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A GENERAL FOURTH–ORDER PARABOLIC EQUATION

CHAO ZHANG AND SHULIN ZHOU

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Abstract. In this paper we establish the existence and uniqueness of weak solutions for the initial-boundary value problem of a general fourth-order parabolic equation. Our assumptions are much weaker than those in the literature.

1. Introduction

Suppose that Ω is a bounded domain of $\mathbb{R}^N (N \geqslant 2)$ with smooth boundary $\partial\Omega$, and T is a positive number. Denote $\Omega_T = \Omega \times (0,T]$, $\Gamma = \partial\Omega \times (0,T]$. In this paper we study the following fourth-order parabolic initial-boundary value problem

$$\begin{cases} u_t + \operatorname{div}\left(D_\xi \Phi(\nabla \Delta u)\right) = f - \operatorname{div} g & \text{in} \quad \Omega_T, \\ u = 0, \Delta u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{1.1}$$

where $\Phi : \mathbb{R}^N \mapsto \mathbb{R}_+$ is a C^1 nonnegative convex function, $D_\xi \Phi : \mathbb{R}^N \to \mathbb{R}$ represents the gradient of $\Phi(\xi)$ with respect to ξ. Without loss of generality we may assume that $\Phi(0) = 0$.

Our main assumptions are that $\Phi(\xi)$ satisfies the super-linear condition (or 1-coercive condition, see [11, Chapter E]),

$$\lim_{|\xi| \to \infty} \frac{\Phi(\xi)}{|\xi|} = \infty, \tag{1.2}$$

and the symmetric condition: there exists a positive number $C > 0$ such that

$$\Phi(-\xi) \leqslant C\Phi(\xi), \quad \xi \in \mathbb{R}^N. \tag{1.3}$$

We assume that

$$u_0 \in H_0^1(\Omega),\ f \in L^N(\Omega_T) \ \text{ and } \ g \in (L^\infty(\Omega_T))^N. \tag{1.4}$$

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There are many interesting papers related to problem (1.1), we refer to [17] and the reference therein. We list some well-known examples of $\Phi(\xi)$ satisfying structure assumptions (1.2) and (1.3):

(i)

$$\Phi(\xi)=\frac{1}{p}|\xi|^p, \quad p>1;$$

(ii)

$$\Phi(\xi)=\frac{1}{p_1}|\xi_1|^{p_1}+\frac{1}{p_2}|\xi_2|^{p_2}+\cdots+\frac{1}{p_N}|\xi_N|^{p_N}, \quad p_i>1,\ i=1,2,\ldots,N,$$

where $\xi=(\xi_1,\xi_2,\ldots,\xi_N)$, see [13, Chapter 2];

(iii)

$$\Phi(\xi)=|\xi|\log(1+|\xi|),$$

see [6] and [2, Chapter 4];

(iv)

$$\Phi(\xi)=|\xi|L_k(|\xi|),$$

where $L_i(s)=\log(1+L_{i-1}(s))\,(i=1,2,\ldots,k)$ and $L_0(s)=\log(1+s)$ for $s\geqslant 0$, see for instance [9];

(v)

$$\Phi(\xi)=e^{\frac{|\xi|^2}{2}}-1,$$

see for instance [7, 12].

In [17], Xu and Zhou have proved the existence and uniqueness of weak solutions of problem (1.1) with Neumann boundary conditions and $f=g=0$ under the main assumption that $\Phi(\xi)$ satisfies the following Δ_2 condition, i.e., there exist a number $K>2$ and a constant $R>0$ such that

$$\Phi(2\xi)\leqslant K\Phi(\xi), \quad |\xi|\geqslant R.$$

This condition implies that $\Phi(\xi)$ would be controlled by a polynomial of $|\xi|$. Therefore, Δ_2 condition is not satisfied by Example 5. This motivates us to weaken the Δ_2 condition. In this paper, we do not assume polynomial or exponential growth for function Φ. Generally speaking, finding solutions for such parabolic problems or deriving the Euler-Lagrangian equations for minimizers of variational problems is not a trivial fact when function $\Phi(\xi)$ does not satisfy the Δ_2 condition. To do this, we need to establish some new estimates under the weaker assumption on $\Phi(\xi)$. We will prove the existence and uniqueness of weak solutions of problem (1.1) under assumptions (1.2), (1.3) and (1.4) by methods of difference and calculus of variations, which have been used in [17, 16]. By the same technique we may obtain the same results for the similar problem with Neumann boundary conditions.

Let $1^*=N/(N-1)$. Now we define weak solutions of problem (1.1).

DEFINITION 1.1. A function $u : \overline{\Omega}\times[0,T]\to\mathbb{R}$ is said to be a weak solution of the problem (1.1) if the following conditions are satisfied:

(1) $u\in C([0,T];L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))\cap L^1(0,T;W^{2,1^*}(\Omega))$, $\Delta u\in L^1(0,T;W_0^{1,1}(\Omega))$ and

$$\int_0^T\int_\Omega D_\xi\Phi(\nabla\Delta u)\cdot\nabla\Delta u\,dxdt<+\infty;$$

(2) for any $\varphi\in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot,T)=0$ and $\varphi(\cdot,t)|_{\partial\Omega}=0$, equality

$$\begin{aligned}-\int_\Omega u_0(x)\varphi(x,0)\,dx-\int_0^T\int_\Omega\left[u\varphi_t+D_\xi\Phi(\nabla\Delta u)\cdot\nabla\varphi\right]dxdt\\=\int_0^T\int_\Omega f\varphi\,dxdt+\int_0^T\int_\Omega g\cdot\nabla\varphi\,dxdt\end{aligned}\tag{1.5}$$

holds.

REMARK 1.1. Recalling (2.4), (1.3) and (2.5), we have

$$\begin{aligned}|D_\xi\Phi(\nabla\Delta u)\cdot\nabla\varphi|&\leqslant\Phi(\nabla\varphi)+\Phi(-\nabla\varphi)+\Psi(D_\xi\Phi(\nabla\Delta u))\\&\leqslant(C+1)\Phi(\nabla\varphi)+D_\xi\Phi(\nabla\Delta u)\cdot\nabla\Delta u.\end{aligned}$$

From (1) in Definition 1.1 and $\varphi\in C^1(\overline{\Omega}_T)$, we can know that $D_\xi\Phi(\nabla\Delta u)\cdot\nabla\varphi$ is an integrable function on Ω_T.

REMARK 1.2. Let u be a weak solution of problem (1.1). By using the approximation technique (see [3, Chapter 3] or [5, Chapter 2]) we have, for each $t\in[0,T]$ and every $\varphi\in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot,t)|_{\partial\Omega}=0$,

$$\begin{aligned}\int_\Omega u\varphi\,dx\Big|_0^t-\int_0^t\int_\Omega\left[u\varphi_t+D_\xi\Phi(\nabla\Delta u)\cdot\nabla\varphi\right]dxd\tau\\=\int_0^t\int_\Omega f\varphi\,dxd\tau+\int_0^t\int_\Omega g\cdot\nabla\varphi\,dxd\tau.\end{aligned}\tag{1.6}$$

REMARK 1.3. By an approximation argument (see [17]), we can formally choose $-\Delta u$ as a test function in (1.6). Indeed we may use the Steklov averages

$$[v]_h(x,t)=\frac{1}{h}\int_t^{t+h}v(x,\tau)\,d\tau\tag{1.7}$$

of the function $v(x,t)$ to replace the corresponding function, and then pass to the limits. Therefore, we obtain from (1.7) an energy type estimate

$$\|\nabla u(t)\|^2_{L^2(\Omega)}+\int_0^t\int_\Omega D_\xi\Phi(\nabla\Delta u)\cdot\nabla\Delta u\,dxd\tau\leqslant\|\nabla u_0\|^2_{L^2(\Omega)}+C,\tag{1.8}$$

where C is a constant depending on $\Phi(\xi)$, $\|f\|_{L^N(\Omega_T)}$ and $\|g\|_{L^\infty(\Omega_T)}$.

Next, we state our main theorem.

THEOREM 1.1. *Under structure assumptions* (1.2), (1.3) *and integrability condition* (1.4)*, the initial-boundary value problem* (1.1) *admits a unique weak solution.*

This paper is organized as follows. In Section 2, we give some crucial lemmas that will be used later. We will prove our main result in Section 3.

In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

In this section, we will list some useful lemmas that will be used later. Let us define the polar function of $\Phi(\xi)$ as

$$\Psi(\eta) = \sup_{\xi \in \mathbb{R}^N} \{\eta \cdot \xi - \Phi(\xi)\}, \tag{2.1}$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is obvious that $\Psi(\eta)$ is a convex function.

LEMMA 2.1. *Suppose that $\Phi(\xi)$ is a convex C^1 function with $\Phi(0) = 0$. Then we have, for all $\xi, \zeta \in \mathbb{R}^N$,*

$$\Phi(\xi) \leqslant \xi \cdot D\Phi(\xi), \tag{2.2}$$

$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \geqslant 0. \tag{2.3}$$

LEMMA 2.2. (see [1]) *Suppose that $\Phi(\xi)$ is a nonnegative convex C^1 function and $\Psi(\eta)$ is its polar function. Then we have, for $\xi, \eta, \zeta \in \mathbb{R}^N$,*

$$\xi \cdot \eta \leqslant \Phi(\xi) + \Psi(\eta), \tag{2.4}$$

$$\Psi(D\Phi(\zeta)) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta. \tag{2.5}$$

LEMMA 2.3. (see [8, Chapter 3]) *Suppose that $\Phi(\xi)$ is a nonnegative convex function with $\Phi(0) = 0$, which satisfies* (1.2)*. Then $\Psi(\eta)$ in* (2.1) *is a well-defined, nonnegative function in $\mathbb{R}^N$, which also satisfies* (1.2).

LEMMA 2.4. (see [14, Chapter 4]) *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure, $f_k, g_k \in L^1(D)\,(k = 1, 2, \ldots)$ and*

$$|f_k(x)| \leqslant g_k(x), \quad a.e.\ x \in D,\ k = 1, 2, \ldots .$$

If

$$\lim_{k\to\infty} f_k(x) = f(x), \quad \lim_{k\to\infty} g_k(x) = g(x),\ a.e.\ x \in D,$$

and

$$\lim_{k\to\infty} \int_D g_k(x)\,dx = \int_D g(x)\,dx < +\infty,$$

then we have

$$\lim_{k\to\infty} \int_D f_k(x)\,dx = \int_D f(x)\,dx.$$

LEMMA 2.5. (see [4, Chapter 3] and [15]) *Suppose that $\Phi(\xi)$ is a nonnegative convex function satisfying* (1.2). *Let $D \subset \mathbb{R}^N$ be a measurable with finite Lebesgue measure $|D|$ and a sequence $\{f_k\} \subset L(D;\mathbb{R}^N)$ satisfies that*

$$\int_D \Phi(f_k)\,dx \leqslant C, \tag{2.6}$$

where C is a positive constant. Then there exist a subsequence $\{f_{k_j}\} \subset \{f_k\}$ and a function $f \in L(D;\mathbb{R}^N)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{weakly in } L(D;\mathbb{R}^N) \text{ as } j \to \infty \tag{2.7}$$

and

$$\int_D \Phi(f)\,dx \leqslant \liminf_{j\to\infty} \int_D \Phi(f_{k_j})\,dx \leqslant C. \tag{2.8}$$

3. Proof of the main result

In this section, we will use the methods of difference and calculus of variations, similar as in [17, 16], to prove the main result.

We first discretize problem (1.1) in the time direction to obtain a sequence of elliptic problems. Let m be a positive number. Denote $h = T/m$. Consider the following elliptic problem

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} + \operatorname{div}\big(D_\xi \Phi(\nabla \Delta u_k)\big) \\ \quad = [f]_h((k-1)h) - \operatorname{div}[g]_h((k-1)h) \quad \text{in } \Omega, \\ u_k|_{\partial\Omega} = 0, \quad \Delta u_k|_{\partial\Omega} = 0, \quad k = 1,2,\ldots,m, \end{cases} \tag{3.1}$$

where the Steklov averages $[f]_h, [g]_h$ of f, g, are, respectively, defined in (1.7). Clearly, we have $[f]_h(\cdot) \in L^N(\Omega)$ and $[g]_h(\cdot) \in (L^\infty(\Omega))^N$.

Set

$$f_0(x) = [f]_h(0) \text{ and } g_0(x) = [g]_h(0).$$

Now, let us consider the existence and uniqueness of weak solutions of the following elliptic problem

$$\begin{cases} \dfrac{u - u_0}{h} + \operatorname{div}\big(D_\xi \Phi(\nabla \Delta u)\big) = f_0(x) - \operatorname{div}(g_0(x)) & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

with $u_0 \in H_0^1(\Omega)$, which is the case of (3.1) when $k = 1$.

We introduce the set

$$V = \left\{ v \in H_0^1(\Omega) \cap W^{2,1^*}(\Omega) \,\middle|\, \Delta v \in W_0^{1,1}(\Omega) \text{ and } \Phi(\nabla \Delta v) \in L^1(\Omega) \right\}.$$

It is easy to verify that V is a closed and convex set.

DEFINITION 3.1. A function $u \in V$ with $D_\xi \Phi(\nabla \Delta u) \cdot \nabla \Delta u \in L^1(\Omega)$ is called a weak solution of problem (3.2) if for any $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \frac{u-u_0}{h} \varphi \, dx - \int_\Omega D_\xi \Phi(\nabla \Delta u) \cdot \nabla \varphi \, dx = \int_\Omega f_0 \varphi \, dx + \int_\Omega g_0 \cdot \nabla \varphi \, dx. \tag{3.3}$$

REMARK 3.1. The requirement $D_\xi \Phi(\nabla \Delta u) \cdot \nabla \Delta u \in L^1(\Omega)$ make it possible to find an energy type estimate and prove the uniqueness of solutions.

PROPOSITION 3.1. *There exists a unique weak solution $u_1 \in V$ for problem* (3.2).

Proof. We consider the variational problem

$$\min\{J(v) | v \in V\},$$

where the functional J is defined by:

$$\begin{aligned} J(v) = & \frac{1}{2h} \int_\Omega |\nabla v - \nabla u_0|^2 \, dx + \int_\Omega \Phi(\nabla \Delta v) \, dx \\ & + \int_\Omega f_0 \Delta v \, dx + \int_\Omega g_0 \cdot \nabla \Delta v \, dx. \end{aligned}$$

We will establish that $J(v)$ has a minimizer $u_1(x)$ in V.

Due to (1.2), for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|\xi| \leqslant \varepsilon \Phi(\xi) + C_\varepsilon. \tag{3.4}$$

By Hölder's and Sobolev's inequalities and (3.4), we have

$$\begin{aligned} \left| \int_\Omega f_0 \Delta v \, dx \right| & \leqslant \|f_0\|_{L^N(\Omega)} \|\Delta v\|_{L^{1^*}(\Omega)} \\ & \leqslant \|f_0\|_{L^N(\Omega)} \|\nabla \Delta v\|_{L^1(\Omega)} \\ & \leqslant C\varepsilon \|\Phi(\nabla \Delta v)\|_{L^1(\Omega)} + C_\varepsilon \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \left| \int_\Omega g_0 \cdot \nabla \Delta v \, dx \right| & \leqslant \|g_0\|_{L^\infty(\Omega)} \|\nabla \Delta v\|_{L^1(\Omega)} \\ & \leqslant C\varepsilon \|\Phi(\nabla \Delta v)\|_{L^1(\Omega)} + C_\varepsilon. \end{aligned} \tag{3.6}$$

Choosing ε sufficiently small, we conclude from (3.5) and (3.6) that

$$\begin{aligned} J(v) & \geqslant \frac{1}{2h} \int_\Omega |\nabla v - \nabla u_0|^2 \, dx + \frac{1}{2} \int_\Omega \Phi(\nabla \Delta v) \, dx \\ & \quad - C(\Phi, \|f_0\|_{L^N(\Omega)}, \|g_0\|_{L^\infty(\Omega)}) \\ & \geqslant -C. \end{aligned}$$

Thus we get

$$-C \leqslant \inf_{v\in V} J(v) \leqslant J(0) = \frac{1}{2h}\int_\Omega |\nabla u_0|^2\,dx.$$

We can choose a minimizing sequence $\{v_n\}_{n=1}^\infty \subset V$ such that

$$J(v_n) \to \inf_{v\in V} J(v).$$

It follows that, for $n = 1, 2, \dots,$

$$\|v_n\|_{H_0^1(\Omega)} \leqslant C \text{ and } \int_\Omega \Phi(\nabla\Delta v_n)\,dx \leqslant C. \tag{3.7}$$

Since $\Delta v_n \in W_0^{1,1}(\Omega)$ with $\Phi(\nabla\Delta v_n) \in L^1(\Omega)$, by using Sobolev's imbedding theorem, we have

$$\begin{aligned}\|\Delta v_n\|_{L^{1^*}(\Omega)} &\leqslant C\|\Delta v_n\|_{W^{1,1}(\Omega)} \leqslant C\|\nabla\Delta v_n\|_{L^1(\Omega)}\\ &\leqslant C\|\Phi(\nabla\Delta v_n)\|_{L^1(\Omega)} + C.\end{aligned}$$

Using $W^{2,p}$-theory of elliptic equations for the function v_n (see [10]), we obtain

$$\|v_n\|_{W^{2,1^*}(\Omega)} \leqslant C\|\Delta v_n\|_{L^{1^*}(\Omega)} \leqslant C\|\Phi(\nabla\Delta v_n)\|_{L^1(\Omega)} + C \leqslant C_1.$$

Thus we obtain

$$\|v_n\|_{H_0^1(\Omega)} + \|v_n\|_{W^{2,1^*}(\Omega)} + \|\Phi(\nabla\Delta v_n)\|_{L^1(\Omega)} \leqslant C. \tag{3.8}$$

By using Lemma 2.5 we may extract a subsequence $\{v_{n_j}\}_{j=1}^\infty \subset \{v_n\}_{n=1}^\infty$ and a function $u_1 \in H_0^1(\Omega) \cap W^{2,1^*}(\Omega)$ with $\Delta u_1 \in W_0^{1,1}(\Omega)$ such that

$$\begin{aligned}&v_{n_j} \rightharpoonup u_1 \quad \text{weakly in } H_0^1(\Omega),\\ &v_{n_j} \rightharpoonup u_1 \quad \text{weakly in } W^{2,1^*}(\Omega),\\ &\nabla\Delta v_{n_j} \rightharpoonup \eta \quad \text{weakly in } \big(L^1(\Omega)\big)^N.\end{aligned}$$

It follows that

$$\Delta v_{n_j} \rightharpoonup \Delta u_1 \quad \text{weakly in } L^{1^*}(\Omega).$$

Then we have $\eta = \nabla\Delta u_1$.

Since

$$\int_\Omega |\nabla u_1 - \nabla u_0|^2\,dx \leqslant \liminf_{j\to\infty}\int_\Omega |\nabla v_{n_j} - \nabla u_0|^2\,dx$$

and

$$\int_\Omega \Phi(\nabla\Delta u_1)\,dx \leqslant \liminf_{j\to\infty}\int_\Omega \Phi(\nabla\Delta v_{n_j})\,dx,$$

we know that $u_1 \in V$, and $J(v)$ is weakly lower semi-continuous on V, which ensures that

$$J(u_1) \leqslant \liminf_{j\to\infty} J(v_{n_j}) = \inf_{v\in V} J(v).$$

This implies that $u_1 \in V$ is a minimizer of the functional $J(v)$ in V, i.e.,

$$J(u_1) = \inf_{v\in V} J(v).$$

Since $u_1 \in V$ is a minimizer, we have $\lambda u_1 \in V, \lambda \in (0,1)$ and

$$J(u_1) \leqslant J(\lambda u_1),$$

which implies

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u_1 - \nabla u_0|^2\,dx + h\int_\Omega \Phi(\nabla\Delta u_1)\,dx\\
&\quad + h\int_\Omega f_0\Delta u_1\,dx + h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\\
&\leqslant \frac{1}{2}\int_\Omega |\lambda\nabla u_1 - \nabla u_0|^2\,dx + h\int_\Omega \Phi(\lambda\nabla\Delta u_1)\,dx\\
&\quad + \lambda h\int_\Omega f_0\Delta u_1\,dx + \lambda h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx.
\end{aligned}$$

Recalling (2.3), we know

$$\Phi(\nabla\Delta u_1) - \Phi(\lambda\nabla\Delta u_1) \geqslant (1-\lambda)D_\xi\Phi(\lambda\nabla\Delta u_1)\cdot\nabla\Delta u_1,$$

then

$$\begin{aligned}
&\frac{1}{2}(1-\lambda^2)\int_\Omega |\nabla u_1|^2\,dx + h(1-\lambda)\int_\Omega D_\xi\Phi(\lambda\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx\\
&\quad + h(1-\lambda)\int_\Omega f_0\Delta u_1\,dx + h(1-\lambda)\int_\Omega g_0\cdot\nabla\Delta u_1\,dx \leqslant (1-\lambda)\int_\Omega \nabla u_1\cdot\nabla u_0\,dx.
\end{aligned}$$

Dividing the above inequality by $1-\lambda$, and passing to limits as $\lambda \to 1$, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_1|^2\,dx + h\liminf_{\lambda\to 1}\int_\Omega D_\xi\Phi(\lambda\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx\\
&\qquad + h\int_\Omega f_0\Delta u_1\,dx + h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx \leqslant \int_\Omega \nabla u_1\cdot\nabla u_0\,dx.
\end{aligned}$$

Since $D_\xi\Phi(\lambda\nabla\Delta u_1)\cdot\nabla\Delta u_1 \geqslant 0$, by Fatou's Lemma we conclude that

$$\int_\Omega D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx \leqslant \liminf_{\lambda\to 1}\int_\Omega D_\xi\Phi(\lambda\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx$$

and

$$\int_\Omega |\nabla u_1|^2\,dx + h\int_\Omega D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx$$

$$+h\int_\Omega f_0\Delta u_1\,dx+h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\leqslant\int_\Omega\nabla u_1\cdot\nabla u_0\,dx. \tag{3.9}$$

Similar to (3.5) and (3.6), using (2.5) we get

$$\begin{aligned}\int_\Omega|\nabla u_1|^2\,dx&+h\int_\Omega\Psi(D_\xi\Phi(\nabla\Delta u_1))\,dx\\&\leqslant\int_\Omega|\nabla u_1|^2\,dx+h\int_\Omega D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx\\&\leqslant\int_\Omega\nabla u_1\cdot\nabla u_0\,dx+Ch\|\Phi(\nabla\Delta u_1)\|_{L^1(\Omega)}+C.\end{aligned}$$

It follows that $D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta u_1\in L^1(\Omega)$ and $\Psi(D_\xi\Phi(\nabla\Delta u_1))\in L^1(\Omega)$.

Now for every $\phi\in V$ and every $\lambda\in(0,1)$, we have

$$J(u_1)\leqslant J(\lambda u_1+(1-\lambda)\phi).$$

Setting

$$\xi_\lambda=\lambda\nabla\Delta u_1+(1-\lambda)\nabla\Delta\phi,$$

which implies from the above inequality that

$$\begin{aligned}&\frac12\int_\Omega|\nabla u_1-\nabla u_0|^2dx+h\int_\Omega\Phi(\nabla\Delta u_1)\,dx\\&\quad+h\int_\Omega f_0\Delta u_1\,dx+h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\\&\leqslant\frac12\int_\Omega|\lambda(\nabla u_1-\nabla u_0)+(1-\lambda)(\nabla\phi-\nabla u_0)|^2+h\int_\Omega\Phi(\xi_\lambda)\,dx\\&\quad+\lambda h\int_\Omega f_0\Delta u_1\,dx+(1-\lambda)h\int_\Omega f_0\Delta\phi\,dx\\&\quad+\lambda h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx+(1-\lambda)h\int_\Omega g_0\cdot\nabla\Delta\phi\,dx.\end{aligned}$$

In view of (2.3), we find

$$\Phi(\nabla\Delta u_1)-\Phi(\xi_\lambda)\geqslant(1-\lambda)D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi).$$

Thus we have

$$\begin{aligned}&(1-\lambda)h\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\,dx\\&\quad+(1-\lambda)h\int_\Omega f_0\Delta u_1\,dx+(1-\lambda)h\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\\&\leqslant\frac{1-\lambda}{2}\int_\Omega\big[-(1+\lambda)|\nabla u_1-\nabla u_0|^2+2\lambda(\nabla u_1-\nabla u_0)\cdot(\nabla\phi-\nabla u_0)\\&\quad+(1-\lambda)|\nabla\phi-\nabla u_0|^2\big]\,dx+(1-\lambda)h\int_\Omega f_0\Delta\phi\,dx+(1-\lambda)h\int_\Omega g_0\cdot\nabla\Delta\phi\,dx.\end{aligned}$$

Dividing the above inequality by $1-\lambda$, we obtain

$$\begin{aligned}&\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\,dx+\int_\Omega f_0\Delta u_1\,dx+\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\\&\leqslant\frac{1}{2h}\int_\Omega\big[-(1+\lambda)|\nabla u_1-\nabla u_0|^2+2\lambda(\nabla u_1-\nabla u_0)\cdot(\nabla\phi-\nabla u_0)\\&\quad+(1-\lambda)|\nabla\phi-\nabla u_0|^2\big]\,dx+\int_\Omega f_0\Delta\phi\,dx+\int_\Omega g_0\cdot\nabla\Delta\phi\,dx.\end{aligned}\tag{3.10}$$

Consider

$$g(\lambda)=\Phi(\xi_\lambda)=\Phi(\lambda\nabla\Delta u_1+(1-\lambda)\nabla\Delta\phi).$$

It is obvious that g is a convex function in $\mathbb{R}$. Then by the monotonicity of a convex function's derivative, we know

$$g'(0)\leqslant g'(\lambda)\leqslant g'(1),\quad \lambda\in(0,1),$$

which yields that

$$\begin{aligned}D_\xi\Phi(\nabla\Delta\phi)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)&\leqslant D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\\&\leqslant D_\xi\Phi(\nabla\Delta u_1)\cdot(\nabla\Delta u_1-\nabla\Delta\phi).\end{aligned}\tag{3.11}$$

Recalling (2.4) and (1.3), we have

$$\begin{aligned}|D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta\phi|&\leqslant\Psi(D_\xi\Phi(\nabla\Delta u_1))+\Phi(\nabla\Delta\phi)+\Phi(-\nabla\Delta\phi)\\&\leqslant\Psi(D_\xi\Phi(\nabla\Delta u_1))+(C+1)\Phi(\nabla\Delta\phi).\end{aligned}\tag{3.12}$$

When $\Psi(D_\xi\Phi(\nabla\Delta u_1))\in L^1(\Omega)$ and $\phi\in V$, then $D_\xi\Phi(\nabla\Delta\phi)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)$ and $D_\xi\Phi(\nabla\Delta u_1)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)$ are in the space $L^1(\Omega)$. By Lebesgue dominated convergence theorem, we have

$$\int_\Omega\lim_{\lambda\to1}D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\,dx=\lim_{\lambda\to1}\int_\Omega D_\xi\Phi(\xi_\lambda)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\,dx.$$

Recalling (3.10), we obtain

$$\begin{aligned}&\int_\Omega D_\xi\Phi(\nabla\Delta u_1)\cdot(\nabla\Delta u_1-\nabla\Delta\phi)\,dx+\int_\Omega f_0\Delta u_1\,dx+\int_\Omega g_0\cdot\nabla\Delta u_1\,dx\\&\quad\leqslant\frac{1}{h}\int_\Omega(\nabla u_1-\nabla u_0)\cdot(\nabla\phi-\nabla u_1)\,dx+\int_\Omega f_0\Delta\phi\,dx+\int_\Omega g_0\cdot\nabla\Delta\phi\,dx.\end{aligned}$$

Denote

$$\begin{aligned}A_0=&\int_\Omega\frac{\nabla u_1-\nabla u_0}{h}\cdot\nabla u_1\,dx+\int_\Omega D_\xi\Phi(\nabla\Delta u_1)\cdot\nabla\Delta u_1\,dx\\&+\int_\Omega f_0\Delta u_1\,dx+\int_\Omega g_0\cdot\nabla\Delta u_1\,dx.\end{aligned}$$

Then we conclude that, for every $\phi \in V$,

$$\int_\Omega \frac{\nabla u_1 - \nabla u_0}{h} \cdot \nabla\phi\, dx + \int_\Omega D_\xi \Phi(\nabla\Delta u_1) \cdot \nabla\Delta\phi\, dx + \int_\Omega f_0 \Delta\phi\, dx + \int_\Omega g_0 \cdot \nabla\Delta\phi\, dx \geqslant A_0. \tag{3.13}$$

It follows form a scaling argument that

$$\int_\Omega \Big[\frac{u_1 - u_0}{h}(-\Delta\phi)\, dx + \int_\Omega D_\xi \Phi(\nabla\Delta u_1) \cdot \nabla\Delta\phi\Big]\, dx = -\int_\Omega f_0 \Delta\phi\, dx - \int_\Omega g_0 \cdot \nabla\Delta\phi\, dx. \tag{3.14}$$

For every fixed $\varphi \in C_0^\infty(\Omega)$, as the problem $\varphi = -\Delta\phi$ is solvable in V, the function u_1 is a weak solution of the corresponding Euler-Lagrange equation of $J(v)$, which is problem (3.2).

Suppose that there exists another weak solution $\tilde{u}_1$ of problem (3.2). Then, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \frac{\tilde{u}_1 - u_0}{h}\varphi\, dx - \int_\Omega D_\xi \Phi(\nabla\Delta\tilde{u}_1) \cdot \nabla\varphi\, dx = \int_\Omega f_0 \varphi\, dx + \int_\Omega g_0 \cdot \nabla\varphi\, dx,$$

which follows that

$$\int_\Omega \frac{u_1 - \tilde{u}_1}{h}\varphi\, dx - \int_\Omega \big[D_\xi \Phi(\nabla\Delta u_1) - D_\xi \Phi(\nabla\Delta\tilde{u}_1)\big] \cdot \nabla\varphi\, dx = 0. \tag{3.15}$$

Recalling (2.4), (1.3) and (2.5), we observe that

$$\begin{aligned}|D_\xi \Phi(\nabla\Delta\tilde{u}_1) \cdot \nabla\Delta u_1| &\leqslant \Phi(\nabla\Delta u_1) + \Phi(-\nabla\Delta u_1) + \Psi(D_\xi \Phi(\nabla\Delta\tilde{u}_1)) \\ &\leqslant (C+1)\Phi(\nabla\Delta u_1) + D_\xi \Phi(\nabla\Delta\tilde{u}_1) \cdot \nabla\Delta\tilde{u}_1 \in L^1(\Omega).\end{aligned}$$

Using the approximation argument, we can take $\varphi = \Delta u_1 - \Delta\tilde{u}_1$ as a test function in (3.15). Thus, we have

$$\frac{1}{h}\int_\Omega |\nabla u_1 - \nabla\tilde{u}_1|^2\, dx + \int_\Omega \big[D_\xi \Phi(\nabla\Delta u_1) - D_\xi \Phi(\nabla\Delta\tilde{u}_1)\big] \cdot (\nabla\Delta u_1 - \nabla\Delta\tilde{u}_1)\, dx = 0.$$

Since the two terms of the left-hand side in the above equality are nonnegative, we have $\nabla u_1 = \nabla\tilde{u}_1$ a.e. in Ω. Recalling $u_1 = \tilde{u}_1 = 0$ on $\partial\Omega$, we conclude that $u_1 = \tilde{u}_1$ a.e. in Ω. Therefore we obtain the uniqueness of weak solutions. This completes the proof of the proposition.

Proof. [*Proof of Theorem* 1.1]

(1) *Existence of weak solutions.*

First we construct an approximation solution sequence $\{u^m\}$ for problem (1.1).

When $k=1$, it implies from Proposition 3.1 that there is a unique solution $u_1 \in V$ satisfying (3.1). By induction, we find weak solutions $u_k \in V$ of (3.1), $k=2,3,\ldots$. It follows that, for every $\eta \in V$,

$$\frac{1}{h}\int_\Omega \nabla(u_k-u_{k-1})\cdot\nabla\eta\,dx - \int_\Omega D_\xi\Phi(\nabla\Delta u_k)\cdot\nabla\Delta\eta\,dx \\ = \int_\Omega [f]_h((k-1)h)\Delta\eta\,dx + \int_\Omega [g]_h((k-1)h)\cdot\nabla\Delta\eta\,dx. \tag{3.16}$$

Next, we take $\eta = u_k$ as a test function in (3.16) to obtain *a priori* estimate for the function $u_k\,(k=1,2,\ldots,m)$. Similar to (3.5) and (3.6), using (2.5) and (1.2) we have

$$\|\nabla u_k\|^2_{L^2(\Omega)} + h\int_\Omega \Phi(\nabla\Delta u_k)\,dx \leqslant \|\nabla u_{k-1}\|^2_{L^2(\Omega)} + Ch, \tag{3.17}$$

where C is a constant depending on $\Phi(\xi)$, $\|[f]_h((k-1)h)\|_{L^N(\Omega)}$, $\|[g]_h((k-1)h)\|_{L^\infty(\Omega)}$. For each $t\in(0,T]$, there exists some $i\in\{1,2,\ldots,m\}$ such that $t\in((i-1)h,ih]$. We add all inequalities (3.17) for $k=1,\ldots,i$, to get

$$\|\nabla u_i\|^2_{L^2(\Omega)} + h\sum_{k=1}^{i}\int_\Omega \Phi(\nabla\Delta u_k)\,dx \leqslant \|\nabla u_0\|^2_{L^2(\Omega)} + CT. \tag{3.18}$$

Now for every $h=T/m$, we define

$$u^m(x,t) = \begin{cases} u_0(x), & t=0,\\ u_1(x), & 0<t\leqslant h,\\ \cdots\cdots, & \cdots\cdots,\\ u_j(x), & (j-1)h<t\leqslant jh,\\ \cdots\cdots, & \cdots\cdots,\\ u_m(x), & (n-1)h<t\leqslant mh=T. \end{cases} \tag{3.19}$$

Thus we have

$$\|\nabla u^m(t)\|^2_{L^2(\Omega)} + \int_0^{ih}\int_\Omega \Phi(\nabla\Delta u^m)\,dxd\tau \leqslant \|\nabla u_0\|^2_{L^2(\Omega)} + C,$$

or

$$\|\nabla u^m(t)\|^2_{L^2(\Omega)} + \int_0^{t}\int_\Omega \Phi(\nabla\Delta u^m)\,dxd\tau \leqslant \|\nabla u_0\|^2_{L^2(\Omega)} + C. \tag{3.20}$$

Therefore, after taking the supremum over $[0,T]$, we get

$$\sup_{0\leqslant t\leqslant T}\int_\Omega |\nabla u^m(x,t)|^2\,dx + \int_0^T\int_\Omega \Phi(\nabla\Delta u^m)\,dxdt \leqslant \|\nabla u_0\|^2_{L^2(\Omega)} + C.$$

Using the same technique as in the proof of (3.8), we have

$$\|u^m\|_{L^\infty(0,T;H^1_0(\Omega))} + \|u^m\|_{L^1(0,T;W^{2,1^*}(\Omega))} + \int_0^T\int_\Omega \Phi(\nabla\Delta u^m)\,dxdt \leqslant C.$$

Therefore, by Lemma 2.5, we may choose a subsequence (we also denote it by the original sequence for simplicity) such that

$$\begin{cases} u^m \rightharpoonup u \quad \text{weakly-}^* \text{ in } \quad L^\infty\big(0,T;H_0^1(\Omega)\big), \\ \nabla\Delta u^m \rightharpoonup h \quad \text{weakly in} \quad \big(L^1(\Omega_T)\big)^N. \end{cases} \tag{3.21}$$

Then we show that $h = \nabla\Delta u$, which implies that

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u\|_{L^1(0,T;W^{2,1^*}(\Omega))} + \int_0^T\int_\Omega \Phi(\nabla\Delta u)\,dxdt \leqslant C. \tag{3.22}$$

Indeed, from (3.21) we have

$$\nabla u^m \rightharpoonup \nabla u \quad \text{weakly-}^* \text{ in } \quad \big(L^\infty\big(0,T;L^2(\Omega)\big)\big)^N.$$

Thus we conclude that $h = \nabla\Delta u$.

Denote

$$\zeta_m = D_\xi \Phi(\nabla\Delta u^m).$$

It follows from (2.5) that

$$\int_0^T\int_\Omega \Psi(\zeta_m)\,dxdt \leqslant \int_0^T\int_\Omega D_\xi\Phi(\nabla\Delta u^m)\cdot\nabla\Delta u^m\,dxdt \leqslant C.$$

Recalling Lemma 2.3 and Lemma 2.5, we conclude that there exists a subsequence $\{\zeta_m\}$ (we also denote it by the original sequence for simplicity) such that

$$\zeta_m \rightharpoonup \zeta \quad \text{weakly in } L^1(\Omega_T) \tag{3.23}$$

and

$$\int_0^T\int_\Omega \Psi(\zeta)\,dxdt \leqslant \liminf_{m\to\infty}\int_0^T\int_\Omega \Psi(\zeta_m)\,dxdt \leqslant C.$$

Recalling inequality (2.4) and (1.3), we have

$$\begin{aligned} |\zeta\cdot\nabla\Delta u| &\leqslant \Psi(\zeta) + \Phi(\nabla\Delta u) + \Phi(-\nabla\Delta u) \\ &\leqslant \Psi(\zeta) + (C+1)\Phi(\nabla\Delta u), \end{aligned}$$

and then conclude that $\zeta\cdot\nabla\Delta u \in L^1(\Omega_T)$.

Next, we prove that the function u is a weak solution of problem (1.1).

For each $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot,T) = 0$ and $\varphi(x,t)|_\Gamma = 0$ and for every $k \in \{1,2,\dots,m\}$, we solve the equation $-\Delta\eta_k(x) = \varphi(x,kh)$ to find a function $\eta_k \in V$ and let it be a text function in (3.16) to have

$$\begin{aligned} &\int_\Omega \frac{u_k(x)-u_{k-1}(x)}{h}\varphi(x,kh)\,dx - \int_\Omega D_\xi\Phi(\nabla\Delta u_k)\cdot\nabla\varphi(x,kh)\,dx \\ &\quad = \int_\Omega [f]_h((k-1)h)\varphi(x,kh)\,dx + \int_\Omega [g]_h((k-1)h)\cdot\nabla\varphi(x,kh)\,dx. \end{aligned} \tag{3.24}$$

Summing up all the equalities and recalling the definition of $u^m(x,t)$ in (3.19) and $\varphi(\cdot,T)=\varphi(\cdot,mh)=0$, we have

$$\begin{aligned} &h\sum_{k=1}^{m-1}\int_\Omega u^m(x,kh)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}\,dx-\int_\Omega u_0(x)\varphi(x,h)\,dx\\ &\quad -h\sum_{k=1}^{m}\int_\Omega D_\xi\Phi\big(\nabla\Delta u^m(x,kh)\big)\cdot\nabla\varphi(x,kh)\,dx\\ &=h\sum_{k=1}^{m}\int_\Omega [f]_h((k-1)h)\varphi(x,kh)\,dx+\int_\Omega [g]_h((k-1)h)\cdot\nabla\varphi(x,kh)\,dx. \end{aligned} \tag{3.25}$$

Passing to the limits as $m\to+\infty$, we obtain from (3.25) that

$$\begin{aligned} &-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\,dxdt-\int_\Omega u_0(x)\varphi(x,0)\,dx-\int_0^T\int_\Omega \zeta\cdot\nabla\varphi\,dxdt\\ &=\int_0^T\int_\Omega (f\varphi+g\cdot\nabla\varphi)\,dxdt. \end{aligned} \tag{3.26}$$

Now we choose $\varphi\in C_0^\infty(\Omega_T)$ to have

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\,dxdt=\int_0^T\int_\Omega (\zeta\cdot\nabla\varphi+f\varphi+g\cdot\nabla\varphi)\,dxdt. \tag{3.27}$$

Since $\zeta\in L^1(\Omega_T)$, we conclude that $u_t\in L\big(0,T;W^{-1,1}(\Omega)\big)$. Thus we find a large positive integer s such that $W^{-1,1}(\Omega)\subset H^{-s}(\Omega)$, and then obtain

$$u_t\in L\big(0,T;H^{-s}(\Omega)\big),$$

which implies (see [18]) that

$$u\in C\big([0,T];H^{-s}(\Omega)\big),$$

where $H^{-s}(\Omega)$ is the dual space of $H_0^s(\Omega)=W_0^{s,2}(\Omega)$. For each $\varepsilon>0$ and all $t,t_0\in[0,T]$, by (3.22) there exists a positive number $\delta>0$ such that

$$\delta\|\nabla u(t)-\nabla u(t_0)\|_{L^2(\Omega)}\leqslant\frac{\varepsilon}{2}.$$

From the compact imbedding relation

$$H_0^1(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow H^{-s}(\Omega).$$

We have, for all $t,t_0\in[0,T]$,

$$\begin{aligned} \|u(t)-u(t_0)\|_{L^2(\Omega)}&\leqslant\delta\|u(t)-u(t_0)\|_{H_0^1(\Omega)}+C(\delta)\|u(t)-u(t_0)\|_{H^{-s}(\Omega)}\\ &\leqslant\delta\|\nabla u(t)-\nabla u(t_0)\|_{L^2(\Omega)}+C(\delta)\|u(t)-u(t_0)\|_{H^{-s}(\Omega)}\\ &\leqslant\frac{\varepsilon}{2}+C(\delta)\|u(t)-u(t_0)\|_{H^{-s}(\Omega)}, \end{aligned}$$

where the first inequality is guaranteed by Lemma 5.1 in Chapter 1 of [13]. It follows from the above inequalities that

$$u \in C\big([0,T];L^2(\Omega)\big).$$

Therefore, the function u satisfies conditions (1) and (2) in Definition 1.1. Using the monotonicity method similar as in [17, 16], we can show that $\zeta = D_\xi \Phi(\nabla\Delta u)$ a.e. in Ω_T. Therefore, we finish the proof of the existence of weak solutions.

(2) *Uniqueness of weak solutions.*

Suppose that there exist two weak solutions u and v of problem (1.1). Denote $w = u - v$. Using Remark 1.2, we have, for each $t \in [0,T]$ and any $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot,t)|_{\partial\Omega} = 0$,

$$\int_\Omega [w\varphi](x,t)\,dx - \int_0^t\int_\Omega \big[w\varphi_t + \big(D_\xi\Phi(\nabla\Delta u) - D_\xi\Phi(\nabla\Delta v)\big)\cdot\nabla\varphi\big]\,dxdt = 0.$$

Using the approximation argument, we choose

$$w_{\varepsilon,h}(x,t) = \frac{1}{2h}\int_{t-h}^{t+h} -\Delta w_\varepsilon(x,\tau)\,d\tau$$

as a test function in the above equality to have

$$\begin{aligned}\int_\Omega [ww_{\varepsilon,h}](x,t)\,dx - \int_0^t\int_\Omega w[w_{\varepsilon,h}]_t\,dt \\ + \int_0^t\int_\Omega \big(D_\xi\Phi(\nabla\Delta u) - D_\xi\Phi(\nabla\Delta v)\big)\cdot\nabla w_{\varepsilon,h}\,dxdt = 0.\end{aligned} \tag{3.28}$$

We denote the sum of the first and second terms on the left side as I_1 and the third term as I_2 in (3.28). We calculate I_1 to have

$$\begin{aligned}I_1 = &\int_\Omega \nabla w(x,t)\cdot\Big(\frac{1}{2h}\int_{t-h}^{t+h}\nabla w_\varepsilon(x,\tau)\,d\tau\Big)\,dx \\ &- \frac{1}{2h}\int_0^t\int_\Omega \nabla w(x,t)\cdot\big(\nabla w_\varepsilon(x,t+h) - \nabla w_\varepsilon(x,t-h)\big)\,dxdt.\end{aligned}$$

Sending $\varepsilon \to 0$, we have that

$$\begin{aligned}I_1 \to &\int_\Omega \nabla w(x,t)\cdot\Big(\frac{1}{2h}\int_{t-h}^{t+h}\nabla w(x,\tau)\,d\tau\Big)\,dx \\ &- \frac{1}{2h}\int_0^t\int_\Omega \nabla w(x,\tau)\cdot\big(\nabla w(x,\tau+h) - \nabla w(x,\tau-h)\big)\,dxd\tau \\ = &\int_\Omega \nabla w(x,t)\cdot\Big(\frac{1}{2h}\int_{t-h}^{t+h}\nabla w(x,\tau)\,d\tau\Big)\,dx \\ &- \frac{1}{2h}\int_t^{t+h}\int_\Omega \nabla w(x,\tau-h)\cdot\nabla w(x,\tau)\,dxd\tau,\end{aligned}$$

since $w(x,t)$ has been extended to be 0 when $t<0$. Next we send $h\to 0$ to have, for a.e. $t\in(0,T]$,

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} I_1=\frac{1}{2}\int_\Omega |\nabla w|^2(x,t)\,dx=\frac{1}{2}\int_\Omega |\nabla u-\nabla v|^2(x,t)\,dx. \tag{3.29}$$

Choosing $\xi=\nabla w_{\varepsilon,h}$, $\zeta=\nabla\Delta u$ and $\eta=D_\xi\Phi(\nabla\Delta u)$ in the inequalities in Lemma 2.2 and recalling (1.3), we have

$$\begin{aligned}|D_\xi\Phi(\nabla\Delta u)\cdot\nabla w_{\varepsilon,h}| &\leqslant \Phi(\nabla w_{\varepsilon,h})+\Phi(-\nabla w_{\varepsilon,h})+\Psi\big(D_\xi\Phi(\nabla\Delta u)\big)\\ &\leqslant D_\xi\Phi(\nabla\Delta u)\cdot\nabla\Delta u+(C+1)\Phi(\nabla w_{\varepsilon,h}).\end{aligned}$$

And we estimate the term

$$|D_\xi\Phi(\nabla\Delta v)\cdot\nabla w_{\varepsilon,h}|\leqslant D_\xi\Phi(\nabla\Delta v)\cdot\nabla\Delta v+(C+1)\Phi(\nabla w_{\varepsilon,h})$$

in the same way. This justifies that

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} I_2=\int_0^t\int_\Omega \big(D_\xi\Phi(\nabla\Delta u)-D_\xi\Phi(\nabla\Delta v)\big)\cdot\nabla\Delta(u-v)\,dxdt. \tag{3.30}$$

Sending first $\varepsilon\to 0$, and then $h\to 0$ in (3.28), and recalling (3.29) and (3.30), we conclude that

$$\int_\Omega\frac{|\nabla u-\nabla v|^2(t)}{2}\,dx+\int_0^t\int_\Omega\big(D_\xi\Phi(\nabla\Delta u)-D_\xi\Phi(\nabla\Delta v)\big)\cdot\nabla(\Delta u-\Delta v)\,dxd\tau=0.$$

Recalling Lemma 2.1, we know that both terms on the left-hand side are nonnegative. Thus, we have $\nabla u=\nabla v$ a.e. in Ω_T. Since $u-v=0$ on Γ, we conclude $u-v=0$ a.e. in Ω_T, which implies $u=v$ a.e. in Ω_T. Therefore we obtain the uniqueness of weak solutions. Thus we complete the proof of Theorem 1.1.

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Chao Zhang
LMAM, School of Mathematical Sciences
Peking University
Beijing 100871
China
e-mail: czhang@math.pku.edu.cn

Shulin Zhou
LMAM, School of Mathematical Sciences
Peking University
Beijing 100871
China
e-mail: szhou@math.pku.edu.cn