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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A SECOND ORDER NEUTRAL EQUATION OF DISCRETE TYPE WITH OSCILLATING COEFFICIENTS

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Abstract. In this paper, we obtain sufficient conditions so that every solution of neutral functional difference equation

$$\Delta^2(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n,$$

oscillates or tends to zero as $n \to \infty$, where the sequence $\{q_n\}$ may change sign. Here Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, $\{\tau_n\}$ and $\{\sigma_n\}$ are increasing sequences, which are less than n and approaches ∞ as n approaches ∞. This paper generalizes and extends some recent results.

1. Introduction

Consider the higher order neutral functional difference equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n, \tag{1.1}$$

where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n, \{p_n\}, \{q_n\}$, $\{f_n\}$ are infinite sequences of real numbers and m is any positive integer. Further, assume that $G \in C(\mathbb{R}, \mathbb{R})$, and $\tau(n) \leqslant n, \sigma(n) \leqslant n$ are monotonic increasing sequences which are unbounded. Recently, the oscillatory and asymptotic behavior of solutions of (1.1) with fixed sign q_n have been investigated by many authors (see [2], [5]–[9], [11, 13, 14, 15, 18]). However, it is difficult to study the oscillation of (1.1) for the general case, when q_n is allowed to change sign, since the difference $\Delta^m(y_n - p_n y_{\tau(n)} - F_n)$ of any non-oscillatory solution $\{y_n\}$ of (1.1) is oscillatory, where $\{F_n\}$ is an infinite sequence such that $\Delta^m F_n = f_n$. Therefore the results on oscillation of (1.1) with oscillating q_n, are relatively scarce; see [16, 17, 10]. Such result for $m = 1$ is found in recently published paper [10], where sufficient conditions are obtained so that every solution of (1.1) oscillates or tends to zero as $n \to \infty$. The technique adopted in the paper, however, fails when applied to (1.1) for $m > 1$. Hence our objective in this work is to extend some of these results and study the asymptotic behaviour of solutions of neutral difference equation

$$\Delta^2(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n, \tag{1.2}$$

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where q_n may change sign.

For the purpose, we define $q_n^+ = \max(q_n, 0)$. and $q_n^- = \max(-q_n, 0)$ then the above equation takes an alternate form

$$\Delta^2(y_n - p_n y_{\tau(n)}) + q_n^+ G(y_{\sigma(n)}) - q_n^- G(y_{\sigma(n)}) = f_n. \tag{1.3}$$

We would in fact, study the neutral difference equation with positive and negative coefficients (1.3) under the following assumptions:

(H_0) for any sequence $\{x_n\}$, if $\liminf_{n\to\infty} |x_n| > 0$ then $\liminf_{n\to\infty} |G(x_n)| > 0$;
(H_1) $xG(x) > 0$ for $x \neq 0$;
(H_2) G is bounded;
(H_3) $\sum_{n=n_0}^{\infty} q_n^+ = \infty$;
(H_4) $\sum_{n=n_0}^{\infty} n q_n^- < \infty$;
(H_5) there exists a bounded sequence $\{F_n\}$ such that $\Delta^2 F_n = f_n$;
(H_6) the sequence $\{F_n\}$ in (H_5) satisfies $\lim_{n\to\infty} F_n = 0$.

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1.2) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geqslant \rho$ and satisfies (1.2) for $n \geqslant n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for} \quad \rho \leqslant n \leqslant n_0, \tag{1.4}$$

is given then the equation (1.2) has a unique solution satisfying given initial condition (1.4). A solution $\{y_n\}$ of (1.2) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \geqslant n_0$ such that $y_n y_{n+1} \leqslant 0$, otherwise $\{y_n\}$ is said to be non-oscillatory. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all n sufficiently large.

2. Main Results

To begin with, we state some lemmas from [1, 4, 12] which would be useful for our work.

LEMMA 2.1. [1] *Let $\{f_n\}, \{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geqslant N_0 > 0$ such that*

$$f_n = q_n - p_n q_{\tau(n)}, \quad n \geqslant N_1 \geqslant N_0,$$

where $\{\tau(n)\}$ is an increasing unbounded sequence such that $\tau(n) \leqslant n$. Suppose that p_n satisfies one of the following three conditions:

$$-1 < -b_1 \leqslant p_n \leqslant 0, \quad -b_2 \leqslant p_n \leqslant -b_3 < -1, \quad \text{and} \quad 0 \leqslant p_n \leqslant b_4 < \infty,$$

$\forall n \in \mathbb{N}$, where b_1, b_2, b_3 and b_4 are constants. If $q_n > 0$ for $n \geqslant N_0$, $\liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists, then $L = 0$.

LEMMA 2.2. [4] *If* $\sum u_n$ *and* $\sum v_n$ *are two positive term series such that*

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=l,$$

where l *is a non-zero finite number, then the two series converge or diverge together. If* $l=0$*, then* $\sum v_n$ *is convergent implies the convergence of* $\sum u_n$*. If* $l=\infty$*, then* $\sum v_n$ *is divergent implies the divergence of* $\sum u_n$*.*

Before we state and prove our next result, we need the following definition and further discussion.

DEFINITION 1. Define the factorial function (cf [3, page-20]) by

$$n^{(k)}:=n(n-1)\dots(n-k+1),$$

where $k\leqslant n$ and $n\in\mathbb{Z}$ and $k\in\mathbb{N}$. Note that $n^{(k)}=0$, if $k>n$.

Then we have

$$\Delta n^{(k)}=kn^{(k-1)},\tag{2.1}$$

where $n\in\mathbb{Z}$, $k\in\mathbb{N}$ and Δ is the forward difference operator. One can show, by summing up (2.1) that

$$\sum_{i=m}^{n-1} i^{(k)}=\frac{1}{k+1}\left(n^{(k+1)}-m^{(k+1)}\right),\tag{2.2}$$

holds. Now set

$$b_k(n,m):=\begin{cases}1, & k=0\\ \sum\limits_{j=m}^{n} b_{k-1}(n,j), & k\in\mathbb{N}.\end{cases}\tag{2.3}$$

Here, we evaluate b_k by recursion. Clearly, for $k=1$ in (2.3), we have

$$b_1(n,m)=\sum_{j=m}^{n} b_0(n,j)=\sum_{j=m}^{n}1=(n+1-m)=(n+1-m)^{(1)}.$$

By (2.2) and for $k=2$ in (2.3), we get

$$\begin{aligned}b_2(n,m) &= \sum_{j=m}^{n} b_1(n,j)=\sum_{j=m}^{n}(n+1-j)^{(1)}\\ &= \sum_{i=1}^{n+1-m} i^{(1)}=\frac{1}{2}(n+2-m)^{(2)}-\frac{1}{2}1^{(2)}=\frac{1}{2}(n+2-m)^{(2)}.\end{aligned}$$

Note that $1^{(2)}=0$. By (2.2) and for $k=3$ in (2.3), we get

$$\begin{aligned}b_3(n,m) &= \sum_{j=m}^{n} b_2(n,j)=\frac{1}{2}\sum_{j=m}^{n}(n+2-j)^{(2)}\\ &= \frac{1}{2}\sum_{i=2}^{n+2-m} i^{(2)}=\frac{1}{6}\left[(n+3-m)^{(3)}-2^{(3)}\right]=\frac{1}{3!}(n+3-m)^{(3)}.\end{aligned}$$

Using a simple induction, we obtain

$$b_k(n,m) = \frac{1}{k!}(n+k-m)^{(k)}. \tag{2.4}$$

LEMMA 2.3. [12] *Let $p \in \mathbb{N}$ and $x(n)$ be a nonoscillatory eventually positive real valued function. If there exists an integer $p_0 \in \{0,1,\dots,p-1\}$ such that $\Delta^{p_0}w(\infty)$ exits (finite) and $\Delta^i w(\infty) = 0$ for all $i \in \{p_0+1,\dots,p-1\}$ then*

$$\Delta^p w(n) = -x(n), \tag{2.5}$$

implies

$$\Delta^{p_0}w(n) = \Delta^{p_0}w(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!}\sum_{i=n}^{\infty}(i+p-p_0-1-n)^{(p-p_0-1)}x(i), \tag{2.6}$$

for all sufficiently large n.

Proof. Summing (2.5) from n to ∞, we get

$$\Delta^{p-1}w(\infty) - \Delta^{p-1}w(n) = -\sum_{i=n}^{\infty}x(i),$$

that is

$$\Delta^{p-1}w(n) = \sum_{i=n}^{\infty}x(i) = \sum_{i=n}^{\infty}b_0(i,n)x(i). \tag{2.7}$$

Summing (2.7) from n to ∞, we get

$$\begin{aligned}\Delta^{p-2}w(n) &= \Delta^{p-2}w(\infty) - \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}b_0(j,i)x(j) = -\sum_{j=n}^{\infty}\sum_{i=n}^{j}b_0(j,i)x(j)\\ &= -\sum_{j=n}^{\infty}b_1(j,n)x(j) = -\sum_{i=n}^{\infty}b_1(i,n)x(i).\end{aligned} \tag{2.8}$$

Further, summing (2.8) from n to ∞, we obtain

$$\begin{aligned}\Delta^{p-3}w(n) &= \sum_{j=n}^{\infty}\sum_{i=j}^{\infty}b_1(i,j)x(i) = \sum_{i=n}^{\infty}\sum_{j=n}^{i}b_1(i,j)x(i)\\ &= \sum_{i=n}^{\infty}b_2(i,n)x(i).\end{aligned}$$

By the emerging pattern, we have

$$\Delta^j w(n) = (-1)^{p-j-1}\sum_{i=n}^{\infty}b_{p-j-1}(i,n)x(i), \quad j \in \{p_0+1,\dots p-1\}.$$

Letting $j = p_0 + 1$, we get

$$\Delta^{p_0+1} w(n) = (-1)^{p-p_0-2} \sum_{i=n}^{\infty} b_{p-p_0-2}(i,n)\, x(i). \tag{2.9}$$

Summing (2.9) from n to ∞ we get

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + (-1)^{p-p_0-1} \sum_{i=n}^{\infty} b_{p-p_0-1}(i,n)\, x(i). \tag{2.10}$$

Clearly, (2.6) follows from (2.4) and (2.10). And the proof of the Lemma is complete.

Now, we state our first main result.

THEOREM 2.4. *Suppose that* (H_0) - (H_5) *hold. Assume that there exists a positive constant* b_1 *such that the sequence* $\{p_n\}$ *satisfies the condition*

$$0 \leqslant p_n \leqslant b_1 < 1, \quad \text{or} \quad -1 < -b_1 \leqslant p_n \leqslant 0. \tag{2.11}$$

Then every non-oscillatory solution of (1.2) *is bounded.*

Proof. Let $y = \{y_n\}$ be any non-oscillatory solution of (1.2) for $n \geqslant N_1$, where N_1 is a fixed positive integer. Then $y_n > 0$ or $y_n < 0$. Suppose $y_n > 0$ eventually. There exits positive integer $n_0 \geqslant N_1 > 0$ such that $y_n > 0, y_{\tau(n)} > 0$, and $y_{\sigma(n)} > 0$ for $n \geqslant n_0$. For $n \geqslant n_0$, let

$$z_n = y_n - p_n y_{\tau(n)}. \tag{2.12}$$

Define for $n \geqslant n_0$

$$T_n = \sum_{i=n}^{\infty} (n-i-1) q_i^- G(y_{\sigma(i)}). \tag{2.13}$$

Due to the assumptions (H_2) and (H_4), $\{T_n\}$ is a well defined real sequence which is convergent. This implies

$$\lim_{n\to\infty} T_n = 0 \tag{2.14}$$

and

$$\Delta^2 T_n = -q_n^- G(y_{\sigma(n)}). \tag{2.15}$$

Set,

$$w_n = y_n - p_n y_{\tau(n)} + T_n - F_n. \tag{2.16}$$

From (1.2), (2.15), and (2.16), it follows due to (H_1) that

$$\Delta^2 w_n = -q_n^+ G(y_{\sigma(n)}) \leqslant 0. \tag{2.17}$$

Then there exists $n_1 \geqslant n_0$ such that $w_n, \Delta w_n,$ are monotonic and of constant sign for $n \geqslant n_1$. For the sake of a contradiction assume that y_n is not bounded. Then there exists a sub sequence $\{y_{n_k}\}$ such that

$$n_k \to \infty, y_{n_k} \to \infty \text{ as } k \to \infty,$$

and

$$y(n_k) = \max\{y_n : n_1 \leqslant n \leqslant n_k\}. \tag{2.18}$$

Since $\tau(n) \to \infty$ and $\sigma(n) \to \infty$ as $n \to \infty$, we may choose k large enough so that for $\tau(n_k) \geqslant n_1$, and $\sigma(n_k) \geqslant n_1$. For $0 < \varepsilon$, because of (2.14) and (H_5), we can find a positive integer n_2 and a constant γ such that, for $k \geqslant n_2 \geqslant n_1$ implies $|T_{n_k}| < \varepsilon$ and $|F_{n_k}| < \gamma$. If the condition $0 \leqslant p_n \leqslant b_1 < 1$ holds, then using (2.16) and (2.18) we obtain

$$w_{n_k} \geqslant y_{n_k}(1 - b_1) - \varepsilon - \gamma,$$

for $k \geqslant n_2$. Similarly, if $-1 < -b_1 \leqslant p_n \leqslant 0$ holds, then for $k \geqslant n_2$, we have

$$w_{n_k} \geqslant y_{n_k} - \varepsilon - \gamma.$$

Taking $k \to \infty$, we find $\lim_{n\to\infty} w_n = \infty$, because of the monotonic nature of w_n. Consequently, $w_n > 0$, and $\Delta w_n > 0$ for $n \geqslant n_2 \geqslant n_1$.

Next we show that y_n is bounded below by a positive constant, which will be used for bounding the G term from below. Using that w_n is positive and increasing, and that $\tau(n) \leqslant n$, we have for sufficiently large n: for the case $0 \leqslant p_n \leqslant b_1 < 1$,

$$\begin{aligned} w_n &\leqslant w_n + p_n w_{\tau(n)} \\ &= y_n + T_n - F_n + p_n\big[-p_{\tau(n)} y_{\tau(\tau(n))} + T_{\tau(n)} - F_{\tau(n)}\big], \end{aligned}$$

and for the case $-1 < -b_1 \leqslant p_n \leqslant 0$,

$$\begin{aligned} (1 - b_1) w_n &\leqslant w_n - b_1 w_{\tau(n)} \\ &\leqslant w_n + p_n w_{\tau(n)} \\ &= y_n + T_n - F_n + p_n\big[-p_{\tau(n)} y_{\tau(\tau(n))} + T_{\tau(n)} - F_{\tau(n)}\big]. \end{aligned}$$

We may note that p_n and $p_{\tau(n)}$ have the same sign in each of the two inequalities above, and $1 - b_1 \leqslant 1$ which implies

$$(1 - b_1) w_n \leqslant y_n + \varepsilon + \gamma + b_1\varepsilon + b_1\gamma, \quad \text{for } n \geqslant n_1.$$

As $\lim_{n\to\infty} w_n = \infty$, it follows that $\lim_{n\to\infty} y_n = \infty$. Then there exists $n_2 \geqslant n_1$ such that for $n \geqslant n_2$: $y_n, y_{\sigma(n)}$ are bounded below by a positive constant. By (H_0), (H_1), for $i \geqslant n_2$, $G(y_{\sigma(i)})$ is bounded below by a positive constant α. Summing (2.17) from $n = n_2$ to $n = k - 1$, we obtain

$$\Delta w_k = \Delta w_{n_2} - \sum_{n=n_2}^{k-1} q_n^+ G(y_{\sigma(n)}) \leqslant \Delta w_{n_2} - \alpha \sum_{n=n_2}^{k-1} q_n^+.$$

Note that by (H_3), the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the non-oscillatory positive solution y_n of (1.2) is bounded.

if y_n is an eventually negative solution of (1.2) for large n then we set $x_n = -y_n$ to obtain $x_n > 0$ and then (1.2) reduces to

$$\Delta^2\big(x_n - p_n x_{\tau(n)}\big) + q_n \tilde{G}(x_{\sigma(n)}) = \tilde{f}_n,$$

where

$$\tilde{f}_n = -f_n, \quad \text{and} \quad \tilde{G}(v) = -G(-v).$$

Further,

$$\tilde{F}_n = -F_n \quad \text{implies} \quad \Delta^2(\tilde{F}_n) = \tilde{f}_n.$$

In view of the above facts, it can be easily verified that $\tilde{G}$ and $\tilde{F}$ satisfy the corresponding conditions satisfied by the functions G and F in the theorem. Proceeding as in the proof for the case $y_n > 0$, we may complete the proof of the theorem.

The following result follows immediately from the above theorem.

COROLLARY 2.5. *Suppose that the hypotheses* (H_0) - (H_5) *and* (2.11) *hold. Then every unbounded solution of* (1.2)*, (if exists) is oscillatory.*

Also note that by setting $p_n = 0$, Theorems 2.4 can be applied to the equation

$$\Delta^2(y_n) + q_n G(y_{\sigma(n)}) = f_n,$$

with oscillating q_n.

2.1. Results for bounded solutions

In this subsection, we study the behaviour of bounded solutions of (1.2) and we do not require the assumption (H_2). However, we need a condition

$$\sum_{n=n_0}^{\infty} n q_n^+ = \infty, \tag{2.19}$$

which is less restrictive than (H_3).

THEOREM 2.6. *Assume that* (H_0), (H_1), (H_4), (H_6) *and* (2.19) *hold. Then every bounded solution of* (1.2) *is oscillatory or tends to zero as* $n \to \infty$*, for each one of the following cases:*

$$0 \leqslant p_n \leqslant b_1 < 1, \ \forall n \in \mathbb{N}, \tag{2.20}$$

$$-1 < -b_1 \leqslant p_n \leqslant 0, \ \forall n \in \mathbb{N}, \tag{2.21}$$

$$b_2 \leqslant p_n \leqslant b_3 < -1, \ \forall n \in \mathbb{N}, \tag{2.22}$$

$$1 < b_4 \leqslant p_n \leqslant b_5, \ \forall n \in \mathbb{N}, \tag{2.23}$$

where b_1, b_2, b_3, b_4, b_5 *are constants.*

Proof. Let $y = y_n$ be a bounded solution of (1.2) for $n \geqslant N_1$. If it oscillates then there is nothing to prove. If it does not oscillate then $y_n > 0$ or $y_n < 0$ eventually. Suppose $y_n > 0$ for large n. There exits positive integer $n_0 \geqslant N_1 > 0$ such that $y_n > 0, y_{\tau(n)} > 0$, and $y_{\sigma(n)} > 0$ for $n \geqslant n_0$. Set z_n, T_n and w_n as in (2.12), (2.13) and (2.16) respectively, to obtain (2.17). T_n is well defined due to the boundedness of y_n

and note that it satisfies (2.14). Then w_n, and Δw_n are monotonic and single sign for $n \geqslant n_1 \geqslant n_0$. Boundedness of y_n implies that of z_n and w_n. Using (2.14), (H_6) and monotonic nature of w_n, we obtain $\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \lambda$, which exists finitely. Then applying Lemma 2.3 to (2.17) for $p = 2$ and $p_0 = 0$ we obtain for $n \geqslant n_1$,

$$w_n = \lambda - \sum_{i=n}^{\infty}(i-n+1)q_i^+ G(y_{\sigma(i)}). \tag{2.24}$$

As $\lim_{n\to\infty} w_n$ exists, from (2.24) it follows that

$$\sum_{i=n}^{\infty}(i-n+1)q_i^+ G(y_{\sigma(i)}) < \infty, \quad n \geqslant n_1. \tag{2.25}$$

Using Lemma 2.2 in (2.25), we obtain

$$\sum_{i=n}^{\infty} i q_i^+ G(y_{\sigma(i)}) < \infty, \quad n \geqslant n_1. \tag{2.26}$$

From (2.26), it follows due to (2.19) that $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. When $\lim_{n\to\infty} \sigma(n) = \infty$, we have $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to (H_0) that $\liminf_{n\to\infty} y_n = 0$. Then using Lemma 2.1, we may obtain $\lim_{n\to\infty} z_n = 0$. If (2.20) holds then

$$\begin{aligned}
0 = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty} (y_n - p_n y_{\tau(n)}) \\
&\geqslant \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} (-p_n y_{\tau(n)}) \\
&\geqslant (1-b_1) \limsup_{n\to\infty} y_n.
\end{aligned}$$

This implies $\limsup_{n\to\infty} y_n = 0$ and consequently $y_n \to 0$ as $n \to \infty$. If (2.21) or (2.22) holds then, since $y_n \leqslant z_n$, it follows that $y_n \to 0$ as $n \to \infty$. If p_n satisfies (2.23), then $z_n \leqslant y_n - b_4 y_{\tau(n)}$, and it follows that

$$\begin{aligned}
0 = \liminf_{n\to\infty} z_n &\leqslant \liminf_{n\to\infty} [y_n - b_4 y_{\tau(n)}] \\
&\leqslant \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} [-b_4 y_{\tau(n)}] \\
&= (1-b_4) \limsup_{n\to\infty} y_n.
\end{aligned}$$

Then $\limsup_{n\to\infty} y_n = 0$, which implies $\lim_{n\to\infty} y_n = 0$.

If y_n is eventually negative for large n, then we may proceed with $x_n = -y_n$.

2.2. Results for bounded or unbounded solutions

Clearly, the condition (H_3) implies (2.19), so we combine the results Corollary 2.5 and Theorem 2.6 to state the following result.

THEOREM 2.7. *Suppose that either* (2.20) *holds or* (2.21) *holds. Further assume* (H_0)-(H_6) *to hold. Then every solution of* (1.2) *is oscillatory or tends to zero as* $n \to \infty$.

For our results in this paper, we need G to be bounded, continuous, and to satisfy (H_0) and (H_1). The proto type of such a function $G(x)$ is $\frac{x^{2n}}{1+x^{2n}}\operatorname{sgn} x$.

To emphasize the need for the condition (H_4) or that of (H_2), for our results we present the following example.

EXAMPLE 1. Consider the equation

$$\Delta^2 y_n + q_n y_{n-m} = 0, \tag{2.27}$$

where m is any positive and even integer and

$$q_n = \begin{cases} -1, & n \text{ is even,} \\ 2, & n \text{ is odd.} \end{cases}$$

Then, q_n is oscillatory but, does not satisfy (H_4). Moreover, $G(u) = u$, does not satisfy (H_2). Note that (H_0), (H_1), (H_3), (H_5) and (H_6) hold, but we cannot apply Theorem 2.4, 2.6 or Theorem 2.7 to the equation (2.27). In fact,

$$y_n = \begin{cases} 1, & n \text{ is odd,} \\ 2, & n \text{ is even,} \end{cases}$$

is a solution of the above delay equation which neither oscillates nor tends to zero as $n \to \infty$.

We present some examples to illustrate, our main results.

EXAMPLE 2. Consider the delay equation

$$\Delta^2(y_n) + q_n y_{n-m} = (5/4)2^{-n}, \tag{2.28}$$

where m is any positive and odd integer and q_n is as given by

$$q_n = \begin{cases} (9/4)2^{-m}, & n \text{ is even,} \\ -2^{-n}, & n \text{ is odd.} \end{cases}$$

It is easily verified that equation(2.28) satisfies all the conditions of Theorem 2.6. Hence every solution (2.28) oscillates or tends to zero as $n \to \infty$. As a result one may find that this equation admits a nonnegative solution given by

$$y_n = \begin{cases} 2^{-n}, & n \text{ is odd,} \\ 0, & n \text{ is even,} \end{cases}$$

which tends to zero as $n \to \infty$.

EXAMPLE 3. Consider the delay equation

$$\Delta^2(y_n) + q_n G(y_{n-m}) = f_n, \tag{2.29}$$

where m is any positive and odd integer, $G(u) = \frac{u^2}{(1+u^2)}\operatorname{sgn} u$, q_n and f_n are as given below:

$$q_n = \begin{cases} (9/4)(1+2^{2n-2m})2^{-n}, & n \text{ is odd}, \\ -(9/4)2^{-n-2m}(1+2^{-2n+2m}), & n \text{ is even}. \end{cases} \tag{2.30}$$

$$f_n = \begin{cases} 0, & n \text{ is odd}, \\ (9/4)(8^{-n}+2^{-n}), & n \text{ is even}. \end{cases}$$

It is easily verified that eq. (2.29) satisfies all the conditions of Theorem 2.7. As such, every solution of (2.29) oscillates or tends to zero as $n \to \infty$ and in fact, this equation admits a oscillatory solution given by

$$y_n = \begin{cases} -2^{-n}, & n \text{ is odd}, \\ 2^{-n}, & n \text{ is even}, \end{cases}$$

which tends to zero as $n \to \infty$.

EXAMPLE 4. Consider the non-linear neutral equation (2.29) with q_n as in (2.30); $G(u)$ and m as defined in the above example. Define

$$f_n = \begin{cases} (13/4)2^{-n}, & n \text{ is odd}, \\ (-9/4)8^{-n} + (1/4)2^{-n}, & n \text{ is even}. \end{cases}$$

Then the non-linear neutral equation (2.29) satisfies all the conditions of Theorem 2.4. Hence every non-oscillatory solution of this difference equation is bounded as per the conclusion of the theorem. One may verify that $y_n = 2^{-n}$ is a non-oscillatory solution of (2.29), which is clearly bounded.

Remark

The authors in [12] assumed the conditions:

$$\liminf_{|u|\to\infty} G(u)/u > 0, \tag{2.31}$$

$$G \quad \text{is non-decreasing}, \tag{2.32}$$

and

$$H \quad \text{is bounded},$$

in order to study the oscillatory behaviour of solutions of the neutral equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + v_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n.$$

Since both the conditions (2.31) and (2.32) are incompatible to the condition (H_2), we removed these conditions and thus, could generalize, improve the work in [12] and apply it to study (1.2) with oscillating q_n. Further, we may add that no result in the cited papers in the reference can be applied to the equations (2.28) or (2.29) due to the reason that either the conditions on G are not satisfied or because q_n is not of constant sign.

Before we close this article, we would like to give our final comments, which might be helpful for further research.

Final Comments

Our results (see Theorems 2.4 and 2.7) of this paper do not hold for $G(u) \equiv u$ because of our assumption (H_2). Hence, it would be interesting to study the oscillation of solutions of (1.2) either by relaxing the condition (H_2) or by considering the corresponding linear equation.

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