 + y + \frac{1}{3}(\sin x)y^3\right] y \\ &= y^2\left[1 + y^2\left(1 + \frac{1}{3}\sin x\right)\right] \\ &\geqslant y^2\left[1 + \frac{2}{3}y^2\right] \\ &> y^2\frac{1-x^2}{(1+x^2)^2} = y^2 g'(x) \;\text{ for }\; y \neq 0.\end{aligned}$$

Then, it is easy to check that all the hypotheses in Corollary 1 are satisfied and so every solution $x(t)$ of Eq. (4.6) satisfies (4.3). However, Theorem C cannot be applied here. For example, observe that $y\psi_x(x,y) = -(\sin x)y^2$ which does not satisfy the condition (vi) assumed in Theorem C.

Next, consider the equation

$$x''' + h(x')x'' + \phi(x') + g(x) = p(t), \tag{4.7}$$

where

$$h, \phi, g' \in C(\mathbb{R}, \mathbb{R}) \;\text{ and }\; p \in C([0,\infty), \mathbb{R}).$$

This equation is a special case of (4.1) with $\psi(x,y) = h(y)$ and so by using Corollary 1, we have the following result.

COROLLARY 3. *Assume that:*

(1) $\dfrac{g(x)}{x} > 0$ *for* $x \neq 0$,

(2) $\dfrac{\phi(y)}{y} > 0$ *for* $y \neq 0$,

(3) $h(y) > a > 0$,

(4) $\int_0^\infty |p(t)|dt < \infty$,
(5) $a\phi(y)y \geqslant g'(x)y^2$.
Then every solution $x(t)$ *of* (4.7) *satisfies* (4.3).

In particular, for the equation

$$x''' + h(x')x'' + \mu(x')x' + k(x)x = p(t), \tag{4.8}$$

where

$$h, \mu, k' \in C(\mathbb{R}, \mathbb{R}) \text{ and } p \in C([0, \infty), \mathbb{R}),$$

the above corollary becomes the following one.

COROLLARY 4. *Assume that:*
(1) $k(x) > 0$ *for* $x \neq 0$,
(2) $h(y) > a > 0$,
(3) $a\mu(y) \geqslant (xk(x))'$,
(4) $\int_0^\infty |p(t)|dt < \infty$.
Then every solution $x(t)$ *of Eq.* (4.8) *satisfies* (4.3).

Finally, when $h(y) = a, \mu(y) = b$ and $k(x) = c$ are all constants, Eq. (4.8) reduces to the linear equation

$$x''' + ax'' + bx' + cx = p(t). \tag{4.9}$$

The following result follows from Corollary 4 immediately.

COROLLARY 5. *Assume that:*
(1) $a > 0, b > 0, c > 0$,
(2) $ab > c$,
(3) $\int_0^\infty |p(t)|dt < \infty$.
Then every solution $x(t)$ *of Eq.* (4.9) *satisfies* (4.3).

Clearly, (1) and (2) are well-known Routh-Hurwitz conditions for the asymptotic stability of the following third-order linear equation:

$$x''' + ax'' + bx' + cx = 0. \tag{4.10}$$

REFERENCES

[1] E. A. BARBASHIN, *Lyapunov Functions*. Izda. 'Nauka', Moscow, 1970.
[2] J. O. C. EZEILO, *An elementary proof of a boundedness theorem for a certain third order differential equation*, J. Lond. Math. Soc., **38** (1963), 11–16.
[3] J. O. C. EZEILO, *Stability results for the solution of some third and fourth order differential equations*, Ann. Mat. Pura Appl., **66** (1964), 233–249.

[4] M. GERA, J. R. GRAEF, M. GREGUS, *On oscillatory and asymptotic properties of solutions of certain nonlinear third order differential equations*, Nonlinear Anal., **32** (1998), 417–425.
[5] C. QIAN, *On global stability of third-order nonlinear differential equations*, Nonlinear Analysis, **42** (2000), 651–661.
[6] C. QIAN, *Asymptotic behavior of a third-order nonlinear differential equation*, J. Math. Anal. Appl., **284** (2003,) 191–205.
[7] M. O. OMEIKE, *Further results on global stability of third-order nonlinear differential equations*, Nonlinear Anal., **67** (2007), 3394–3400.
[8] R. REISSIG, G. SANSONE, R. CONTI, *Nonlinear Differential Equations of Higher Order*, Noordhoff Inter. Pub. Leyden, 1974.
[9] C. TUNC, *On the ultimate boundedness result of the solutions of certain fourth order differential equation*, Ann. Differential Equations, **14** (1998), 475–485.
[10] T. YOSHIZAWA, *On the evaluation of the derivatives of solutions of $y'' = f(x,y,y')$*, Mem. Coll. Sc. Univ. Kyoto, Series A, **28** (1953), 27–32.
[11] T. YOSHIZAWA, *Asymptotic behavior of solutions of a system of differential equations*, Contrib. Differential Equations, **1** (1963), 371–387.

(Received August 16, 2008)

Mathew Omeike
Department of Mathematics
University of Agriculture
Abeokuta
Nigeria
e-mail: moomeike@yahoo.com