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SOLUTIONS FOR SINGULAR ELLIPTIC SYSTEMS INVOLVING HARDY–SOBOLEV CRITICAL NONLINEARITY

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Abstract. In this paper, we deal with a class of singular elliptic system with Hardy-Sobolev critical nonlinearity. The existence and multiplicity of solutions for this system are obtained by the variational methods and some analysis techniques.

1. Introduction and main results

Elliptic systems have extensive practical backgrounds. They can be used to describe the multiplicative chemical reaction catalyzed by the catalyst grains under constant or variate temperature, a correspondence of the stable station of dynamical system determined by the reaction-diffusion system. In recent years, much attention has been payed to the existence of nontrivial solutions for nonvariational systems, potential systems and hamiltonian systems, see, for instance, [1, 7, 9, 10, 12] and their references. In particular, some elliptic systems with critical exponents have been studied in [7, 9, 12] and the references therein.

In this paper, we consider the following elliptic systems,

$$\begin{cases} -\Delta u-\mu\frac{u}{|x|^2}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s}+\lambda\frac{\partial}{\partial u}F(x,u,v), & x\in\Omega\setminus\{0\},\\ -\Delta v-\mu\frac{v}{|x|^2}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}+\lambda\frac{\partial}{\partial v}F(x,u,v), & x\in\Omega\setminus\{0\},\\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^N (N\geqslant 3)$ with smooth boundary $\partial\Omega$ and $0\in\Omega$, $0\leqslant\mu<\overline{\mu}\triangleq((N-2)/2)^2$, $\lambda>0$, $\alpha,\ \beta>1$ satisfy $\alpha+\beta=2^*(s)=2(N-s)/(N-2)\,(0\leqslant s<2)$, which is the critical Hardy-Sobolev exponent and $2^*=2^*(0)=2N/(N-2)$ is the Sobolev critical exponent. F is a real function satisfying some assumptions.

We shall work with the space $(H_0^1)^2:=H_0^1(\Omega)\times H_0^1(\Omega)$ endowed with the norm

$$\|(u,v)\|_{(H_0^1)^2}=\left(\|u\|^2_{H_0^1(\Omega)}+\|v\|^2_{H_0^1(\Omega)}\right)^{\frac{1}{2}},$$

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where the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx\right)^{1/2},$$

which is equivalent to the usual norm of $H_0^1(\Omega)$. Denote

$$\tilde{A}_{\mu,s}(\Omega) = \inf_{(u,v)\in(H_0^1)^2\setminus\{0\}} \frac{\|(u,v)\|^2_{(H_0^1)^2}}{\left(\int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^s} dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{2}$$

Modifying the proof of Theorem 5 in [2], we can easily deduce that

$$\tilde{A}_{\mu,s}(\Omega) = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right] A_{\mu,s}(\Omega), \tag{3}$$

where

$$A_{\mu,s}(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2_{H_0^1(\Omega)}}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}}}.$$

From Lemma 2.2 in [8], we know that $A_{\mu,s}(\Omega)$ is attained when $\Omega = \mathbb{R}^N$ by the functions

$$y_\varepsilon(x) = \frac{\left[\frac{2\varepsilon(N-s)(\overline{\mu}-\mu)}{\sqrt{\overline{\mu}}}\right]^{\frac{\sqrt{\overline{\mu}}}{2-s}}}{|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}\left(\varepsilon + |x|^{\frac{(2-s)\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}}}\right)^{\frac{N-2}{2-s}}}$$

for all $\varepsilon > 0$ and $A_{\mu,s}(\Omega)$ is independent of Ω, so we denote $A_{\mu,s}$ instead of $A_{\mu,s}(\Omega)$. The statement (3) implies that the constant $\tilde{A}_{\mu,s}(\Omega)$ is achieved and independent of Ω when $\alpha+\beta = 2^*(s)$, so we denote $\tilde{A}_{\mu,s}$ instead of $\tilde{A}_{\mu,s}(\Omega)$.

In recent years, the existence of solutions of the problem (1) with $\mu = 0$ and $s = 0$ has been payed much attention. Alves, Filho and Souto in [2] proved the existence of least energy solutions for any $\lambda \in (0,\lambda_1)$ and generalized the corresponding results [3] with $\mu = s = 0$, $\frac{\partial}{\partial u}F(x,u,v) = u$ and $\frac{\partial}{\partial v}F(x,u,v) = v$. Subsequently, in this case, Han in [5, 6] studied the existence of multiple positive solutions for the problem (1). The existence of a positive solution for the problem (1) is studied by Liu and Han in [9] with $s = 0$, $\frac{\partial}{\partial u}F(x,u,v) = u$ and $\frac{\partial}{\partial v}F(x,u,v) = v$ for $\lambda \in (0,\lambda_1)$ and $\mu \in (0,\overline{\mu}-1)$.

However, as far as we know, there are few results on the problem (1) with Hardy terms, critical Hardy-Sobolev exponents and general form F. Due to the lack of compactness of embedding of $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2}dx)$ and $H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |x|^{-s}dx)$, we can not use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale ((PS) in short) condition in $H_0^1(\Omega)$. However, we use argument of Brezis and Nirenberg [3] to verify that the associated functional satisfies the Palais-Smale condition on a given interval of the real line. Then the existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to Rabinowitz [11].

Here are the main results of this paper.

THEOREM 1. *Suppose that* $N \geqslant 3$, $0 \leqslant \mu < \overline{\mu}$, $0 \leqslant s < 2$ *and* F *satisfies:*

$(F1)$ $F \in C^1(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ *and* $F(x,0,0) = \frac{\partial F(x,0,t)}{\partial u} = \frac{\partial F(x,z,0)}{\partial v} = 0$;

$(F2)$ *there exist* $1 < p_i < p_0$ $($*here* $p_0 \in (2,2^*])$, $i = 1,2$, $R_0 > 0$ *and* $T > 0$ *such that*

$$z\frac{\partial}{\partial u}F(x,z,t) + t\frac{\partial}{\partial v}F(x,z,t) \leqslant T\left(z^{p_1} + t^{p_2}\right),\ \textit{if}\ z+t \geqslant R_0$$

for all $(z,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ *and for almost every* $x \in \overline{\Omega}$;

$(F3)$ *there exist* $\theta_i \in (\frac{1}{2^*(s)}, \frac{1}{2})(i = 1,2)$ *such that*

$$0 < F(x,z,t) \leqslant \theta_1 z\frac{\partial}{\partial u}F(x,z,t) + \theta_2 t\frac{\partial}{\partial v}F(x,z,t),\ (z,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus (0,0),\ x \in \overline{\Omega};$$

$(F4)$ *let* $b_0 := \inf_{|(z,t)|=1} F(x,z,t) > 0$, $(z,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus (0,0)$, $x \in \overline{\Omega}$.

Assume that

$$\eta \stackrel{\triangle}{=} \frac{1}{\max\{\theta_1,\theta_2\}} > max\left\{2,\ \frac{N}{\sqrt{\overline{\mu}} + \sqrt{\overline{\mu}-\mu}},\ \frac{N - 2\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}}\right\} \stackrel{\triangle}{=} r_0. \tag{4}$$

Then there exists $\lambda^* > 0$ *such that the problem* (1) *possesses one positive solution for every* $\lambda \in (0,\lambda^*)$.

COROLLARY 1. *Suppose that* $N \geqslant 4$, $0 \leqslant \mu \leqslant \overline{\mu} - 1$ *and* $0 \leqslant s < 2$. *Assume that* $(F1)$-$(F4)$ *hold. Then the problem* (1) *has at least a positive solution for every* $\lambda \in (0,\lambda^*)$.

THEOREM 2. *Suppose that* $N \geqslant 3$, $0 \leqslant \mu < \overline{\mu}$, $0 \leqslant s < 2$ *and* F *satisfies:*

$(F1')$ $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *and* $F(x,0,0) = \frac{\partial F(x,0,v)}{\partial u} = \frac{\partial F(x,u,0)}{\partial v} = 0$;

$(F2')$ *there exist* $1 < p_i < p_0$ $($*here* $p_0 \in (2,2^*])$, $i = 1,2$, $R_0 > 0$ *and* $T > 0$ *such that*

$$\left|z\frac{\partial}{\partial u}F(x,z,t) + t\frac{\partial}{\partial v}F(x,z,t)\right| \leqslant T\left(|z|^{p_1} + |t|^{p_2}\right),\ \textit{if}\ |z|+|t| \geqslant R_0$$

for all $(z,t) \in \mathbb{R}^2$ *and for almost every* $x \in \overline{\Omega}$*;*

$(F3')$ *there exist* $\theta_i \in (\frac{1}{2^*(s)}, \frac{1}{2})(i = 1,2)$ *such that*

$$0 < F(x,z,t) \leqslant \theta_1 z\frac{\partial}{\partial u}F(x,z,t) + \theta_2 t\frac{\partial}{\partial v}F(x,z,t),\ (z,t) \in R^2 \setminus (0,0),\ x \in \overline{\Omega};$$

$(F4')$ *let* $b_0 := \inf_{|(z,t)|=1} F(x,z,t) > 0$, $(z,t) \in \mathbb{R}^2 \setminus (0,0)$, $x \in \overline{\Omega}$.

Assume that (4) *holds. Then the problem* (1) *possesses two distinct nontrivial solutions for every* $\lambda \in (0,\lambda^*)$.

COROLLARY 2. *Suppose that* $N \geqslant 4,\ 0 \leqslant \mu \leqslant \overline{\mu}-1$ *and* $0 \leqslant s < 2$. *Assume that* $(F1')$-$(F4')$ *hold. Then the problem* (1) *has at least two distinct nontrivial solutions for every* $\lambda \in (0,\lambda^*)$.

REMARK 1. Theorems 1, 2 are supplements to Theorem 1.3 in [9]. The case of $s \neq 0$(the critical Hardy-Sobolev exponents) and general nonlinearity perturbation which is suplinear at zero is not considered in [9], where the authors only studied the case of $s=0$(the Sobolev exponent) and the perturbation of the linear at zero.

In the sequel, we shall give the proof of theorems. $|\Omega|$ and $C_i (i=1,2,3,...)$ will denote the measure of Ω and various positive constants, respectively.

2. Proofs of theorems

It is obvious that the values of $F(x,z,t)$ for z or $t<0$ are irrelevant in our theorems and we may define

$$F(x,z,t)=0 \ \text{ for } x\in\Omega,\ \ z\leqslant 0 \text{ or } t\leqslant 0.$$

Let $u^{\pm}=\max\{\pm u,0\}$. The energy functional corresponding to the problem (1) is defined on $(H_0^1)^2$ by

$$J((u,v))=\frac{1}{2}\int_\Omega\left(|\nabla u|^2+|\nabla v|^2-\mu\frac{|u|^2}{|x|^2}-\mu\frac{|v|^2}{|x|^2}\right)dx-\lambda\int_\Omega F(x,u^+,v^+)dx \\ -\frac{2}{\alpha+\beta}\int_\Omega\frac{(u^+)^\alpha(v^+)^\beta}{|x|^s}dx. \tag{5}$$

According to the Hardy, Hardy-Sobolev inequalities, $J\in C^1((H_0^1)^2,\mathbb{R})$. Now it is well known that there exists a one to one correspondence between the nonegative solutions of the problem (1) and the critical points of J on $(H_0^1)^2$. More precisely we say that $(u,v)\in(H_0^1)^2$ is a weak solution of the problem (1), if for any $(\varphi_1,\varphi_2)\in(H_0^1)^2$, there holds

$$\langle J'((u,v)),(\varphi_1,\varphi_2)\rangle=\int_\Omega\Big[\nabla u\nabla\varphi_1+\nabla v\nabla\varphi_2-\mu\frac{u\varphi_1+v\varphi_2}{|x|^2} \\ -\lambda\frac{\partial}{\partial u}F(x,u^+,v^+)\varphi_1-\lambda\frac{\partial}{\partial v}F(x,u^+,v^+)\varphi_2\Big]dx \\ -\frac{2\alpha}{\alpha+\beta}\int_\Omega\frac{(u^+)^{\alpha-1}(v^+)^\beta}{|x|^s}\varphi_1dx-\frac{2\beta}{\alpha+\beta}\int_\Omega\frac{(u^+)^\alpha(v^+)^{\beta-1}}{|x|^s}\varphi_2dx=0. \tag{6}$$

LEMMA 2.1. *Suppose that* $N\geqslant 3,\ 0\leqslant\mu<\overline{\mu},\ 0\leqslant s<2$ *and* $\lambda>0$. *Assume that* $(F1)$-$(F3)$ *and* (4) *hold. Then* J *satisfies* $(PS)_c$ *condition with*

$$c<\overline{c}\triangleq\frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}(\Omega)}{2}\right)^{\frac{N-s}{2-s}}.$$

Proof. Suppose that $\{(u_j,v_j)\} \subset (H_0^1)^2$ satisfies

$$J((u_j,v_j)) \to c < \overline{c} \text{ and } J'((u_j,v_j)) \to 0 \text{ as } j \to \infty.$$

Together with (5), (6) and $(F3)$, we get as $j \to \infty$ the following:

$$\begin{aligned}
c{+}1 &+ o(1)\|u_j\|_{H_0^1(\Omega)} + o(1)\|v_j\|_{H_0^1(\Omega)} \\
&\geqslant J((u_j,v_j)) - \langle J'((u_j,v_j)),(\theta_1 u_j, \theta_2 v_j)\rangle \\
&= \left(\frac{1}{2}-\theta_1\right)\|u_j\|^2_{H_0^1(\Omega)} + \left(\frac{1}{2}-\theta_2\right)\|v_j\|^2_{H_0^1(\Omega)} \\
&\quad + \lambda \int_\Omega \left(\theta_1 u_j^+ \frac{\partial F}{\partial u}(x,u_j^+,v_j^+) + \theta_2 v_j^+ \frac{\partial F}{\partial v}(x,u_j^+,v_j^+) - F(x,u_j^+,v_j^+)\right) dx \\
&\quad + \frac{2(\alpha\theta_1+\beta\theta_2-1)}{\alpha+\beta}\int_\Omega \frac{(u_j^+)^\alpha (v_j^+)^\beta}{|x|^s} dx \\
&\geqslant \left(\frac{1}{2}-\theta_1\right)\|u_j\|^2_{H_0^1(\Omega)} + \left(\frac{1}{2}-\theta_2\right)\|v_j\|^2_{H_0^1(\Omega)} \\
&\geqslant \min\left\{\frac{1}{2}-\theta_1, \frac{1}{2}-\theta_2\right\}\|(u_j,v_j)\|^2_{(H_0^1)^2},
\end{aligned}$$

which implies $\|(u_j,v_j)\|$ is bounded in $(H_0^1)^2$. Going if necessary to a subsequence, we can assume that

$$\begin{cases}
(u_j,v_j) \rightharpoonup (u,v) \text{ weakly in } (H_0^1)^2, \\
u_j \to u, \text{ in } L^\gamma(\Omega),\ 1<\gamma<2^*(s), \\
v_j \to v, \text{ in } L^\gamma(\Omega),\ 1<\gamma<2^*(s), \\
(u_j,v_j) \to (u,v) \text{ a.e. in } \Omega
\end{cases}$$

as $j \to \infty$. By $(F1)$ and $(F2)$, there exists a positive constant $M > 0$ such that

$$F(x,u_j^+,v_j^+) \leqslant \frac{T}{2}\left((u_j^+)^{p_1} + (v_j^+)^{p_2}\right) + M. \tag{7}$$

According to the absolutely continuity of integral, for any $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{2M} > 0$, when $E \subset \Omega$, $\text{mes}(E) < \delta$, we have

$$\int_E \left((u_j^+)^{p_1} + (v_j^+)^{p_2}\right) dx < \frac{\varepsilon}{T}.$$

Together with (7), we deduce that

$$\begin{aligned}
\int_E F(x,u_j^+,v_j^+)dx &\leqslant \frac{T}{2}\int_E \left((u_j^+)^{p_1} + (v_j^+)^{p_2}\right) dx + M\text{mes}(E) \\
&\leqslant \frac{T}{2}\frac{\varepsilon}{T} + M\delta = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $\left\{\int_\Omega F(x,u_j^+,v_j^+)dx, j \in N\right\}$ is equi-absolutely-continuous. It follows easily from the Vitali Convergence Theorem, we deduce that

$$\int_\Omega F(x,u_j^+,v_j^+)dx \to \int_\Omega F(x,u^+,v^+)dx.$$

By the same method, we have

$$\int_\Omega \frac{\partial F(x,u_j^+,v_j^+)}{\partial u} u_j^+ dx \to \int_\Omega \frac{\partial F(x,u^+,v^+)}{\partial u} u^+ dx,$$

$$\int_\Omega \frac{\partial F(x,u_j^+,v_j^+)}{\partial v} v_j^+ dx \to \int_\Omega \frac{\partial F(x,u^+,v^+)}{\partial v} v^+ dx,$$

as $j \to \infty$.

Let $\tilde{u}_j = u_j - u$, $\tilde{v}_j = v_j - v$. Then, we have

$$\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} = \|(u_j,v_j)\|^2_{(H_0^1)^2} - \|(u,v)\|^2_{(H_0^1)^2} + o(1).$$

Using the similar method of Lemma 2.1 in [6], one gets

$$\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s} dx = \int_\Omega \frac{(u_j^+)^\alpha(v_j^+)^\beta}{|x|^s} dx - \int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^s} dx + o(1).$$

Since

$$\begin{aligned} o(1) &= \langle J'((u_j,v_j)),(u_j,v_j)\rangle \\ &= \|(u_j,v_j)\|^2_{(H_0^1)^2} - 2\int_\Omega \frac{(u_j^+)^\alpha(v_j^+)^\beta}{|x|^s} dx \\ &\quad -\lambda \int_\Omega \left(u_j^+ \frac{\partial}{\partial u} F(x,u_j^+,v_j^+) + v_j^+ \frac{\partial}{\partial v} F(x,u_j^+,v_j^+) \right) dx, \end{aligned}$$

we deduce

$$\begin{aligned} &\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} + \|(u,v)\|^2_{(H_0^1)^2} \\ &\quad - 2\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s} dx - 2\int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^s} dx \\ &\quad - \lambda \int_\Omega \left(u^+ \frac{\partial}{\partial u} F(x,u^+,v^+) + v^+ \frac{\partial}{\partial v} F(x,u^+,v^+) \right) dx = o(1). \end{aligned} \tag{8}$$

Furthermore, we have

$$\begin{aligned} &\lim_{j\to\infty} \langle J'(u_j,v_j),(u,v)\rangle \\ &\quad = \|(u,v)\|^2_{(H_0^1)^2} - 2\int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^s} dx \\ &\qquad - \lambda \int_\Omega \left(u^+ \frac{\partial}{\partial u} F(x,u^+,v^+) + v^+ \frac{\partial}{\partial v} F(x,u^+,v^+) \right) dx = 0. \end{aligned} \tag{9}$$

It yields

$$\begin{aligned} J((u,v)) &= \left(1 - \frac{2}{\alpha+\beta}\right) \int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^s} dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{2} \left(u^+ \frac{\partial}{\partial u} F(x,u^+,v^+) + v^+ \frac{\partial}{\partial v} F(x,u^+,v^+) \right) - F(x,u^+,v^+) \right] dx. \end{aligned}$$

Together with $(F3)$, we conclude that

$$J((u,v)) \geqslant 0. \tag{10}$$

Since $J((u,v)) \to c\ (j \to \infty)$, we obtain

$$\begin{aligned} J((u_j,v_j)) &= \frac{1}{2}\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} + \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}dx \\ &\quad - \frac{2}{\alpha+\beta}\int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^s}dx - \lambda\int_\Omega F(x,u^+,v^+)dx + o(1) \\ &= J((u,v)) + \frac{1}{2}\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}dx + o(1) \\ &= c + o(1). \end{aligned}$$

Therefore, one gets

$$J((u,v)) + \frac{1}{2}\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}dx = c + o(1). \tag{11}$$

From (8) and (9), we have

$$\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} - 2\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}dx = o(1),$$

then $\|(\tilde{u}_j,\tilde{v}_j)\|^2 \to 0$ as $j \to \infty$. Otherwise, there exists a subsequence (still denoted by $(\tilde{u}_j,\tilde{v}_j)$) such that

$$\lim_{j\to\infty}\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} = k, \quad \lim_{j\to\infty} 2\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}dx = k, \tag{12}$$

where k is a positive constant. By (2), we deduce that

$$\|(\tilde{u}_j,\tilde{v}_j)\|^2_{(H_0^1)^2} \geqslant \tilde{A}_{\mu,s}\left(\int_\Omega \frac{(\tilde{u}_j^+)^\alpha(\tilde{v}_j^+)^\beta}{|x|^s}\right)^{\frac{2}{\alpha+\beta}} \quad \text{for all} \ \ j \in N,$$

then $k \geqslant \tilde{A}_{\mu,s}(\frac{k}{2})^{\frac{2}{2^*(s)}}$, i.e., $k \geqslant 2(\frac{\tilde{A}_{\mu,s}}{2})^{\frac{N-s}{2-s}}$, which, together with (11) (12), shows that

$$J((u,v)) = c - \frac{1}{2}k + \frac{1}{2^*(s)}k \leqslant c - \frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}}{2}\right)^{\frac{N-s}{2-s}} < 0,$$

which contradicts (10). Therefore, we get

$$\|(\tilde{u}_j,\tilde{v}_j)\|^2 \to 0 \text{ as } j \to \infty.$$

This proves $(u_j,v_j) \to (u,v)$ in $(H_0^1)^2$ as $j \to \infty$.

From the discussion above, J satisfies $(PS)_c$ condition. □

Let

$$C_\varepsilon = \left(\frac{2\varepsilon(N-s)(\overline{\mu}-\mu)}{\sqrt{\overline{\mu}}}\right)^{\frac{N-2}{2(2-s)}} \quad \text{and} \quad U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}.$$

Define a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leqslant r$, $\varphi(x) = 0$ for $|x| \geqslant 2r$, $0 \leqslant \varphi(x) \leqslant 1$, where $B_{2r}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$ and

$$v_\varepsilon(x) = u_\varepsilon(x) / \left(\int_\Omega |u_\varepsilon|^{2^*(s)}|x|^{-s}dx\right)^{1/2^*(s)},$$

so that $\int_\Omega |v_\varepsilon|^{2^*(s)}|x|^{-s}dx = 1$. Then we can get the following results by the methods used in [4]:

$$A_{\mu,s} + C_1\varepsilon^{\frac{N-2}{2-s}} \leqslant \|v_\varepsilon\|^2_{H_0^1(\Omega)} \leqslant A_{\mu,s} + C_2\varepsilon^{\frac{N-2}{2-s}}, \tag{13}$$

and

$$\begin{cases} C_3\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q} \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}, & 1 \leqslant q < \frac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \\ C_3\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}|\ln\varepsilon| \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2-s}q}|\ln\varepsilon|, & q = \frac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \\ C_3\varepsilon^{\frac{\sqrt{\overline{\mu}}(N-q\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}} \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_4\varepsilon^{\frac{\sqrt{\overline{\mu}}(N-q\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}}, & \frac{N}{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}} < q < 2^*. \end{cases} \tag{14}$$

LEMMA 2.2. *Suppose that $N \geqslant 3$, $0 \leqslant \mu < \overline{\mu}$, $0 \leqslant s < 2$. Assume that $(F1)$-$(F4)$ hold. Then there exist $(u_0, v_0) \in (H_0^1)^2$, $(u_0, v_0) \not\equiv 0$ and $\lambda_1^* > 0$ such that*

$$\sup_{t \geqslant 0} J((tu_0, tv_0)) < \frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}}{2}\right)^{\frac{N-s}{2-s}},$$

for every $\lambda \in (0, \lambda_1^*)$.

Proof. Let $u = \sqrt{\alpha}v_\varepsilon$, $v = \sqrt{\beta}v_\varepsilon$, then we have

$$\begin{aligned} h(t) &:= J((tu,tv)) = J((t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)) \\ &= \frac{t^2}{2}(\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)} - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}} - \lambda\int_\Omega F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)dx. \end{aligned}$$

Let

$$\tilde{h}(t) := \frac{t^2}{2}(\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)} - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}.$$

Note that $\lim\limits_{t\to+\infty} h(t)=-\infty$, $h(0)=0$, $h(t)>0$ for $t\to 0^+$, so $\sup\limits_{t\geqslant 0} h(t)$ is attained for some $t_\varepsilon>0$. Since $(F3)$ and

$$\begin{aligned}
0 =& h'(t_\varepsilon)=t_\varepsilon(\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)}\\
&-\lambda\int_\Omega\left(\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon+\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\right)dx\\
&-2t_\varepsilon^{\alpha+\beta-1}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}},
\end{aligned}$$

we have

$$\begin{aligned}
\|v_\varepsilon\|^2_{H_0^1(\Omega)}=&\frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2}\\
&+\frac{\lambda}{t_\varepsilon(\alpha+\beta)}\int_\Omega\Big(\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon\\
&+\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\Big)dx\geqslant\frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2}.
\end{aligned}$$

Therefore, one has

$$t_\varepsilon\leqslant\left[\frac{(\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)}}{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}\right]^{\frac{1}{\alpha+\beta-2}}\triangleq t_\varepsilon^0. \tag{15}$$

By (13) and (14), we get

$$\|v_\varepsilon\|^2_{H_0^1(\Omega)}\to A_{\mu,s},\ \int_\Omega v_\varepsilon^{p_1}dx\to 0 \text{ and } \int_\Omega v_\varepsilon^{p_2}dx\to 0 \tag{16}$$

as $\varepsilon\to 0$. From $(F1)$ and $(F2)$, we deduce that

$$\begin{aligned}
&\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon+\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\\
&\qquad\leqslant T\left(t_\varepsilon^{p_1-1}\alpha^{\frac{p_1}{2}}v_\varepsilon^{p_1}+t_\varepsilon^{p_2-1}\alpha^{\frac{p_2}{2}}v_\varepsilon^{p_2}\right)+C_5t_\varepsilon
\end{aligned}$$

for some constant $C_5 > 0$. According to (15), (16) and the Hölder inequality, we obtain

$$\begin{aligned}\|v_\varepsilon\|^2_{H_0^1(\Omega)} &= \frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2} \\ &\quad + \frac{\lambda}{t_\varepsilon(\alpha+\beta)}\int_\Omega\Big(\frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon \\ &\qquad + \frac{\partial F(x,t_\varepsilon\sqrt{\alpha}v_\varepsilon,t_\varepsilon\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\Big)dx \\ &\leqslant \frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2} + \frac{\lambda}{(\alpha+\beta)}T\Big((t_\varepsilon^0)^{p_1-2}\alpha^{\frac{p_1}{2}}\int_\Omega v_\varepsilon^{p_1}dx \\ &\qquad + (t_\varepsilon^0)^{p_2-2}\beta^{\frac{p_2}{2}}\int_\Omega v_\varepsilon^{p_2}dx\Big) + \frac{\lambda C_5|\Omega|}{(\alpha+\beta)} \\ &= \frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2} + \frac{\lambda}{(\alpha+\beta)}(T+C_5|\Omega|) + o(1)\end{aligned}$$

as $\varepsilon \to 0$. So there exists $\lambda_1^* = \frac{\alpha+\beta}{2(T+C_5|\Omega|)}A_{\mu,s} > 0$ such that

$$t_\varepsilon \geqslant \left(\frac{A_{\mu,s}}{2}\frac{\alpha+\beta}{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}\right)^{\frac{1}{\alpha+\beta-2}} \triangleq T_0 \tag{17}$$

for every $\lambda \in (0,\lambda_1^*)$.

On the first hand, from (13), we get

$$\|v_\varepsilon\|_{H_0^1(\Omega)}^{\frac{2(N-s)}{2-s}} \leqslant A_{\mu,s}^{\frac{N-s}{2-s}} + C_6\varepsilon^{\frac{N-2}{2-s}}. \tag{18}$$

Furthermore, from $(F3)$, we get

$$\begin{aligned}F(x,u,v) &\leqslant \theta_1 u\frac{\partial}{\partial u}F(x,u,v) + \theta_2 v\frac{\partial}{\partial v}F(x,u,v) \\ &\leqslant \max\{\theta_1,\theta_2\}\langle\nabla F(x,u,v),(u,v)\rangle \\ &= \frac{1}{\eta}\langle\nabla F(x,u,v),(u,v)\rangle.\end{aligned} \tag{19}$$

Consider the function $h : [1,\infty) \longrightarrow R$ defined by

$$h(t) = F(x,t^{-1}u,t^{-1}v)t^\eta,$$

clearly, this function is nonincreasing by (19). Thus for any $|(u,v)| \geqslant 1$, we have $h(1) \geqslant h(|(u,v)|)$. Together with $(F4)$, it yields

$$\begin{aligned}F(x,u,v) &\geqslant F(x,(u,v)/|(u,v)|)|(u,v)|^\eta \\ &\geqslant \inf_{|(u,v)|=1} F(x,u,v)|(u,v)|^\eta = b_0|(u,v)|^\eta.\end{aligned} \tag{20}$$

If $|(u,v)| \leqslant 1$, by the continuity of F, one has,

$$F(x,u,v) \geqslant b_0|(u,v)|^{\eta} - C_7,$$

where $C_7 \geqslant \max\{0, b_0 - \min_{|(u,v)|\leqslant 1} F(x,u,v)\}$. Together with (20), we deduce that

$$F(x,u,v) \geqslant b_0|(u,v)|^{\eta} - C_7 \tag{21}$$

for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

On the other hand, the function $\widetilde{h}(t)$ attains it's maximum at t_ε^0 and is increasing in the interval $[0,t_\varepsilon^0]$, together with (14), (17), (18) and (21), we deduce that

$$\begin{aligned}
h(t_\varepsilon) &\leqslant \widetilde{h}(t_\varepsilon^0) - \lambda \int_\Omega F(x, t_\varepsilon\sqrt{\alpha}v_\varepsilon, t_\varepsilon\sqrt{\beta}v_\varepsilon)dx \\
&\leqslant \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\left[\frac{(\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)}}{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}\right]^{\frac{2}{\alpha+\beta-2}} (\alpha+\beta)\|v_\varepsilon\|^2_{H_0^1(\Omega)} \\
&\quad - \lambda b_0(\alpha+\beta)^{\eta/2} t_\varepsilon^{\eta} \int_\Omega v_\varepsilon^{\eta} dx - \lambda C_7|\Omega| \\
&\leqslant 2\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\left[\frac{(\alpha+\beta)}{2\alpha^{\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}}\right]^{\frac{\alpha+\beta}{\alpha+\beta-2}} (\alpha+\beta)\|v_\varepsilon\|^{\frac{2(N-s)}{2-s}}_{H_0^1(\Omega)} - \lambda C_7|\Omega| \\
&\quad - \lambda b_0(\alpha+\beta)^{\eta/2} T_0^{\eta} C_3 \varepsilon^{\frac{\sqrt{\overline{\mu}}(N-\eta\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}} \\
&\leqslant \frac{2-s}{N-s}\left[\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right)\frac{A_{\mu,s}(\Omega)}{2}\right]^{\frac{N-s}{2-s}} + C_8\varepsilon^{\frac{N-2}{2-s}} \\
&\quad - C_9\varepsilon^{\frac{\sqrt{\overline{\mu}}(N-\eta\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}} - \lambda C_7|\Omega|,
\end{aligned} \tag{22}$$

where

$$C_8 = \frac{2-s}{N-s}\left[\frac{1}{2}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right)\right]^{\frac{N-s}{2-s}} C_6 \text{ and } C_9 = \lambda b_0(\alpha+\beta)^{\eta/2} T_0^{\eta} C_3.$$

By (4), we obtain that

$$\frac{N-2}{2-s} > \frac{\sqrt{\overline{\mu}}(N-\eta\sqrt{\overline{\mu}})}{(2-s)\sqrt{\overline{\mu}-\mu}}.$$

Choosing ε small enough, by (3) and (22), we have

$$\sup_{t\geqslant 0} J((tu,tv)) = h(t_\varepsilon) < \frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}(\Omega)}{2}\right)^{\frac{N-s}{2-s}}. \quad \square$$

Proof of Theorem 1. For any $\varepsilon > 0$, fix $\lambda_2^* \in (0,\varepsilon)$. If $\lambda \in (0,\lambda_2^*)$, from (2), $(F3)$, (7) and the continuity of embedding, for any $(u,v) \in (H_0^1)^2$, we have

$$\begin{aligned}
&J((u,v))\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\lambda\int_\Omega F(x,u^+,v^+)dx\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\lambda\int_\Omega\left(\theta_1\frac{\partial F(x,u^+,v^+)}{\partial u}u+\theta_2\frac{\partial F(x,u^+,v^+)}{\partial v}v\right)dx\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\frac{\lambda T}{2}\int_\Omega\left((u^+)^{p_1}+(v^+)^{p_2}\right)dx-\lambda M|\Omega|\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\frac{\lambda T}{2}C_{10}\left(\|u^+\|^{p_1}_{H_0^1(\Omega)}+\|v^+\|^{p_2}_{H_0^1(\Omega)}\right)-\lambda M|\Omega|\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\frac{\lambda_2^* T}{2}C_{10}\left(\|(u,v)\|^{p_1}_{(H_0^1)^2}+\|(u,v)\|^{p_2}_{(H_0^1)^2}\right)-\lambda_2^* M|\Omega|\\
&\geqslant \frac{1}{2}\|(u,v)\|^2_{(H_0^1)^2} - \frac{2}{\alpha+\beta}(\tilde{A}_{\mu,s}(\Omega))^{-\frac{2^*(s)}{2}}\|(u,v)\|^{2^*(s)}_{(H_0^1)^2}\\
&\quad -\frac{\varepsilon T}{2}C_{10}\left(\|(u,v)\|^{p_1}_{(H_0^1)^2}+\|(u,v)\|^{p_2}_{(H_0^1)^2}\right)-\varepsilon M|\Omega|.
\end{aligned}$$

As ε small enough, there exists $\beta' > 0$ such that $J((u,v)) \geqslant \beta'$ for all $J((u,v)) \in \partial B_\rho = \{(u,v) \in (H_0^1)^2, \|(u,v)\|_{(H_0^1)^2} = \rho\}$, where $\rho > 0$ small enough. Let $\lambda^* = \min\{\lambda_1^*,\lambda_2^*\}$. By Lemma 2.2, for $\lambda \in (0,\lambda^*)$, there exists $(u_0,v_0) \in (H_0^1)^2$, $(u_0,v_0) \not\equiv 0$, such that

$$\sup_{t\geqslant 0} J((tu_0,tv_0)) < \frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}(\Omega)}{2}\right)^{\frac{N-s}{2-s}}.$$

In addition, by the nonegativity of F, we get

$$\begin{aligned}
J((tu_0,tv_0)) &= \frac{1}{2}t^2\|(u_0,v_0)\|^2_{(H_0^1)^2} - \lambda\int_\Omega F(x,tu_0,tv_0)dx - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega\frac{(u_0^+)^\alpha(u_0^+)^\beta}{|x|^s}dx\\
&\leqslant \frac{1}{2}t^2\|(u_0,v_0)\|^2_{(H_0^1)^2} - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega\frac{(u_0^+)^\alpha(u_0^+)^\beta}{|x|^s}dx,
\end{aligned}$$

which implies that $\lim_{t\to+\infty} J((tu_0,tv_0)) \to -\infty$. Hence we can choose $t_0>0$ such that $\|(t_0u_0,t_0v_0)\|>\rho$ and $J((tu_0,tv_0))\leqslant 0$. Applying the Mountain Pass Lemma in [11], there is a sequence $\{(u_n,v_n)\}\subset (H_0^1)^2$ satisfying $J(u_n,v_n)\to c\geqslant \beta'$ and $J'(u_n,v_n)\to 0$, where

$$c=\inf_{\eta\in\tau}\max_{t\in[0,1]} J(\eta(t)),$$

$$\tau=\{\eta\in([0,1],(H_0^1)^2)|\eta(0)=(0,0),\eta(1)=(t_0u_0,t_0v_0)\}.$$

Note that

$$\begin{aligned}0<\beta'\leqslant c=\inf_{\eta\in\tau}\max_{t\in[0,1]} J(\eta(t))&\leqslant \max_{t\in[0,1]} J((tt_0u_0,tt_0v_0))\\ &\leqslant c\sup_{t\geqslant 0} J((tu_0,tv_0))<\frac{2-s}{N-s}\left(\frac{\tilde{A}_{\mu,s}(\Omega)}{2}\right)^{\frac{N-s}{2-s}}.\end{aligned}$$

Now Lemma 2.1 suggests $\{(u_n,v_n)\}\subset (H_0^1)^2$ has a convergent subsequence, still denoted by $\{(u_n,v_n)\}$. Assume that $\{(u_n,v_n)\}$ converges to $(u,v)\subset (H_0^1)^2$. From the continuity of J' we know that (u,v) is a solution of the problem (1). Then

$$\langle J'((u,v)),(u^-,v^-)\rangle=0,$$

where $u^-=\min\{u,0\}$ and $v^-=\min\{v,0\}$. It yields $\|(u^-,v^-)\|=0$ together with $(F1)$. So (u,v) is a nonnegative solution of the problem (1). Then $(u,v)>0$ in Ω by the Strong Maximum Principle. □

Proof of Theorem 2. First, let us consider the following truncated problem:

$$\begin{cases}-\Delta u-\mu\frac{u}{|x|^2}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s}+\lambda\frac{\partial F_1(x,u,v)}{\partial u}, & x\in\Omega\setminus\{0\},\\ -\Delta v-\mu\frac{v}{|x|^2}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}+\lambda\frac{\partial F_1(x,u,v)}{\partial v}, & x\in\Omega\setminus\{0\},\\ u=v=0, & x\in\partial\Omega,\end{cases}\tag{23}$$

where $F_1(x,z,t)=F(x,z,t)|_{(z,t)\geqslant 0}$. For this problem, it is easy to see that $F_1(x,z,t)$ satisfies the conditions of Theorem 1. Therefore, by Theorem 1, there exists $\lambda^*>0$ such that the problem (23) has a positive solution (u_1,v_1) for each $\lambda\in(0,\lambda^*)$ and it is also a positive solution of the problem (1) by the definition of $F_1(x,z,t)$. Next we consider the following truncated problem:

$$\begin{cases}-\Delta u-\mu\frac{u}{|x|^2}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s}+\lambda\frac{\partial F_2(x,u,v)}{\partial u}, & x\in\Omega\setminus\{0\},\\ -\Delta v-\mu\frac{v}{|x|^2}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}+\lambda\frac{\partial F_2(x,u,v)}{\partial v}, & x\in\Omega\setminus\{0\},\\ u=v=0, & x\in\partial\Omega,\end{cases}\tag{24}$$

where $F_2(x,z,t)=F(x,z,t)|_{(z,t)\leqslant 0}$. Set $G(x,u,v)=-F_2(x,-u,-v)$ for $(u,v)\in\mathbb{R}^2$. Then the problem (24) is equivalent to the following problem:

$$\begin{cases}-\Delta u-\mu\frac{u}{|x|^2}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s}+\lambda\frac{\partial G(x,u,v)}{\partial u}, & x\in\Omega\setminus\{0\},\\ -\Delta v-\mu\frac{v}{|x|^2}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}+\lambda\frac{\partial G(x,u,v)}{\partial v}, & x\in\Omega\setminus\{0\},\\ u=v=0, & x\in\partial\Omega,\end{cases}\tag{25}$$

it is easy to see that $G(x,z,t)$ satisfies the conditions of Theorem 1. Hence, there exists $\lambda'^{*} > 0$ such that the problem (25) has a positive solution (u,v) for each $\lambda \in (0,\lambda'^{*})$. Let $(u_2,v_2) = -(\underline{u},v)$, then (u_2,v_2) is a solution of (24) and it is also a solution of the problem (1). Set $\overline{\lambda} = \min\{\lambda^{*},\lambda'^{*}\}$. It is obvious that $(u_1,v_1) \neq (0,0),(u_2,v_2) \neq (0,0)$ and $(u_1,v_1) \neq (\underline{u_2},v_2)$. So the equation (1) has at least two distinct nontrivial solutions for every $\lambda \in (0,\overline{\lambda})$. Therefore, Theorem 2 holds. □

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