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UNIFORM ATTRACTORS FOR THE NON–AUTONOMOUS PARABOLIC EQUATION WITH NONLINEAR LAPLACIAN PRINCIPAL PART IN UNBOUNDED DOMAIN

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Abstract. In this paper, we are concerned with the asymptotic behavior of the solution for the non-autonomous parabolic equation with nonlinear Laplacian principal part in $\mathbb{R}^n$. The existence of the $(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$-uniform attractor, the $(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))$-uniform attractor and the $(L^2(\mathbb{R}^n),W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n))$-uniform attractor will be proved.

1. Introduction

The main purpose of this paper is to prove the existence of uniform attractors for the following non-autonomous parabolic equation:

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u + f(u) = g(t,x) \quad \text{in } \mathbb{R}^n \times [\tau,+\infty), \tag{1.1}$$

with initial data

$$u(x,\tau) = u_\tau, \quad \forall \tau \in \mathbb{R}, \tag{1.2}$$

where $p>2,\ \lambda>0,\ g(t,x)\in L^\infty(\mathbb{R},L^2(\mathbb{R}^n))$, and $f\in C^1(\mathbb{R})$ satisfies the following conditions:

$$f(0)=0, \qquad f'(u)\geqslant -l, \tag{1.3}$$

$$\alpha_1|u|^q-\beta_1|u|^p+\gamma_1|u|^2 \leqslant f(u)u \leqslant \alpha_2|u|^q+\beta_2|u|^p+\gamma_2|u|^2, \quad q\geqslant 2, \tag{1.4}$$

for some positive constants $l,\ \alpha_i,\ \beta_i,\ \gamma_i\ (i=1,2)$ and $\lambda>\beta_1$.

There is a large literature on the existence of global attractors of solutions to the autonomous problem (1.1)-(1.2) in bounded domain Ω. In [2], A. V. Babin and M. I. Vishik studied the abstract evolution equations with monotone principal part, they proved the existence of $(L^2(\Omega),L^2(\Omega))$-global attractor and the $(L^2(\Omega),W_0^{1,p}(\Omega)\cap$

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$L^q(\Omega))_w$-global attractor for this problem. After that, the authors of [5] obtained the associated semigroup has an $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-global attractor under some additional conditions (see [5] for details). In [15], by use of the asymptotic a priori estimate initiated in [17], the authors proved the existence of the $(L^2(\Omega), W_0^{1,p}(\Omega)\cap L^q(\Omega))$-global attractor, where $\lambda = 0$ and the nonlinear term $f(u)$ is arbitrary polynomial growth. For some other results concerning this problem in bounded domains, see [6] and the references therein.

The existence of global attractor for nondegenerate parabolic equations in unbounded domains was investigated in [1,8] in some weighted spaces. Later, these results were extended to degenerate parabolic equation in unbounded domains (see [9,10]). However, when working in weighted spaces, the initial data are always assumed to be in the same spaces. In order to solve this problem, the authors of [3,14] used a suitable cut-off function to prove the asymptotic compactness of the associated semigroup for reaction-diffusion equation; in the following, applying this method, the authors in [12] proved the existence of the $(L^2(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$-global attractor when $n \leqslant p$, and the $(L^2(\mathbb{R}^n), L^{\frac{np}{n-p}}(\mathbb{R}^n))$-global attractor when $n > p$. And in [16], the authors proved the existence of the $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n))$-global attractor by combining cut-off function method and asymptotic a priori estimate.

For the non-autonomous problems in bounded domain Ω, in [18], the authors proved that the associated family of processes for p-Laplacian equation (i.e. $\lambda = 0$ in (1.1)) possesses uniform attractors in the spaces $L^q(\Omega)$ $(\forall q \geqslant 2)$ and $W_0^{1,p}(\Omega)\cap L^q(\Omega)$, and even that these uniform attractors are coincident with each other.

In this paper, we are interested in the asymptotic behavior of the solution for (1.1-(1.2). We use the notion of uniform attractors with respect to the initial instant $\tau \in \mathbb{R}$, which was introduced by V.V.Chepyzhov and M.I.Vishik in [4], rather than the concept of uniform attractors with respect to the time symbol σ. This is one of the several related notions to study the asymptotic behavior for non-autonomous dynamical systems (see [4,7] and the references therein), and it means that whenever the obits start going, wherever the obits come from in the phase space, all these obits are attracted by this attractor. So in our work, we must handle the initial instant carefully, at the same time, we should overcome the difficulties brought by noncompactness of Sobolev embedding.

In order to get the uniform attractors of solution for (1.1)-(1.2), first of all, we provide some necessary and sufficient conditions for the existence of the uniform attractors with respect to initial instant (see section 2.1). And then, by applying the cut-off function method, extended asymptotic a priori estimate (see section 2.2) and differentiating skills, from section 3 to section 5, we obtain the existence of the $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-uniform attractor, the $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniform attractor and the $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n))$-uniform attractor, respectively.

For convenience, hereafter let $|u|$ be the modulus of u, $m(e)$ (sometimes we write it as $|e|$) the Lebesgue measure of $e \subset \mathbb{R}^n$, $\mathbb{R}^n(u \geqslant M) = \{x \in \mathbb{R}^n : u(x) \geqslant M\}$ and $\mathbb{R}^n(u \leqslant -M) = \{x \in \mathbb{R}^n : u(x) \leqslant -M\}$, C and c arbitrarily positive constants, which may be different from line to line and even in the same line.

2. Preliminaries and abstract results

2.1. Preliminaries

In this subsection, we give some basic definitions (see [2,4] for details) and the abstract results about the existence of bi-space uniform (with respect to (w. r. t.) $\tau \in \mathbb{R}$) attractors. Let X, Y be two Banach spaces, we consider a family of processes $\{U(t,\tau)\}$ defined on X, i.e., a family $\{U(t,\tau) : -\infty < \tau \leqslant t < \infty\}$ of mappings $U(t,\tau) : X \to X$, such that,

$$
\begin{aligned}
&U(\tau,\tau) = Id \quad \text{(identity)},\\
&U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau \leqslant r \leqslant t.
\end{aligned}
$$

DEFINITION 2.1. A set $B_0 \subset Y$ is called to be (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing for $\{U(t,\tau)\}$, if for any bounded subset $B \subset X$, there exists a positive constant $t_0 = t_0(B)$, such that:

$$
\bigcup_{\tau\in\mathbb{R}} U(t+\tau,\tau)B \subset B_0 \text{ for all } t \geqslant t_0.
$$

A set $P \subset Y$ is said to be (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) attracting for $\{U(t,\tau)\}$, if:

$$
\sup_{\tau\in\mathbb{R}} \mathrm{dist}_Y(U(t+\tau,\tau)B,P) \to 0 \ \ (t \to \infty) \text{ for any bounded set } B \subset X.
$$

DEFINITION 2.2. A closed set $\mathscr{A} \subset Y$ is said to be an (X,Y)-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor for the family of processes $\{U(t,\tau)\}$, if it is (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) attracting and it is contained in any closed (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) attracting set $\mathscr{A}'$ for $\{U(t,\tau)\}$: $\mathscr{A} \subset \mathscr{A}'$.

DEFINITION 2.3. A family of process $\{U(t,\tau)\}$ is called (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotically compact if for any bounded subset $B \subset X$ and any sequences $\{\tau_n\} \subset \mathbb{R}$, $t_n \to +\infty$ as $n \to +\infty$ and $\{x_n\} \subset B$, $\{U(t_n+\tau_n,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in Y.

LEMMA 2.4. *Let $\{U(t,\tau)\}$ be a family of processes acting on X. If it is (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotically compact, then for any bounded subset $B \subset X$, the set $\bigcap_{t\geqslant 0} \overline{\bigcup_{\tau\geqslant t} \bigcup_{h\in\mathbb{R}} U(\tau+h,h)B}^{Y}$ is nonempty, compact and it is the minimal set which uniformly (w. r. t. $\tau \in \mathbb{R}$) attracts B in the topology of Y.*

Proof. Denote by

$$
\omega(B) := \bigcap_{t\geqslant 0} \overline{\bigcup_{\tau\geqslant t} \bigcup_{h\in\mathbb{R}} U(\tau+h,h)B}^{Y},
$$

it is obvious that $y \in \omega(B)$ if and only if there exist sequences $\{\tau_n\} \subset \mathbb{R}$, $t_n \to +\infty$ as $n \to +\infty$ and $\{x_n\} \subset B$, such that $U(t_n+\tau_n,\tau_n)x_n \to y$ in Y. Thus from the assumption that $\{U(t,\tau)\}$ is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) asymptotically compact, we deduce that $\omega(B)$ is nonempty.

In order to prove the compactness of $\omega(B)$, we need to verify that for any sequence $\{y_n\}\subset\omega(B)$, there exists a convergent subsequence of $\{y_n\}$ in Y. In fact, by the definition of $\omega(B)$, there exist sequences $\{\tau_n^i\}_{i=1}^\infty$, $\{t_n^i\}_{i=1}^\infty$ and $\{x_n^i\}_{i=1}^\infty$, such that for any $n\in\mathbb{N}$, $U(t_n^i+\tau_n^i,\tau_n^i)x_n^i\to y_n$ as $i\to+\infty$. Therefore, for the subsequences $\{t_i^i\}_{i=1}^\infty$, $\{\tau_i^i\}_{i=1}^\infty$ and $\{x_i^i\}$, the (X,Y)-asymptotic compactness of $U(t,\tau)$ implies that there exists subsequence of $\{U(t_i^i+\tau_i^i,\tau_i^i)x_i^i\}_{i=1}^\infty$ which converges to some $y_0\in\omega(B)$, and it is easy to see that y_0 is a cluster point of $\{y_n\}$.

Finally, we prove the minimal property of $\omega(B)$. Let P be another closed set, which uniformly (w. r. t. $\tau\in\mathbb{R}$) attracts B in the topology of Y, we will show that $\omega(B)\subset P$. For any $y\in\omega(B)$, there exist sequences $\{\tau_n\}\subset\mathbb{R}$, $t_n\to\infty$ as $n\to\infty$, and $\{x_n\}\subset B$, such that $U(t_n+\tau_n,\tau_n)x_n\to y$ in Y. On the other hand, P is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) attracting B, i.e.,

$$\lim_{t\to\infty}\sup_{\tau\in\mathbb{R}}\operatorname{dist}_Y(U(t+\tau,\tau)B,P)=0,$$

then, we have

$$\lim_{t_n\to\infty}\sup_{\tau_n\in\mathbb{R}}\operatorname{dist}_Y(U(t_n+\tau_n,\tau_n)x_n,P)=0,$$

from the closeness of P, we conclude that $y\in P$, therefore, $\omega(B)\subset P$. □

THEOREM 2.5. *Let $\{U(t,\tau)\}$ be a family of processes acting on X. Then $\{U(t,\tau)\}$ possesses a compact (X,Y)-uniform (w. r. t. $\tau\in\mathbb{R}$) attractor $\mathscr{A}_0$, if and only if the following conditions hold:*

1.) *there exists $B_0\subset Y$, which is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) absorbing for $\{U(t,\tau)\}$,*

2.) *$\{U(t,\tau)\}$ is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) asymptotically compact.*

Moreover,

$$\mathscr{A}_0=\bigcap_{t\geqslant 0}\overline{\bigcup_{\tau\geqslant t}\bigcup_{h\in\mathbb{R}}U(\tau+h,h)B_0}^{Y}.$$

Proof. $\Leftarrow$) From lemma 2.4, we know that $\mathscr{A}_0$ is nonempty, compact, and A_0 is the minimal set which uniformly attracts B_0 in the topology of Y, therefore, it suffices to show that $\mathscr{A}_0$ is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) attracting any bounded subset $B\subset X$.

Since B_0 is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) absorbing, then for any $\tau\in\mathbb{R}$ and $B\subset X$ bounded, there is a positive constant $t_1=t_1(B)$, such that $\bigcup_{\tau\in\mathbb{R}}U(t+\tau,\tau)B\subset B_0$ for all $t\geqslant t_1$. Thus, we have

$$U(t+\tau+h,\tau)B=U(t+\tau+h,t+\tau)U(t+\tau,\tau)B\subset U(t+\tau+h,t+\tau)B_0,$$

for any $\tau\in\mathbb{R}$, $h\geqslant 0$ and any $t\geqslant t_1$. On the other hand, $\mathscr{A}_0$ is (X,Y)-uniformly (w. r. t. $\tau\in\mathbb{R}$) attracting for B_0, therefore,

$$\lim_{h\to\infty}\sup_{\tau\in\mathbb{R}}\operatorname{dist}_Y(U(t+\tau+h,\tau)B,\mathscr{A}_0)=0,$$

for any $t \geqslant t_1$. Therefore, $\mathscr{A}_0$ is an (X,Y)-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor for the family of processes $\{U(t,\tau)\}$.

$\Rightarrow$) Assume that $\mathscr{A}_0$ is the (X,Y)-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor, then it is easy to see that any ε-neighborhood $\mathscr{N}_\varepsilon(\mathscr{A}_0)$ of $\mathscr{A}_0$ in Y is (X,Y)-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing.

Now, we prove that $\{U(t,\tau)\}$ is (X,Y)-uniformly asymptotically compact. If not, there exist sequences $\{\tau_n\} \subset \mathbb{R}$, $t_n \to \infty$ $(n \to \infty)$ and $\{x_n\} \subset B$, such that $\{U(t_n + \tau_n, \tau_n)x_n\}$ is not precompact, then there exists some $\varepsilon_0 > 0$, such that:

$$d(U(t_i+\tau_i,\tau_i)x_i, U(t_j+\tau_j,\tau_j)x_j) \geqslant \varepsilon_0 > 0 \text{ for } i \neq j.$$

On the other hand, since $\mathscr{A}_0$ is uniformly attracting for $\{U(t,\tau)\}$, we have

$$\lim_{n\to\infty} \sup_{\tau_n} \operatorname{dist}_Y(U(t_n+\tau_n,\tau_n)x_n, \mathscr{A}_0) = 0,$$

then from the compactness of $\mathscr{A}_0$, there exists some $y \in \mathscr{A}_0$, which is a cluster point of $\{U(t_n+\tau_n,\tau_n)x_n\}$, this will lead a contradiction. □

2.2. Abstract results

In this subsection, we will extend the ideas of asymptotic a priori estimate in [13,16,17] to non-autonomous dynamical systems, which is useful to prove the existence of uniform (w. r. t. $\tau \in \mathbb{R}$) attractors in $L^p(\mathbb{R}^n)$ $(p \geqslant 2)$ for the family of processes $\{U(t,\tau)\}$.

LEMMA 2.6. (see [13]) *Let $B \subset L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ be bounded in both $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. Then for any $\varepsilon > 0$, B has a finite ε-net in $L^p(\mathbb{R}^n)$ if there exists a positive constant $M = M(\varepsilon)$ which depends on ε, such that:*
1.) *B has a finite $(3M)^{(2-p)/2}(\varepsilon/2)^{p/2}$-net in $L^2(\mathbb{R}^n)$;*
2.) *for all $u \in B$,*

$$\left(\int_{\mathbb{R}^n(|u|\geqslant M)} |u|^p dx\right)^{1/p} < 2^{-(2p+2)/p}\varepsilon.$$

LEMMA 2.7. *Assume $\{U(t,\tau)\}$ be a family of processes acting on $L^2(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ $(p \geqslant 1)$, $\{U(t,\tau)\}$ possesses a bounded $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing set. Then for any $\tau \in \mathbb{R}$ and any bounded subset $B \subset L^2(\mathbb{R}^n)$, there exist positive constants $t_0 = t_0(B,\varepsilon)$ and $M = M(\varepsilon)$, such that*

$$m(\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)) < \varepsilon \quad \text{for any } u_\tau \in B \text{ and } t \geqslant t_0 + \tau.$$

Proof. Let B_0 be a bounded $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing set, which is bounded by M_1. Then, for any bounded subset $B \subset L^2(\mathbb{R}^n)$, there is a positive constant $t_0(B)$ such that

$$\bigcup_{\tau\in\mathbb{R}} U(t+\tau,\tau)B \subset B_0, \quad t \geqslant t_0,$$

thus, for any $\tau \in \mathbb{R}$ and $u_\tau \in B$, we have

$$|U(t,\tau)u_\tau|_p^p \leqslant M_1, \quad t \geqslant t_0+\tau,$$

that is

$$M_1 \geqslant \int_{\mathbb{R}^n} |U(t,\tau)u_\tau|^p dx \geqslant \int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)} |U(t,\tau)u_\tau|^p dx$$
$$\geqslant \int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)} M^p dx \geqslant M^p \cdot m(\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)),$$

taking $M \geqslant (M_1/\varepsilon)^{1/p}$, we have $m(\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)) < \varepsilon$. □

LEMMA 2.8. (see [13]) *Let B be a bounded subset in $L^p(\mathbb{R}^n)$ $(p \geqslant 1)$. If B has a finite ε-net in $L^p(\mathbb{R}^n)$, then there exists an $M = M(B,\varepsilon)$ such that for any $u \in B$, the following estimate is valid:*

$$\int_{\mathbb{R}^n(|u| \geqslant M)} |u|^p dx \leqslant 2^{p+1}\varepsilon^p.$$

From theorem 2.5 and lemma 2.6, we can get the abstract result on the existence of the $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor, i.e. the following theorem.

THEOREM 2.9. *Let $\{U(t,\tau)\}$ be a family of processes acting on $L^2(\mathbb{R}^n)$ and on $L^p(\mathbb{R}^n)$, where $2 \leqslant p < \infty$. Suppose that $\{U(t,\tau)\}$ has a compact $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor. Then $\{U(t,\tau)\}$ has a compact $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniform (w. r. t. $\tau \in \mathbb{R}$) attractor provided that the following conditions hold:*
1.) *$\{U(t,\tau)\}$ has a bounded $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing the set B_0;*
2.) *for any $\varepsilon > 0$, $\tau \in \mathbb{R}$ and any bounded subset $B \subset L^2(\mathbb{R}^n)$, there exist positive constants $M = M(\varepsilon)$ and $t_0 = t_0(\varepsilon)$, such that*

$$\int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \geqslant M)} |U(t,\tau)u_\tau|^p dx \leqslant \varepsilon, \quad \textit{for any } u_\tau \in B \textit{ and } t \geqslant t_0+\tau. \tag{2.1}$$

3. The existence of $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-uniformly absorbing set and the $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-uniform attractor

We start with the following general existence and uniqueness of solution which can be obtained by the standard Faedo-Galerkin methods (see [2,11,14]). Here we only state the result:

THEOREM 3.1. *Assume that f satisfies (1.3) and (1.4), $g(t) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$. Then for $\forall \tau \in \mathbb{R}$, $\forall u_\tau \in L^2(\mathbb{R}^n)$ and $\forall T \geqslant \tau$, there exists a unique solution $u = u(T,\tau;u_\tau)$ for problem (1.1)-(1.2). Moreover,*

$$u \in L^\infty(\tau,T;L^2(\mathbb{R}^n)) \cap L^p(\tau,T;W^{1,p}(\mathbb{R}^n)) \cap L^q(\tau,T;L^q(\mathbb{R}^n)),$$

and the mapping $u_\tau \to u(t,\tau;u_\tau)$ is weakly continuous in $L^2(\mathbb{R}^n)$.

From theorem 3.1, we can define a family of processes $\{U(t,\tau): -\infty < \tau \leqslant t < \infty\}$ in $L^2(\mathbb{R}^n)$ which is weakly continuous,

$$U(t,\tau)u_\tau = u(t) := u(t,\tau;u_\tau), \qquad \text{for all } t \geqslant \tau, \tag{3.1}$$

where $u(t)$ is the solution of (1.1) with initial data $u(\tau) = u_\tau \in L^2(\mathbb{R}^n)$.

THEOREM 3.2. *Assume that* (1.3)-(1.4) *hold and* $g(t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then for any bounded subset* $B \subset L^2(\mathbb{R}^n)$ *and any* $\tau \in \mathbb{R}$, *there exists* $t_1(B) > 0$, *such that*

$$|u|_2^2 + |u|_q^q + |u|_p^p + |\nabla u|_p^p \leqslant R_0 \quad \text{for any } t \geqslant t_1(B) + \tau \text{ and } u_\tau \in B,$$

where $R_0 > 0$ *is independent of* B.

Proof. Multiplying (1.1) by u and integrating on $\mathbb{R}^n$, after the standard integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_p^p + \lambda |u|_p^p + \int_{\mathbb{R}^n} f(u)u dx = \int_{\mathbb{R}^n} g(t)u dx,$$

applying (1.4) and Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \int_{\mathbb{R}^n} \left(|\nabla u|^p + (\lambda - \beta_1)|u|^p + \alpha_1 |u|^q + \gamma_1 |u|^2\right) dx \leqslant c\int_{\mathbb{R}^n} |g(t)|^2 dx + \frac{\gamma_1}{2}\int_{\mathbb{R}^n} |u|^2 dx, \tag{3.2}$$

that is

$$\frac{d}{dt}|u|_2^2 + \gamma_1 |u|_2^2 \leqslant c\int_{\mathbb{R}^n} |g(t)|^2 dx,$$

Gronwall's inequality yields that

$$\begin{aligned} |u(t)|_2^2 &\leqslant e^{-\gamma_1(t-\tau)}|u_\tau|_2^2 + c\int_\tau^t e^{-\gamma_1(t-s)}|g(s)|_2^2 ds \\ &\leqslant e^{-\gamma_1(t-\tau)}|u_\tau|_2^2 + C(1 - e^{-\gamma_1(t-\tau)}), \end{aligned}$$

from which, we can find a $t_0(B) > 0$ large enough, such that $|u(t)|_2^2 \leqslant R_1$ for $\forall u_\tau \in B$ and $t \geqslant t_0(B) + \tau$.

In addition, integrating (3.2) about s from $t-1$ to t $(t \geqslant t_0(B) + \tau + 1)$, we know that

$$\frac{1}{2}|u(t)|_2^2 + c\int_{t-1}^t \left(|\nabla u(s)|_p^p + |u(s)|_p^p + |u(s)|_q^q + |u|_2^2\right) ds \leqslant \frac{1}{2}|u(t-1)|_2^2 + c\int_{t-1}^t \int_{\mathbb{R}^n} |g(t)|^2 dx ds \leqslant C(R_1). \tag{3.3}$$

Set $F(u)=\int_0^u f(s)ds$, after precise calculation we can deduce from (1.3) and (1.4) that

$$\frac{1}{2}\gamma_1|u|^2+\frac{\alpha_1}{q}|u|^q-\frac{\beta_1}{p}|u|^p\leqslant F(u)\leqslant \alpha_3|u|^q+\beta_3|u|^p+\gamma_3|u|^2, \tag{3.4}$$

where $\alpha_3,\ \beta_3,\ \gamma_3>0$, therefore,

$$\frac{1}{2}\gamma_1|u|_2^2+\frac{\alpha_1}{q}|u|_q^q-\frac{\beta_1}{p}|u|_p^p\leqslant \int_{\mathbb{R}^n}F(u)dx\leqslant \alpha_3|u|_q^q+\beta_3|u|_p^p+\gamma_3|u|_2^2. \tag{3.5}$$

Combining (3.3) and (3.5), we have

$$\int_{t-1}^{t}\left(|\nabla u(s)|_p^p+|u(s)|_p^p+\int_{\mathbb{R}^n}F(u(s))\right)ds\leqslant C. \tag{3.6}$$

On the other hand, taking inner product of (1.1) by u_t, we obtain

$$\begin{aligned}|u_t|_2^2+\frac{d}{dt}\left(\frac{1}{p}|\nabla u|_p^p+\frac{\lambda}{p}|u|_p^p+\int_{\mathbb{R}^n}F(u)dx\right)&=\int_{\mathbb{R}^n}g(t)u_t dx\\&\leqslant\frac{1}{2}\int_{\mathbb{R}^n}|g(t)|^2dx+\frac{1}{2}\int_{\mathbb{R}^n}|u_t|^2dx,\end{aligned} \tag{3.7}$$

which implies that,

$$\frac{d}{dt}\left(\frac{1}{p}|\nabla u|_p^p+\frac{\lambda}{p}|u|_p^p+\int_{\mathbb{R}^n}F(u)dx\right)\leqslant\frac{1}{2}\int_{\mathbb{R}^n}|g(t)|^2dx,$$

then inequality (3.6) and uniform Gronwall's inequality yield that

$$\frac{1}{p}|\nabla u(t)|_p^p+\frac{\lambda}{p}|u(t)|_p^p+\int_{\mathbb{R}^n}F(u(t))\leqslant C,$$

notice (3.5) and the fact that $\lambda>\beta_1$, we obtain

$$|u(t)|_2^2+|u(t)|_q^q+|u(t)|_p^p+|\nabla u(t)|_p^p\leqslant R_0\quad \text{for any } u_\tau\in B, \tag{3.8}$$

the proof is completed. □

REMARK 3.3. In the proof of theorem 3.2, we take inner product of (1.1) by u_t to obtain (3.7), in fact, we must be in the face of the regularity of solution there. However, by applying Galerkin approximation, the required regularity of solution is not difficult to prove, therefore, we only give formal deduction.

From theorem 3.2, we set B_0 by

$$B_0:=\{v\in L^2(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n):|u|_2^2+|u|_q^q+|u|_p^p+|\nabla u|_p^p\leqslant R_0\}, \tag{3.9}$$

it is easy to see that B_0 is $(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$-uniformly (w. r. t. $\tau\in\mathbb{R}$) absorbing, $(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))$-uniformly (w. r. t. $\tau\in\mathbb{R}$) absorbing and $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n))$ -uniformly (w. r. t. $\tau\in\mathbb{R}$) absorbing for the family of processes $\{U(t,\tau)\}$.

LEMMA 3.4. *Assume that* (1.3)-(1.4) *hold and* $g(t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then for any* $\varepsilon > 0, \tau \in \mathbb{R}$ *and any bounded subset* $B \subset L^2(\mathbb{R}^n)$, *there exist positive constants* $t_0(\varepsilon, B)$ *and* $R(\varepsilon)$, *such that for* $\forall r > R(\varepsilon)$,

$$\int_{|x|\geqslant r} |u(t)|^2 dx \leqslant C\varepsilon, \quad for\ \forall u_\tau \in B\ and\ t \geqslant t_0 + \tau,$$

in which $u(t) = U(t,\tau)u_\tau$, *and* C *is independent of* B.

Proof. Choosing a smooth function $\theta(s)$, such that $0 \leqslant \theta(s) \leqslant 1$ for any $s \geqslant 0$, and

$$\theta(s) = 0 \ \text{ for } 0 \leqslant s \leqslant 1, \qquad \theta(s) = 1 \ \text{ for } s \geqslant 2, \tag{3.10}$$

then there exists a constant M, such that $|\theta'(s)| \leqslant M$ for $s \in \mathbb{R}^+$.

Denote $\theta_r(x) := \theta(|x|^2/r^2)$. Multiplying (1.1) by $\theta_r^p(x)u$ and integrating on $\mathbb{R}^n$, we get that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx - \langle \mathrm{div}(|\nabla u|^{p-2}\nabla u), \theta_r^p u\rangle + \lambda \int_{\mathbb{R}^n} \theta_r^p |u|^p dx + \int_{\mathbb{R}^n} \theta_r^p f(u) u dx \\ = \int_{\mathbb{R}^n} \theta_r^p g(t) u dx \leqslant \frac{\gamma_1}{2}\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx + c\int_{\mathbb{R}^n} \theta_r^p |g(t)|^2 dx. \end{aligned} \tag{3.11}$$

Now, we bounded the second term of (3.11) as follows,

$$\langle -\mathrm{div}(|\nabla u|^{p-2}\nabla u), \theta_r^p u\rangle = \int_{\mathbb{R}^n} \theta_r^p |\nabla u|^p dx + \int_{\mathbb{R}^n} \frac{2px}{r^2}\theta_r' \theta_r^{p-1} |\nabla u|^{p-2} u \nabla u dx, \tag{3.12}$$

and

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \frac{2px}{r^2}\theta_r'\theta_r^{p-1}|\nabla u|^{p-2}u\nabla u dx\right| &\leqslant \frac{M}{r}\int_{r\leqslant|x|\leqslant\sqrt{2}r} \theta_r^{p-1}|\nabla u|^{p-1}|u| dx \\ &\leqslant \frac{C}{r}(|u|_p^p + \theta_r^p |\nabla u|_p^{p/(p-1)}) \\ &\leqslant \frac{C_1(M,R_0)}{r}, \text{ as } t > t_1(B) + \tau, \end{aligned} \tag{3.13}$$

here, $t_1(B)$ is given as in theorem 3.2.

In addition, from (1.4) we have

$$\begin{aligned} \gamma_1 \int_{\mathbb{R}^n} \theta_r^p |u|^2 dx + \alpha_1 \int_{\mathbb{R}^n} \theta_r^p |u|^q dx - \beta_1 \int_{\mathbb{R}^n} \theta_r^p |u|^p dx \leqslant \int_{\mathbb{R}^n} \theta_r^p f(u) u dx \\ \leqslant \alpha_2 \int_{\mathbb{R}^n} \theta_r^p |u|^q dx + \beta_2 \int_{\mathbb{R}^n} \theta_r^p |u|^p dx + \gamma_2 \int_{\mathbb{R}^n} \theta_r^p |u|^2 dx, \end{aligned} \tag{3.14}$$

combining (3.12)-(3.14) and (3.11) implies

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx + \frac{\gamma_1}{2}\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx \leqslant c\int_{|x|\geqslant r} \theta_r^p |g(t)|^2 dx + \frac{C_1}{r}, \tag{3.15}$$

since $g(t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$, we get that

$$\int_{|x|\geqslant r} \theta_r^p |g(t)|^2 dx \to 0 \qquad \text{as } r \to +\infty,$$

so for arbitrary $\varepsilon > 0$, there exists $R_1 > 0$, such that $\int_{|x|\geqslant r} \theta_r^p |g(t)|^2 \leqslant C\varepsilon$ for $r \geqslant R_1$. Therefore, taking $R = \max\{R_1, C_1/\varepsilon\}$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx + \gamma_1 \int_{\mathbb{R}^n} \theta_r^p |u|^2 dx \leqslant C\varepsilon, \quad t > t_1(B) + \tau, \tag{3.16}$$

by use of the Gronwall's inequality, we obtain that

$$\int_{\mathbb{R}^n} \theta_r^p |u|^2 dx \leqslant C\varepsilon, \tag{3.17}$$

the proof is completed. □

Thanks to the theorem 3.2 and lemma 3.4, according to the method of cut-off function used in [3,12,14], we know that $\{U(t,\tau)\}$ is $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotically compact, so we have the corollary below:

COROLLARY 3.5. *Suppose* (1.3)-(1.4) *hold and* $g(t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then there exists an* $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$*-uniform* (*w. r. t.* $\tau \in \mathbb{R}$) *attractor* $\mathscr{A}_2$ *for the family of processes* $\{U(t,\tau)\}$ *associated with problem* (1.1)-(1.2), *and*

$$\mathscr{A}_2 = \bigcap_{t\geqslant 0} \overline{\bigcup_{h\in\mathbb{R}} \bigcup_{\tau \geqslant t} U(h+\tau, h) B_0}^{L^2},$$

here B_0 *is the* $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$*-uniformly* (*w. r. t.* $\tau \in \mathbb{R}$) *absorbing set defined by (3.9).*

4. Asymptotic a priori estimate, existence of the $(L^2(\mathbb{R}^n), L^q(\mathbb{R}^n))$-uniform attractor and the $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniform attractor

In this section, we will give asymptotic a priori estimates for the unbounded part of the modulus $|u|$ for the solution of problem (1.1)-(1.2) in spaces $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, they are important for us to get the $(L^2(\mathbb{R}^n), L^q(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotic compactness and the $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotic compactness for the family of processes $\{U(t,\tau)\}$ associated with (1.1)-(1.2).

THEOREM 4.1. *Assume that* (1.3)-(1.4) *hold and* $g(t) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then for any* $\tau \in \mathbb{R}$, $\varepsilon > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^n)$, *there exist positive constants* $t_0 = t_0(B, \varepsilon)$ *and* $M = M(\varepsilon)$, *such that*

$$\int_{\mathbb{R}^n(|u|\geqslant M)} (|u(t)|^q + |u(t)|^p) dx \leqslant C\varepsilon, \quad \text{for any } t \geqslant t_0 + \tau \text{ and } u_\tau \in B, \tag{4.1}$$

here $u(t) = U(t,\tau)u_\tau$, C *is independent of* B.

Proof. From the absolute continuity of integrable function, for any fixed $\varepsilon > 0$, there exists a $\delta > 0$, such that if $e \subset \mathbb{R}^n$ and $m(e) < \delta$, then

$$\int_e |g|^2 dx < \varepsilon. \tag{4.2}$$

On the other hand, from lemma 2.7, lemma 2.8 and theorem 3.2, we know that there exist $t_1 = t_1(B,\varepsilon)$ and $M_1 = M_1(\varepsilon)$ such that for any $u_\tau \in B$ and $t \geqslant t_1 + \tau$, we have

$$m(\mathbb{R}^n(|u(t)| \geqslant M_1)) \leqslant min\{\varepsilon, \delta\}, \tag{4.3}$$

and

$$\int_{\mathbb{R}^n(|u(t)|\geqslant M_1)} |u(t)|^2 dx \leqslant \varepsilon. \tag{4.4}$$

Moreover, from (1.3) and (1.4) we can take M_0 large enough such that

$$\begin{aligned}\alpha_1|u|^{q-1} - \beta_1|u|^{p-1} + \gamma_1|u| \\ \leqslant f(u) \leqslant \alpha_2|u|^{q-1} + \beta_2|u|^{p-1} + \gamma_2|u| \ \text{ in } \mathbb{R}^n(u \geqslant M_0),\end{aligned} \tag{4.5}$$

Now, take $M_2 = max\{M_0, M_1\}$ and $t \geqslant t_1 + \tau$.

Let $(u - M_2)_+$ denote the positive part of $u - M_2$, that is

$$(u-M_2)_+ = \begin{cases} u - M_2, & u \geqslant M_2 \\ 0, & u \leqslant M_2. \end{cases}$$

Multiplying (1.1) by $(u - M_2)_+$ and integrating on $\mathbb{R}^n$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_2} |(u-M_2)_+|^2 dx + \int_{\Omega_2} |\nabla u|^p dx + \lambda \int_{\Omega_2} |u|^{p-2}u(u-M_2)_+ dx \\ + \int_{\Omega_2} f(u)(u-M_2)_+ dx = \int_{\Omega_2} g(t)(u-M_2)_+ dx,\end{aligned}$$

where $\Omega_2 \triangleq \mathbb{R}^n(u \geqslant M_2)$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_2} |(u-M_2)_+|^2 dx + \int_{\Omega_2} |\nabla u|^p dx + \lambda \int_{\Omega_2} |u|^p dx + \int_{\Omega_2} f(u) u dx \\ \leqslant \int_{\Omega_2} g(t)(u-M_2)_+ dx + M_2\left[\lambda \int_{\Omega_2} |u|^{p-1} dx + \int_{\Omega_2} f(u) dx\right],\end{aligned}$$

from (4.3)-(4.5) and Young's inequality, it becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_2} |(u-M_2)_+|^2 dx + \int_{\Omega_2} |\nabla u|^p dx + (\lambda - \beta_1)\int_{\Omega_2} |u|^p dx$$

$$
\begin{aligned}
&+\alpha_1\int_{\Omega_2}|u|^q dx+\gamma_1\int_{\Omega_2}|u|^2dx\\
\leqslant{}& C\int_{\Omega_2}|g(t)|^2dx+\frac{\gamma_1}{4}\int_{\Omega_2}|(u-M_2)_+|^2dx+C\varepsilon+\frac{\lambda-\beta_1}{2}\int_{\Omega_2}|u|^pdx\\
&+\frac{\gamma_1}{4}\int_{\Omega_2}|u|^2dx,
\end{aligned}
$$

since (4.2) holds, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_2}|(u-M_2)_+|^2dx\\
&\quad+\int_{\Omega_2}\left(|\nabla u|^p+\frac{\lambda-\beta_1}{2}|u|^pdx+\frac{\gamma_1}{2}|u|^2+\alpha_1|u|^q\right)dx\leqslant C\varepsilon, \qquad (4.6)
\end{aligned}
$$

from the fact that $\int_{\Omega_2}|u|^2dx\geqslant\int_{\Omega_2}|(u-M_2)_+|^2dx$, and Gronwall's inequality, we get that

$$
\int_{\Omega_2}|(u(t)-M_2)_+|^2dx<C\varepsilon,\ \ \text{for } t-\tau\geqslant\tau_2(\varepsilon). \qquad (4.7)
$$

In addition, for any $r\geqslant t+\tau_2$, integrating (4.6) from r to $r+1$ with respect to t, and applying (4.7), we get that

$$
\begin{aligned}
&\int_{\Omega_2}|(u(r+1)-M_2)_+|^2dx\\
&\quad+\int_r^{r+1}\int_{\Omega_2}\left(|\nabla u(t)|^p+\frac{\lambda-\beta_1}{2}|u(t)|^pdx+\frac{\gamma_1}{2}|u(t)|^2+\alpha_1|u(t)|^q\right)dxdt\leqslant C\varepsilon,
\end{aligned}
$$

notice that (3.5), we have

$$
\int_r^{r+1}\int_{\Omega_2}\big(|\nabla u(t)|^p+|u(t)|^pdx+|u(t)|^2+|u(t)|^q+F(u(t))\big)\,dxdt\leqslant C\varepsilon. \qquad (4.8)
$$

On the other hand, multiplying (1.1) by $(u-M_2)_{+t}$ and integrating on $\mathbb{R}^n$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|(u-M_2)_{+t}|^2dx+\int_{\mathbb{R}^n}|\nabla u|^{p-2}\nabla u\nabla(u-M_2)_{+t}dx+\lambda\int_{\mathbb{R}^n}|u|^{p-2}u\cdot(u-M_2)_{+t}dx\\
&\qquad+\int_{\mathbb{R}^n}f(u)\cdot(u-M_2)_{+t}dx=\int_{\mathbb{R}^n}g(t)\cdot(u-M_2)_{+t}dx,
\end{aligned}
$$

from Young's inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|(u-M_2)_{+t}|^2dx+\frac{1}{p}\frac{d}{dt}\int_{\Omega_2}|\nabla u|^pdx+\frac{\lambda}{p}\frac{d}{dt}\int_{\Omega_2}|u|^pdx+\frac{d}{dt}\int_{\Omega_2}F(u)dx\\
&\qquad=\int_{\Omega_2}g(t)\cdot(u-M_2)_{+t}dx\leqslant\frac{1}{2}\int_{\Omega_2}|g(t)|^2dx+\frac{1}{2}\int_{\Omega_2}|(u-M_2)_{+t}|^2dx,
\end{aligned}
$$

then we have

$$\frac{1}{2}\int_{\mathbb{R}^n}|(u-M_2)_{+t}|^2dx+\frac{d}{dt}\int_{\Omega_2}\left(\frac{1}{p}|\nabla u|^p+\frac{\lambda}{p}|u|^p+F(u)\right)dx \leqslant \frac{1}{2}\int_{\Omega_2}|g(t)|^2dx\leqslant C\varepsilon,$$

recalling (4.8) and applying uniform Gronwall's inequality, we obtain

$$\int_{\Omega_2}\left(\frac{1}{p}|\nabla u(t)|^p+\frac{\lambda}{p}|u(t)|^p+F(u(t))\right)dx\leqslant C\varepsilon,$$

employing (3.5) once again, we get that

$$\int_{\Omega_2}\left(|\nabla u(t)|^p+|u(t)|^p+|u|^q+|u|^2\right)\right)dx\leqslant C\varepsilon. \tag{4.9}$$

Repeating the same steps above, just taking $|(u+M_2)_-|$ instead of $(u-M_2)_+$, and replacing $(u-M_2)_{+t}$ with $(u+M_2)_{-t}$, we obtain

$$\int_{\mathbb{R}^n(u\leqslant -M_2)}\left(|\nabla u(t)|^p+|u(t)|^p+|u|^q+|u|^2\right)\right)dx\leqslant C\varepsilon, \tag{4.10}$$

combining (4.9) with (4.10), we conclude that for any $M\geqslant M_2$,

$$\int_{\mathbb{R}^n(|u|\geqslant M)}\left(|\nabla u(t)|^p+|u(t)|^p+|u|^q+|u|^2\right)\right)dx\leqslant C\varepsilon,$$

and the proof is completed. □

Collecting theorem 4.1, corollary 3.5 and theorem 2.9, the $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n))$-uniform (w. r. t. $\tau\in\mathbb{R}$) attractor and the $(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))$-uniform (w. r. t. $\tau\in\mathbb{R}$) attractor for $\{U(t,\tau)\}$ can be constructed, to be precise, the following theorems hold:

THEOREM 4.2. *Suppose* (1.3) *and* (1.4) *hold,* $g(t)\in L^\infty(\mathbb{R},L^2(\mathbb{R}^n))$. *Then there exists an* $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n))$*-uniform* (*w. r. t.* $\tau\in\mathbb{R}$) *attractor* $\mathscr{A}_q$ *for the family of processes* $\{U(t,\tau)\}$ *associated with* (1.1)-(1.2). *And*

$$\mathscr{A}_q=\bigcap_{t\geqslant 0}\overline{\bigcup_{h\in\mathbb{R}}\bigcup_{\tau\geqslant t}U(h+\tau,h)B_0}^{L^q},$$

here B_0 *is the* $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n))$*-uniformly* (*w. r. t.* $\tau\in\mathbb{R}$) *absorbing set defined by* (3.9).

THEOREM 4.3. *Under the same assumptions of preceding theorem 4.2, there exists an* $(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))$*-uniform* (*w. r. t.* $\tau\in\mathbb{R}$) *attractor* $\mathscr{A}_p$ *for* $\{U(t,\tau)\}$ *associated with problem* (1.1)-(1.2), *and*

$$\mathscr{A}_p=\bigcap_{t\geqslant 0}\overline{\bigcup_{h\in\mathbb{R}}\bigcup_{\tau\geqslant t}U(h+\tau,h)B_0}^{L^p},$$

where B_0 *is defined by* (3.9).

5. The existence of $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n))$-uniform attractor

In this section, we prove the existence of the $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n))$-uniform (w. r. t. $\tau\in\mathbb{R}$) attractor. Firstly, we give a priori estimates about u_t in L^2-norm.

LEMMA 5.1. *Assume that f satisfies (1.3)-(1.4), $g\in W^{1,\infty}(\mathbb{R}, L^2(\mathbb{R}^n))$. Then for any $\tau\in\mathbb{R}$ and any bounded subset $B\subset L^2(\mathbb{R}^n)$, there is a positive constant $t_0 = t_0(B)$, such that*

$$|u_t(t)|_2^2 \leqslant M, \quad \text{for any } u_\tau\in B \text{ and } t\geqslant t_0+\tau.$$

where $u_t(t)=\frac{d}{dt}(U(t,\tau)u_\tau)$, and M is a positive constant which is independent of B.

Proof. We give the formal calculations, the rigorous proof can be obtained by Galerkin approximation. Differentiating (1.1) and setting $v=u_t$, we have

$$v_t - \operatorname{div}(|\nabla u|^{p-2}\nabla v) - (p-2)\operatorname{div}(|\nabla u|^{p-4}(\nabla u\cdot\nabla v)\nabla u) \\ + \lambda(p-1)|u|^{p-2}v + f'(u)v = g'(t), \quad (5.1)$$

where "$\cdot$" denotes the dot product in $\mathbb{R}^n$.

Multiplying (5.1) by v and integrating on $\mathbb{R}^n$, we obtain that

$$\frac{1}{2}\frac{d}{dt}|v(t)|_2^2 + \int_{\mathbb{R}^n}|\nabla u|^{p-2}|\nabla v|^2dx + (p-2)\int_{\mathbb{R}^n}|\nabla u|^{p-4}|(\nabla u\cdot\nabla v)|^2dx \\ + \lambda(p-1)\int_{\mathbb{R}^n}|u|^{p-2}|v|^2dx + \int_{\mathbb{R}^n}f'(u)|v|^2dx \\ \leqslant C\int_{\mathbb{R}^n}|g'(t)|^2dx + l\int_{\mathbb{R}^n}|v|^2dx, \quad (5.2)$$

from (1.3), we have

$$\frac{1}{2}\frac{d}{dt}|v(t)|_2^2 \leqslant C\int_{\mathbb{R}^n}|g'(t)|^2dx + 2l\int_{\mathbb{R}^n}|v(t)|^2dx. \quad (5.3)$$

Now, employing (3.7) once again, by integrating about s from $t-1$ to t, $t\geqslant t_0(B)+\tau+1$ ($t_0(B)$ is as given in theorem 3.2), we have

$$\int_{t-1}^{t}|u_t|_2^2ds \leqslant C,$$

then inequality (5.3) and uniform Gronwall's inequality tell us

$$\int_{\mathbb{R}^n}|u_t(t)|^2dx \leqslant M,$$

for any $u_\tau\in B$ and $t\geqslant t_0(B)+\tau+1$, M is independent of B. □

Let $\xi(\cdot)\in C^\infty(\mathbb{R}^n)$ be such that $0\leqslant\xi(s)\leqslant 1$, for any $s\geqslant 0$, and

$$\xi(s)=1 \ \text{ for } 0\leqslant s\leqslant 1, \qquad \xi(s)=0 \ \text{ for } s\geqslant 2.$$

Furthermore, define $\xi_k(x) = \xi(\frac{|x|^2}{k^2})$ for any $k \in \mathbb{R}^+$.

Now, we prove that $\{U(t,\tau)\}$ is in fact $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotically compact. From theorem 4.2, it is sufficient to prove that for any fixed k, $\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}$ is precompact in $W^{1,p}(\mathbb{R}^n)$ for any sequences $\{u_{\tau_n}\} \subset B_0$, $\{\tau_n\} \subset \mathbb{R}$ and $t_n \to +\infty$ $(n \to \infty)$, where B_0 is $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing for $\{U(t,\tau)\}$.

THEOREM 5.2. *Assume that f satisfies* (1.3)-(1.4), $g \in W^{1,\infty}(\mathbb{R}, L^2(\mathbb{R}^n))$. *Then the family of processes $\{U(t,\tau)\}$ is $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) asymptotically compact.*

Proof. Let B_0 be the $(L^2(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$-uniformly (w. r. t. $\tau \in \mathbb{R}$) absorbing set obtained in (3.9), then according to theorem 2.5, we need only to show that

$$\begin{aligned}&\text{for any } \{u_{\tau_n}\} \subset B_0,\ \{\tau_n\} \subset \mathbb{R} \text{ and } t_n \to +\infty \text{ as } n \to \infty,\\ &\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}_{n=1}^{\infty} \text{ is precompact in } W^{1,p}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n),\end{aligned} \tag{5.4}$$

thanks to theorem 4.2, it is sufficient to verify that for every fixed k, we have

$$\begin{aligned}&\text{for any } \{u_{\tau_n}\} \subset B_0,\ \{\tau_n\} \subset \mathbb{R} \text{ and } t_n \to +\infty \text{ as } n \to \infty,\\ &\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}_{n=1}^{\infty} \text{ is precompact in } W^{1,p}(\mathbb{R}^n).\end{aligned} \tag{5.5}$$

In fact, from corollary 3.5, theorem 4.2 and theorem 4.3, we know that for every fixed k, $\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}_{n=1}^{\infty}$ is precompact in $L^2(\mathbb{R}^n)$, $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$. Without loss of generality, we assume that $\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$.

Now, we prove that for every fixed k, $\{\xi_k U(t_n+\tau_n,\tau_n)u_{\tau_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$. Denote by $u^n_{\tau_n}(t_n) := U(t_n+\tau_n,\tau_n)u_{\tau_n}$, and recall the property of p-Laplacian operator for $p \geqslant 2$: there exists a positive constant δ, such that for all $u_1, u_2 \in W^{1,p}(\mathbb{R}^n)$,

$$\begin{aligned}\langle Au_1 - Au_2, \xi_k^p(u_1-u_2)\rangle \geqslant{}& \delta\|(u_1-u_2)\|^p_{W^{1,p}}\\ &+\int_{\mathbb{R}^n}\frac{2px}{k^2}\xi_k'\xi_k^{p-1}(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(u_1-u_2)dx,\end{aligned} \tag{5.6}$$

where $Au = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u$ and $\langle\cdot,\cdot\rangle$ is the L^2-inner product. Then from equation (1.1), we have

$$\begin{aligned}&\langle Au^n_{\tau_n}(t_n) - Au^m_{\tau_m}(t_m), \xi_k^p(u^n_{\tau_n}(t_n) - u^m_{\tau_m}(t_m))\rangle\\ &= \langle g(t_n+\tau_n) - \frac{d}{dt}u^n_{\tau_n}(t_n) - f(u^n_{\tau_n}(t_n)) + \frac{d}{dt}u^m_{\tau_m}(t_m) + f(u^m_{\tau_m}(t_m)) - g(t_m+\tau_m),\end{aligned}$$

$$
\begin{aligned}
&\xi_k^p(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))\rangle\\
&\quad\leqslant \int_{\mathbb{R}^n}\left|\frac{d}{dt}u_{\tau_n}^n(t_n)-\frac{d}{dt}u_{\tau_m}^m(t_m)\right|\cdot|\xi_k^p(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|dx\\
&\qquad+\int_{\mathbb{R}^n}|f(u_{\tau_n}^n(t_n))-f(u_{\tau_m}^m(t_m))|\cdot|\xi_k^p(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|dx\\
&\qquad+\int_{\mathbb{R}^n}|g(t_n+\tau_n)-g(t_m+\tau_m)|\cdot|\xi_k^p(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|dx\\
&\quad\leqslant\left|\frac{d}{dt}u_{\tau_n}^n(t_n)-\frac{d}{dt}u_{\tau_m}^m(t_m))\right|_2\cdot|\xi_k(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|_2\\
&\qquad+C(1+|u_{\tau_n}^n(t_n)|_q^q+|u_{\tau_m}^m(t_m)|_q^q)\cdot|\xi_k(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|_q^q\\
&\qquad+C(1+|u_{\tau_n}^n(t_n)|_p^p+|u_{\tau_m}^m(t_m)|_p^p)\cdot|\xi_k(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|_p^p\\
&\qquad+C(1+|u_{\tau_n}^n(t_n)|_2^2+|u_{\tau_m}^m(t_m)|_2^2)\cdot|\xi_k(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|_2^2\\
&\qquad+C|g(t_n+\tau_n)-g(t_m+\tau_m)|_2\cdot|\xi_k(u_{\tau_n}^n(t_n)-u_{\tau_m}^m(t_m))|_2, \qquad (5.7)
\end{aligned}
$$

which, combining with lemma 5.1, yields (5.4) immediately. □

COROLLARY 5.3. *Under the same assumptions in theorem 5.2, there exists a subset* $\mathscr{A}_{p,q}$*, which is compact in* $W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$*, and uniformly* ($w.\ r.\ t.\ \tau\in\mathbb{R}$) *attracts all the bounded subset of* $L^2(\mathbb{R}^n)$ *in the topology of* $W^{1,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$*. Moreover,*

$$
\mathscr{A}_{p,q}=\bigcap_{t\geqslant 0}\overline{\bigcup_{h\in\mathbb{R}}\bigcup_{\tau\geqslant t}U(t+h,h)B_0}^{W^{1,p}\cap L^q},
$$

in which B_0 *is given by* (3.9).

REMARK 5.4. From the prove above, we can deduce that $\mathscr{A}_2$, $\mathscr{A}_p$, $\mathscr{A}_q$ and $\mathscr{A}_{p,q}$ coincide with each other. Furthermore, from the relationship among uniform attractors, uniform pullback attractors and uniform forward attractors (see [4,7]), we know that problem (1.1)-(1.2) possesses $(L^2(\mathbb{R}^n),L^p(\mathbb{R}^n))$-uniform forward (pullback) attractor and $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n))$-uniform forward (pullback) attractor under the same conditions in corollary 5.3, and these attractors are included in the uniform attractor (see [4,7] for detail). However, when we relax the conditions, especially the conditions on $g(t)$, the existence of $(L^2(\mathbb{R}^n),L^q(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n))$-uniform attractor is unknown.

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