Differential
Equations
& Applications
Volume 2, Number 4 (2010), 465–486

QUADRATIC PERTURBATIONS OF PERIODIC BOUNDARY VALUE PROBLEMS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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(*Communicated by R. L. Pouso*)

Abstract. In this paper, we describe a systematic development of the different types of perturbations methods in the theory of differential and integral equations. A special quadratic perturbation of the periodic boundary value problems of second order ordinary differential equations is studied in detail for different aspects of the solutions. An existence theorem is proved under mixed generalized Lipschitz and Carathéodory conditions and the existence results for extremal positive solutions are established for Carathéodory as well as discontinuity conditions. Our results include some known existence results for periodic boundary value problems of second order ordinary nonlinear differential equations as special cases.

1. Introduction

Perturbation techniques or methods are very much useful in the subject of nonlinear analysis for studying the dynamical systems represented by nonlinear differential and integral equations in a nice way. Sometimes a differential equation representing a certain dynamical system is not easily solvable or analyzed, however, the perturbation of such problem in someone manner makes it possible to study the problem with available methods for different aspects of the solutions. To be more specific, for any closed and bounded interval $J=[0,T]$ of the real line $\mathbb{R}$, consider the initial value problem of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} x'(t) &= f(t,x(t)) \text{ a.e. } t\in J \\ x(0) &= x_0\in\mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where $f: J\times\mathbb{R}\to\mathbb{R}$.

The IVP (1.1) is fundamental or core part of nonlinear analysis and widely studied in the literature over the years for different aspects of the solutions. It is not wrong if one says that the subject of nonlinear analysis starts with this nonlinear differential equations. Now it may happen that the nonlinearity f involved in the above equation (1.1) is not smooth or regular to discuss for existence or some other characterizations of the solutions. However, if we split the function f into sum of two functions f_1 and f_2,

Mathematics subject classification (2010): 34K10.
Keywords and phrases: Banach algebra, periodic boundary value problem, existence theorem.

that is, $f = f_1 + f_2$, then these functions have some nice properties and the nonlinear differential equation

$$\left.\begin{array}{c} x'(t) = f_1(t,x(t)) + f_2(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R} \end{array}\right\} \tag{1.2}$$

is easily solvable with the available functional theoretic techniques. The method of doing so is called perturbation method and the differential equation (1.2) is called a perturbation of the differential equation

$$\left.\begin{array}{c} x'(t) = f_1(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.3}$$

The above differential equation (1.2) is obtained by perturbing the nonlinearity f_1 from (1.3) and is called a perturbed differential equation. Now the perturbed differential equations are classified into two main categories, namely (1) perturbed differential equation of first type and (2) perturbed differential equation of first type. If the free unknown function in a differential equation is perturbed in someone way, then it is called a *perturbation differential equation of first type*. Similarly, if the the unknown function under derivative is perturbed in some fashion, then it is called a *perturbation differential equation of second type*. The nonlinear differential equation (1.2) is itself in fact an implicit perturbation of the first type corresponding to well-known initial value problems of a first order linear differential equation

$$\left.\begin{array}{l} x'(t) = x(t) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.4}$$

Now consider the perturbed differential equation related to (1.3), viz.,

$$\left.\begin{array}{c} \dfrac{d}{dt}[x(t) - f_2(t,x(t))] = f_1(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.5}$$

In this perturbation of the differential equation (1.5), the term under derivative is perturbed and such type of perturbation is called a *perturbation of second type*. A perturbation of a nonlinear equation which involves the addition or subtraction of a term is called a *linear perturbation* and a perturbation which involves the multiplication or division by a term is called a *quadratic perturbation* of the equations in question. Similarly, if the unknown function in the differential equation (1.1) is perturbed by a function, then it is called an *implicit perturbation* of the differential equation (1.3). Again, an implicit perturbation may be of first or second type. The differential equation (1.5) is a linear perturbation of second type for the differential equation (1.3) whereas the differential equation,

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{x(t)}{f_2(t,x(t))}\right] = f_1(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array}\right\} \tag{1.6}$$

is a quadratic perturbation for the differential equations (1.3) of second type. Similarly, the differential equation

$$\left.\begin{aligned} x'(t) = f_1\Big(t, \int_0^t x(s)\,ds\Big) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.7}$$

is an implicit perturbation of differential equation (1.3) of the first type and the differential equation

$$\left.\begin{aligned} \frac{d}{dt}[f(t,x(t))] = f_1(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.8}$$

is an implicit perturbation of the differential equation (1.3) of second type.

In a similar way, we may have perturbations of nonlinear integral equations. For example, the nonlinear integral equations,

$$x(t) = q(t) + \int_0^t f((s,x(s))\,ds + \int_0^t g((s,x(s))\,ds, \tag{1.9}$$

$$x(t) = q(t) + f((t,x(t)) + \int_0^t g((s,x(s))\,ds, \tag{1.10}$$

and

$$x(t) = \big[f((t,x(t))\big]\left(q(t) + \int_0^t g((s,x(s))\,ds\right), \tag{1.11}$$

are the perturbations of the nonlinear integral equation

$$x(t) = q(t) + \int_0^t g((s,x(s))\,ds. \tag{1.12}$$

Note that integral equation (1.9) is a linear perturbation of second kind, equation (1.10) is a linear perturbation of first type, whereas integral equation (1.13) is a quadratic perturbation of first type for the nonlinear integral equation (1.12). The nonlinear integral equation (1.2) is itself in fact an implicit perturbation of the second type corresponding to the well-known linear integral equation

$$x(t) = q(t) + \int_0^t x(s)\,ds. \tag{1.13}$$

Similarly, the nonlinear integral equations

$$f(t,x(t)) = q(t) + \int_0^t g(s,x(s))\,ds \tag{1.14}$$

and

$$x(t) = q(t) + \int_0^t g((s,f(s,x(s))))\,ds. \tag{1.15}$$

are the implicit perturbations of the integral equation (1.12) of first and second type respectively.

Similarly, there may be other perturbations of nonlinear differential or integral equations or integro-differential equations involving the linear and quadratic perturbations of first as well as second type. Then such type of perturbation is called *perturbation of mixed type* for the nonlinear differential and integral equations. See Dhage and O'Regan [9], Dhage *et al.* [10] and the references therein.

Almost all perturbed nonlinear differential or integral equations are generally tackled with the use of hybrid fixed point theory, or reciprocally, the study of nonlinear perturbed equations is the origin or the motivation for the development of the hybrid fixed point theory in abstract spaces. It is well-known that the inversion of a linearly perturbed differential equations gives rise to the operator equation involving the sum of two operators like $Ax+Bx=x$ and the inversion of a quadratically perturbed differential equations gives rise to the operator equation involving the product of two operators like $AxBx=x$ or $AxBx+Cx=x$ in the appropriate function spaces. Therefore, linear perturbations are usually handled using the hybrid fixed point theory developed on the lines of a fixed point theorem of Krasnoselskii [11], whereas quadratic perturbations are usually handled with the hybrid fixed point theory developed on the lines of a fixed point theorem of Dhage [2]. The hybrid fixed point theory along the first direction is quite old an dates back to Krasnoselskii [11], while the hybrid fixed point theory in the later direction is relatively new and may be found in the works of Dhage [2, 3, 4, 5, 6, 7, 8] etc.

Similarly, the unknown function in the implicit perturbations of first kind and second kind for the nonlinear differential equations is equivalent respectively to the operator equations $x=A(Bx)$ and $Ax=Bx$ in a suitable function space. The studies along the first line has already been exploited in the literature, however, the theory of implicit perturbation of of second kind did not progress much. This is because of the fact that the fixed point theory for the operator equation $Ax=Bx$ requires certain kind of inclusion property of the operators A and B, namely, $A(X)\subset B(X)$ which is difficult to verify in the practical applications to the differential equations.

In this article, we study a quadratic perturbation of a periodic boundary value problem of nonlinear second order ordinary quadratic differential equations by applying a perturbation of first type for existence as well as existence results for extremal solutions under mixed Lipschitz, Carathéodory and monotonic conditions. The main tools used in the study are the hybrid fixed point theorems of Dhage [3, 4, 6, 7]. We claim that the nonlinear differential equation as well as the existence results of this paper are new to the literature on the theory of nonlinear ordinary differential equations.

The rest of the paper is organized as follows: In Section 2 we state the perturbation problem which is to be discussed in the subsequent part of the paper. Section 3 deals with preliminaries and definitions and auxiliary results needed in the sequel. The main existence result is given in Section 4, while the results on extremal solutions are given in Section 5. Finally, in Section 6, an example is presented to illustrate the abstract result developed in Section 4.

2. Second order Periodic Boundary Value Problem

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0,2\pi]$ in $\mathbb{R}$, consider the periodic boundary value problems (in short PBVP) of first order ordinary differential equations

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x(t)) \ \text{ a.e. } t \in J \\ x(0) = x(2\pi),\ x'(0) &= x'(2\pi), \end{aligned}\right\} \tag{2.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ and $g : J \times \mathbb{R} \to \mathbb{R}$.

By a *solution* of the PBVP (2.1) we mean a function $x \in AC^1(J,\mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{d}{dt}\Big(\dfrac{x(t)}{f(t,x(t))}\Big)$ is absolutely continuous on J, and

(ii) x satisfies the equations in (2.1),

where $AC^1(J,\mathbb{R})$ is the space of continuous functions whose first derivative exists and is absolutely continuous real-valued functions defined on J.

When $f(t,x) = 1$ for all $t \in J$ and $x \in \mathbb{R}$, the PBVP (1.1) reduces to a PBVP

$$\left.\begin{aligned} -x''(t) &= g(t,x(t)) \ \text{ a.e. } t \in J \\ x(0) = x(2\pi),\ x'(0) &= x'(2\pi), \end{aligned}\right\} \tag{2.2}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$.

Note that PBVP (2.1) is a quadratic perturbation of second type for the PBVP (2.2) on the closed and bounded interval J. A study of PBVP (1.2) has been made in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela [15], Leela [16], Nieto [17, 18], Yao [19], and the references therein. In this paper, we discuss the PBVP (1.1) for existence as well as for existence of extremal solutions under some suitable conditions of the nonlinearity f and g which thereby generalize several existence results of the PBVP (1.2) proved in the above mentioned papers. Our analysis rely on a nonlinear alternative of Leray-Schauder type (see Dhage [3, 5]) and an algebraic fixed point theorem of Dhage [3] in Banach algebras. Our method of study is to convert the PBVP (1.1) into an equivalent integral equation and apply the hybrid fixed point theorems of Dhage [3, 4, 6, 7] under suitable conditions on the nonlinearities f and g involved in it.

In the following section we describe some basic tools from nonlinear functional analysis which will be used in subsequent part of the paper.

3. Auxiliary Results

In this section, we give some basic definitions and the hybrid fixed point theorems that will be used in the subsequent part of the paper.

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called $\mathscr{D}$-*Lipschitz* if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leqslant \psi(\|x - y\|) \tag{3.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ ($\alpha > 0$), A is called a *Lipschitz* with the Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with the contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a *nonlinear* $\mathscr{D}$*-contraction* on X. Sometimes we call the function ψ a $\mathscr{D}$*-function* of A on X for convenience.

An operator $B : X \to X$ is called *compact* if $\overline{B(S)}$ is a compact subset of X for any $S \subset X$. Similarly $B : X \to X$ is called *totally bounded* if B maps a bounded subset of X into a relatively compact subset of X. Finally $B : X \to X$ is called *completely continuous* operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true, however, both the notions coincide on bounded subsets of X. A nonlinear alternative of Schaefer type recently proved by Dhage [7] is embodied in the following theorem.

THEOREM 3.1. (Dhage[7]). *Let $\mathscr{B}_r(0)$ and $\overline{\mathscr{B}_r(0)}$ be respectively open and closed balls in a Banach algebra X centered at origin 0 and of radius r. Let $A, B : \overline{\mathscr{B}_r(0)} \to X$ be two operators satisfying*

(a) A is Lipschitz with a Lipschitz constant α,

(b) B is compact and continuous, and

(c) $\alpha M < 1$, where $M = \|B(\overline{\mathscr{B}_r(0)})\| := \sup\{\|Bx\| : x \in \overline{\mathscr{B}_r(0)}\}$.

Then either

(i) the equation $\lambda[AxBx] = x$ has a solution for $\lambda = 1$, or

(ii) there exists $u \in X$ such that $\|u\| = r$ satisfying $\lambda[AuBu] = u$ for some $0 < \lambda < 1$.

To exploit the monotonic nature of the nonlinearities involved in a quadratic nonlinear problem, we need some hybrid fixed point theorems from ordered Banach algebras. In the fellowing we state some useful fixed point theorems for the purpose of our study.

A non-empty closed set K in a Banach algebra X is called a *cone* if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geqslant 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be *positive* if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leqslant$ in X as follows. Let $x, y \in X$. Then $x \leqslant y$ if and only if $y - x \in K$. A cone K is called to be *normal* if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leqslant N\|y\|$ for all $x, y \in K$ with $x \leqslant y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [13].

LEMMA 3.1. (Dhage [3]) *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leqslant v_1$ and $u_2 \leqslant v_2$. Then $u_1 u_2 \leqslant v_1 v_2$.*

For any $a,b \in X, a \leqslant b$, the order interval $[a,b]$ is a set in X given by

$$[a,b] = \{x \in X : a \leqslant x \leqslant b\}.$$

DEFINITION 3.1. A mapping $Q : [a,b] \to X$ is said to be *nondecreasing* or *monotone increasing* if $x \leqslant y$ implies $Qx \leqslant Qy$ for all $x,y \in [a,b]$.

We use the following fixed point theorems of Dhage [3, 4, 6] for proving the existence of extremal solutions for the BVP (2.1) under certain monotonicity conditions.

THEOREM 3.2. (Dhage [3]). *Let K be a cone in a Banach algebra X and let $a,b \in X$. Suppose that $A,B : [a,b] \to K$ are two operators such that*

(a) *A is a Lipschitz with the Lipschitz constant α,*

(b) *B is completely continuous,*

(c) *$AxBx \in [a,b]$ for each $x \in [a,b]$, and*

(d) *A and B are nondecreasing.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a,b]$, whenever $\alpha M < 1$, where $M = \|B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.

THEOREM 3.3. (Dhage [8]). *Let K be a cone in a Banach algebra X and let $a,b \in X$. Suppose that $A,B : [a,b] \to K$ are two operators such that*

(a) *A is completely continuous,*

(b) *B is totally bounded,*

(c) *$AxBy \in [a,b]$ for each $x,y \in [a,b]$, and*

(d) *A and B are is nondecreasing.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a,b]$.

THEOREM 3.4. (Dhage [8]). *Let K be a cone in a Banach algebra X and let $a,b \in X$. Suppose that $A,B : [a,b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*

(b) *B is totally bounded,*

(c) *$AxBy \in [a,b]$ for each $x,y \in [a,b]$, and*

(d) *A and B are is nondecreasing.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a,b]$, whenever $\alpha M < 1$, where $M = \|B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.

REMARK 3.1. Note that hypothesis (c) of Theorems 3.2, 3.3, and 3.4 holds if the operators A and B are positive, monotone increasing and there exist elements a and b in X such that $a \leqslant AaBa$ and $AbBb \leqslant b$. Again, each of these hybrid fixed point theorems has some advantages and disadvantages over the others.

In the following sections we prove the main existence results of this paper.

4. Existence Theory

Let $B(J,\mathbb{R})$ denote the space of bounded real-valued functions defined on J. Let $C(J,\mathbb{R})$, denote the space of all continuous real-valued functions defined on J. Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|$$

and

$$(x.y)(t) = (xy)(t) = x(t)y(t)$$

for $t \in J$. Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to above norm and multiplication. By $L^1(J,\mathbb{R})$ we denote the vector space of Lebesgue integrable functions defined on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^{2\pi} |x(t)|\,ds.$$

The following lemma appears in Nieto [17] and which is useful in the study of second order periodic boundary value problems of ordinary differential equations.

LEMMA 4.1. *For any real number $m > 0$ and $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} -x''(t) + m^2 x(t) = \sigma(t)\ a.e.\ t \in J \\ x(0) = x(2\pi), x'(0) = x'(2\pi) \end{aligned}\right\} \tag{4.1}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^{2\pi} G_m(t,s)\sigma(s)\,ds, \tag{4.2}$$

where

$$G_m(t,s) = \begin{cases} \dfrac{1}{2m(e^{2m\pi}-1)}\left[e^{m(t-s)} + e^{m(2\pi-t+s)}\right], & 0 \leqslant s \leqslant t \leqslant 2\pi, \\ \dfrac{1}{2m(e^{2m\pi}-1)}\left[e^{m(s-t)} + e^{m(2\pi-s+t)}\right], & 0 \leqslant t < s \leqslant 2\pi. \end{cases} \tag{4.3}$$

Notice that the Green's function G_m is continuous and nonnegative on $J \times J$ and the numbers

$$\alpha = \min\{\,|G_m(t,s)|\,:\,t,s \in [0,2\pi]\,\} = \frac{e^{m\pi}}{m(e^{2m\pi}-1)}$$

and

$$\beta = \max\{\,|G_m(t,s)|\,:\,t,s \in [0,2\pi]\,\} = \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}$$

exist for all positive real number m.

We need the following definition in the sequel.

DEFINITION 4.1. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \mapsto \beta(t,x)$ is continuous almost everywhere for $t \in J$.

Again, a Carathéodory function $\beta(t,x)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \leqslant h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leqslant r$.

Finally, a Carathéodory function $\beta(t,x)$ is called $L^1_{\mathbb{R}}$-Carathéodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \leqslant h(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$. For convenience, the function h is referred to as a bound function of β.

We will use the following hypotheses in the sequel.

(A_0) The functions $t \mapsto f(t,x)$, $t \mapsto f_t(t,x)$ and $t \mapsto f_x(t,x)$ are periodic of period 2π for all $x \in \mathbb{R}$.

(A_1) The function $x \mapsto \dfrac{x}{f(0,x)}$ is injective in $\mathbb{R}$.

(A_2) $f(0,x) \neq x f_x(0,x)$ for all $x \in \mathbb{R}$, where $f_x(0,x) = \left.\dfrac{\partial f(t,x)}{\partial x}\right|_{t=0}$.

(A_3) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A_4) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell \in C(J,\mathbb{R})$ such that

$$|f(t,x) - f(t,y)| \leqslant \ell(t)\,|x-y|$$

for all $t \in J$ and $x,y \in \mathbb{R}$. Moreover, we assume that $L = \max_{t \in J} \ell(t)$.

(A_5) The function g is Carathéodory.

REMARK 4.1. Note that hypotheses (A_3) through (A_5) are much common in the literature on the theory of nonlinear differential equations. Similarly, there do exist functions satisfying the hypotheses (A_0) through (A_2). Indeed, it is easy to verify that the function $f : J\times\mathbb{R}\to\mathbb{R}$ defined by

$$f(t,x)=\begin{cases} a+bx & \text{if } x\geqslant 0,\\ a & \text{if } x<0,\end{cases}$$

for some $a,b\in\mathbb{R}$ with $a>0$ and $b>0$ satisfies the hypotheses (A_0)-(A_4) mentioned above.

Now consider the linear perturbation of the PBVP (2.1) of first type,

$$\left.\begin{aligned} -\Big(\frac{x(t)}{f(t,x(t))}\Big)''+m^2\Big(\frac{x(t)}{f(t,x(t))}\Big)=g_m(t,x(t))\ \text{a.e. } t\in J,\\ x(0)=x(2\pi),\ x'(0)=x'(2\pi),\end{aligned}\right\} \tag{4.4}$$

where $m>0$ is a real number and the function $g_m : J\times\mathbb{R}\to\mathbb{R}$ is defined by

$$g_m(t,x)=g(t,x)+m^2\Big(\frac{x}{f(t,x)}\Big). \tag{4.5}$$

REMARK 4.2. Note that the PBVP (2.1) is equivalent to the PBVP (4.4) and a solution of the PBVP (2.1) is a solution for the PBVP (4.4) defined on J and vice versa.

REMARK 4.3. Assume that hypotheses (A_3) and (A_5) hold. Then the function g_m defined by (4.5) is Carathéodory on $J\times\mathbb{R}$.

LEMMA 4.2. *Assume that hypothesis (A_0)-(A_4) holds. Then for any real number $m>0$ and $g_m(.,x(.))\in L^1(J,\mathbb{R})$, x is a solution to the differential equation* (4.4) *if and only if it is a solution of the integral equation*

$$x(t)=\big[f(t,x(t))\big]\left(\int_0^{2\pi}G_m(t,s)g_m(s,x(s))\,ds\right), \tag{4.6}$$

where the Green's function $G_m(t,s)$ is defined by (4.3).

Proof. Let $y(t)=\dfrac{x(t)}{f(t,x(t))}$. Since $f(t,x)$ is periodic in t of period 2π for all $x\in\mathbb{R}$, we have

$$y(0)=\frac{x(0)}{f(0,x(0))}=\frac{x(2\pi)}{f(2\pi,x(2\pi))}=y(2\pi).$$

Similarly, we have

$$y'(0)=\frac{f(0,x(0))x'(0)-x(0)\big[f_t(0,x(0))+f_x(0,x(0))x'(0)\big]}{[f(0,x(0))]^2}$$

$$= \frac{f(2\pi,x(2\pi))x'(2\pi) - x(2\pi)\big[f_t(2\pi,x(2\pi)) + f_x(2\pi,x(2\pi))x'(2\pi)\big]}{[f(2\pi,x(2\pi))]^2}$$
$$= y'(2\pi).$$

Now an application of Lemma 4.1 yields that the solution to differential equation (4.4) is the solution to integral equation (4.6). Conversely, suppose that x is any solution to the integral equation (4.6), then

$$y(0) = \frac{x(0)}{f(0,x(0))} = y(2\pi) = \frac{x(2\pi)}{f(2\pi,x(2\pi))} = \frac{x(2\pi)}{f(0,x(2\pi))}.$$

Since the function $x \mapsto \dfrac{x}{f(0,x)}$ is injective, one has $x(0) = x(2\pi)$. Again, assume that $y'(0) = y'(2\pi)$ and $x(0) = x(2\pi)$. Then, we obtain

$$\begin{aligned}
&\frac{f(0,x(0))x'(0) - x(0)\big[f_t(0,x(0)) + f_x(0,x(0))x'(0)\big]}{[f(0,x(0))]^2}\\
&\quad = \frac{f(2\pi,x(2\pi))x'(2\pi) - x(2\pi)\big[f_t(2\pi,x(2\pi)) + f_x(2\pi,x(2\pi))x'(2\pi)\big]}{[f(2\pi,x(2\pi))]^2}\\
&\quad = \frac{f(0,x(0))x'(2\pi) - x(0)\big[f_t(0,x(0)) + f_x(0,x(0))x'(2\pi)\big]}{[f(0,x(0))]^2}
\end{aligned}$$

which implies that

$$\big[f(0,x(0)) - x(0)f_x(0,x(0))\big]x'(0) = \big[f(0,x(0)) - x(0)f_x(0,x(0))\big]x'(2\pi). \tag{4.7}$$

Since $f(0,x) - xf_x(0,x) \neq 0$ for all $x \in \mathbb{R}$, one has $x'(0) = x'(2\pi)$. Therefore, x is a solution to PBVP (2.1). The proof is complete.

We make use of the following hypothesis in the sequel.

(A_6) There exists a continuous and nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $\gamma \in L^1(J,\mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ satisfying

$$|g_m(t,x)| \leqslant \gamma(t)\psi\big(|x|\big) \text{ a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

THEOREM 4.1. *Assume that the hypotheses* (A_0) - (A_2) *and* (A_4) - (A_6) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{F_0\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r)}{1 - L\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r)}, \tag{4.8}$$

where

$$L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r) < 1$$

and $F_0 = \sup_{t\in[0,2\pi]} |f(t,0)|$. *Then the PBVP* (2.1) *has a solution defined on* J.

Proof. Let $X = C(J,\mathbb{R})$. Define an open ball $\mathscr{B}_r(0)$ in X centered at origin 0 of radius r, where the real number r satisfies the inequality (4.8). Define two mappings A and B on $\overline{\mathscr{B}_r(0)}$ by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{4.9}$$

and

$$Bx(t) = \int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds, \quad t \in J. \tag{4.10}$$

Obviously A and B define the operators $A,B : \overline{\mathscr{B}_r(0)} \to X$. Then the integral equation (4.6) is equivalent to the operator equation

$$Ax(t)\,Bx(t) - x(t), \quad t \in J. \tag{4.11}$$

We shall show that the operators A and B satisfy all the hypotheses of Theorem 3.1.

We first show that A is a Lipschitz on $\overline{\mathscr{B}_r(0)}$. Let $x,y \in X$. Then by (A_3),

$$\begin{aligned}|Ax(t) - Ay(t)| &\leqslant |f(t,x(t)) - f(t,y(t))|\\ &\leqslant \ell(t)\,|x(t) - y(t)|\\ &\leqslant L\,\|x - y\|\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \leqslant L\|x - y\|$$

for all $x,y \in \overline{\mathscr{B}_r(0)}$. So A is a Lipschitz on $\overline{\mathscr{B}_r(0)}$ with the Lipschitz constant L. Next we show that B is completely continuous on X. Using the standard arguments as in Granas et. al. [12], it is shown that B is a continuous operator on $\overline{\mathscr{B}_r(0)}$. We shall show that $B(\overline{\mathscr{B}_r(0)})$ is a uniformly bounded and equicontinuous set in X. Let $x \in \overline{\mathscr{B}_r(0)}$ be arbitrary. Since g is Carathéodory, we have

$$\begin{aligned}|Bx(t)| &\leqslant \left|\int_0^{2\pi} G_k(t,s)g_m(s,x(s))\,ds\right|\\ &\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\int_0^{2\pi}\left[\gamma(s)\,\psi(|x(s)|)\right]ds\\ &\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\int_0^{2\pi}\gamma(s)\,\psi(|x(s)|)\,ds\\ &\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\,\psi(r).\end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leqslant M$ for all $x \in \overline{\mathscr{B}_r(0)}$, where

$$M = \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\,\psi(r).$$

This shows that $B(\overline{\mathscr{B}_r(0)})$ is a uniformly bounded set in X. Next, we show that $B(\overline{\mathscr{B}_r(0)})$ is an equi-continuous set in X. Let $x \in \overline{\mathscr{B}}_r(0)$ be arbitrary. Then for any $t_1,t_2 \in J$, one has

$$|Bx(t_1) - Bx(t_1)|$$

$$\begin{aligned}
&\leqslant \int_0^{2\pi} |G_m(t_1,s)-G_m(t_2,s)|\,|g_m(s,x(s))|\,ds \\
&\leqslant \int_0^{2\pi} |G_m(t_1,s)-G_m(t_2,s)|\,\gamma(s)\,\psi(|x(s)|)\,ds \\
&\leqslant \int_0^{2\pi} |G_m(t_1,s)-G_m(t_2,s)|\,\gamma(s)\,\psi(r)\,ds \\
&\leqslant \left(\int_0^{2\pi} |G_m(t_1,s)-G_m(t_2,s)|^2\,ds\right)^{1/2}\left(\int_0^{2\pi}|\gamma(s)|^2\,ds\right)^{1/2}\psi(r). \qquad (4.12)
\end{aligned}$$

Hence for all $t_1,t_2 \in J$,

$$|Bx(t_1)-Bx(t_1)| \to 0 \text{ as } t_1 \to t_2$$

uniformly for all $x \in \overline{\mathscr{B}_r(0)}$. Therefore, $B(\overline{\mathscr{B}_r(0)})$ is a equi-continuous set in X. Now $B(\overline{\mathscr{B}_r(0)})$ is a uniformly bounded and equi-continuous set in X, so it is compact by Arzelà-Ascoli theorem. As a result B is a compact and continuous operator on $\overline{\mathscr{B}_r(0)}$. Thus, all the conditions of Theorem 3.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be a solution to the operator equation $\lambda\,[AuBu] = u$ for some $0<\lambda<1$ satisfying $\|u\| = r$. Then we have, for any $\lambda \in (0,1)$,

$$u(t) = \lambda\,\big[f(t,x(t))\big]\left(\int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds\right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|u(t)| &\leqslant \lambda\,|f(t,u(t))|\left(\left|\int_0^{2\pi} G_m(t,s)g_m(s,u(s))\,ds\right|\right) \\
&\leqslant \lambda\Big(\big|f(t,x(t))-f(t,0)\big| + \big|f(t,0)\big|\Big) \\
&\qquad \times\left(\int_0^{2\pi} G_m(t,s)|g_m(s,u(s))|\,ds\right) \\
&\leqslant [\ell(t)\,|u(t)|+F_0]\left(\int_0^{2\pi}\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]|g_m(s,u(s))|\,ds\right) \\
&\leqslant L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]|u(t)|\left(\int_0^{2\pi}\gamma(s)\psi(|u(s)|)\,ds\right) \\
&\qquad + F_0\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\left(\int_0^{2\pi}\gamma(s)\psi(|u(s|)\,ds\right) \\
&\leqslant L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(\|u\|)|u(t)| \\
&\qquad + F_0\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(\|u\|). \qquad (4.13)
\end{aligned}$$

Taking the supremum in the above inequality (4.13),

$$\|u\| \leqslant \frac{F_0\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(\|u\|)}{1-L\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(\|u\|)}.$$

Substituting $\|u\| = r$ in above inequality,

$$r \leqslant \frac{F_0\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r)}{1-L\left[\dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r)}.$$

This is a contradiction to to inequality (4.9). Hence the conclusion (ii) of Corollary 3.1 does not hold. Therefore, the operator equation $AxBx = x$ and consequently the PBVP (2.1) has a solution defined on J. This completes the proof.

REMARK 4.4. We note that in Theorem 4.1, we only require for the hypotheses (A_1) and (A_2) to hold in the interval $[-r,r]$.

It is desirable that one should be interested to know the realistic behavior of the solutions for a given dynamical system in question. Therefore, in the following section we prove the existence of positive solutions for the PBVP (2.1) defined on J.

5. Existence of Extremal Positive Solutions

We equip the space $C(J,\mathbb{R})$ with the order relation $\leqslant$ with the help of the cone defined by

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geqslant 0, \forall t \in J\}. \tag{5.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. We need the following definitions in the sequel.

DEFINITION 5.1. A function $a \in AC^1(J,\mathbb{R})$ is called a lower solution of the PBVP (2.1) defined on J if the function $t \mapsto \dfrac{d}{dt}\Big(\dfrac{a(t)}{f(t,a(t))}\Big)$ is absolutely continuous and satisfies

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\Big[\frac{a(t)}{f(t,a(t))}\Big] \leqslant g(t,a(t)) \quad \text{a.e. } t \in J \\ a(0) \leqslant a(2\pi),\ a'(0) = a'(2\pi). \end{aligned}\right\}$$

Again, a function $b \in AC^1(J,\mathbb{R})$ is called an upper solution of the PBVP (2.1) defined on J if the function $t \mapsto \frac{d}{dt}\Big(\frac{b(t)}{f(t,b(t))}\Big)$ is absolutely continuous and satisfies

$$\left.\begin{aligned} -\frac{d^2}{dt^2}\Big[\frac{b(t)}{f(t,b(t))}\Big] &\geqslant g(t,b(t)) \ \text{ a.e. } t \in J \\ b(0) \geqslant b(2\pi),\ b'(0) &= b'(2\pi). \end{aligned}\right\}$$

A solution to the PBVP (2.1) is a lower as well as an upper solution of the PBVP (2.1) defined on J.

DEFINITION 5.2. A solution x_M of the PBVP (2.1) is said to be maximal if for any other solution x to the PBVP (2.1) one has $x(t) \leqslant x_M(t)$, for all $t \in J$. Again, a solution x_m of the PBVP (2.1) is said to be minimal if $x_m(t) \leqslant x(t)$, for all $t \in J$, where x is any solution of the PBVP (2.1) on J.

REMARK 5.1. The upper and lower solutions of the PBVP (2.1) are respectively the upper and lower solutions of the PBVP (4.4) and vice-versa. Similarly the maximal and minimal solutions of the PBVP (2.1) are respectively the upper and lower solutions of the PBVP (4.4) and vice-versa.

5.1. Carathéodory case

We need the following definition in the sequel.

DEFINITION 5.3. A function $f : \mathbb{R} \to \mathbb{R}$ is called nondecreasing if $f(x) \leqslant f(y)$ for all $x, y \in \mathbb{R}$ for which $x \leqslant y$. Similarly, $f(x)$ is called increasing in x if $f(x) < f(y)$ for all $x, y \in \mathbb{R}$ for which $x < y$.

We consider the following set of assumptions.

(B_0) The function g_m is nonnegative, i.e., $g_m : J \times \mathbb{R} \to \mathbb{R}^+$.
(B_1) The PBVP (2.1) has a lower solution a and an upper solution b defined on J with $a \leqslant b$.
(B_2) The function $x \mapsto \dfrac{x}{f(0,x)}$ is increasing and the hypothesis (A_2) holds in the interval $\big[\min_{t\in J} a(t), \max_{t\in J} b(t)\big]$.
(B_3) The functions $f(t,x)$ and $g_m(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.
(B_4) The function $h : J \to \mathbb{R}$ defined by

$$h(t) = g_m(t,b(t)),$$

is Lebesgue integrable.

We remark that hypothesis (B_4) holds in particular if f is continuous and g is L^1-Carathéodory on $J \times \mathbb{R}$.

REMARK 5.2. If the hypothesis (B_1) -(B_3) holds, then the map $x \mapsto \dfrac{x}{f(0,x)}$ is injective and

$$\frac{a(0)}{f(0,a(0))} \leqslant \frac{a(2\pi)}{f(2\pi,a(2\pi))}$$

and

$$\frac{b(0)}{f(0,b(0))} \geqslant \frac{b(2\pi)}{f(2\pi,b(2\pi))}$$

which guarantee that $a \leqslant AaBa$ and $AbBa \leqslant b$.

REMARK 5.3. Assume that hypotheses (B_0) through (B_4) hold. Then the function $t \mapsto g_m(t,x(t))$ is Lebesgue integrable defined on J and

$$|g_m(t,x(t))| = g_m(t,x(t)) \leqslant h(t) \quad \text{a.e. } t \in J,$$

for all $x \in [a,b]$.

THEOREM 5.1. *Suppose that the assumptions (A_0) through (A_5) and (B_0) through (B_4) hold. Further if*

$$L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1} < 1,$$

where h is given in Remark 5.1 *and* $L = \max_{t\in J}\ell(t)$, *then the PBVP* (2.1) *has a minimal and a maximal positive solution defined on J.*

Proof. Now the PBVP (2.1) is equivalent to integral equation (4.6) defined on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (4.9) and (4.10) respectively. Then integral equation (4.6) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_1) implies $A,B : [a,b] \to K$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitz with the Lipschitz constant L and B is completely continuous operator on $[a,b]$. Again, the hypothesis (B_2) implies that A and B are nondecreasing on $[a,b]$. To see this, let $x,y \in [a,b]$ be such that $x \leqslant y$. Then by (B_2),

$$Ax(t) = f(t,x(t)) \leqslant f(t,y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned}
Bx(t) &= \int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds \\
&\leqslant \int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds \\
&= By(t)
\end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[a,b]$. Again, Lemma 3.1, Remark 5.2 and hypothesis (B_3) together imply that

$$a(t) \leqslant [f(t,a(t))]\left(\int_0^{2\pi} G_m(t,s)g_m(s,a(s)))\,ds\right)$$

$$\leqslant [f(t,x(t))]\left(\int_0^{2\pi} G_m(t,s)g_m(s,x(s))\,ds\right)$$
$$\leqslant [f(t,b(t))]\left(\int_0^{2\pi} G_m(t,s)g_m(s,b(s)))\,ds\right)$$
$$\leqslant b(t),$$

for all $t \in J$ and $x \in [a,b]$. As a result $a(t) \leqslant Ax(t)Bx(t) \leqslant b(t)$, for all $t \in J$ and $x \in [a,b]$. Hence, $AxBx \in [a,b]$ for all $x \in [a,b]$. Again,

$$\begin{aligned} M &= \|B([a,b])\| \\ &= \sup\{\|Bx\| : x \in [a,b]\} \\ &\leqslant \sup\left\{\sup_{t\in J}\int_0^{2\pi} G_m(t,s)|g_m(s,x(s))|\,ds \;\middle|\; x \in [a,b]\right\} \\ &\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\int_0^{2\pi} h(s)\,ds \\ &= \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1}. \end{aligned}$$

Since $\alpha M \leqslant L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1} < 1$, we apply Theorem 3.2 to the operator equation $AxBx = x$ to yield that the PBVP (2.1) has a minimal and a maximal positive solution defined on J. This completes the proof.

5.2. Discontinuous case

We need the following definition in the sequel.

DEFINITION 5.4. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Chandrabhan if
(i) $t \mapsto \beta(t,x(t))$ is measurable for each $x \in C(J,\mathbb{R})$, and
(ii) $x \mapsto \beta(t,x)$ is nondecreasing almost everywhere for $t \in J$.

Again, a Chandrabhan function $\beta(t,x)$ is called L^1-Chandrabhan if
(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \leqslant h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leqslant r$.

Finally, a Chandrabhan function $\beta(t,x)$ is called $L^1_{\mathbb{R}}$-Chandrabhan if
(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \leqslant h(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$. For convenience, the function h is referred to as a bound function of β.

We consider the following hypotheses in the sequel.

(C_1) The function $f(t,x)$ is nondecreasing in x almost everywhere for $t \in J$.
(C_2) The function g_m defined by (4.5) is Chandrabhan.

THEOREM 5.2. *Suppose that the hypotheses* (A_0), (A_2), (A_3), (B_0)-(B_2) *and* (C_1)-(C_2) *hold. Then the PBVP* (2.1) *has a minimal and a maximal positive solution defined on* J.

Proof. Now the PBVP (2.1) is equivalent to integral equation (4.6) defined on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (4.9) and (4.10) respectively. Then the integral equation (4.6) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A,B : [a,b] \to K$. Note that the conditions (B_0) and (B_2) provides $a \leqslant AaBa$ and $AbBb \leqslant b$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X.

Step I: First we show that A is completely continuous on $[a,b]$. Now the cone K in X is normal, so the order interval $[a,b]$ is norm-bounded in X. Hence there exists a constant $r > 0$ such that $\|x\| \leqslant r$ for all $x \in [a,b]$. As f is continuous on compact $J \times [-r,r]$, it attains its maximum, say M. Therefore for any subset S of $[a,b]$ we have

$$\begin{aligned}
\|A(S)\|_{\mathscr{P}} &= \sup\{\|Ax\| : x \in S\} \\
&= \sup\Big\{ \sup_{t\in J} |f(t,x(t))| : x \in S \Big\} \\
&\leqslant \sup\Big\{ \sup_{t\in J} |f(t,x)| : x \in [-r,r] \Big\} \\
&\leqslant M.
\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t,x)$ is uniformly continuous on $[0,2\pi] \times [-r,r]$. Therefore for any $t,\tau \in [0,2\pi]$ we have

$$|f(t,x) - f(\tau,x)| \to 0 \text{ as } t \to \tau$$

uniformly for all $x \in [-r,r]$. Similarly for any $x,y \in [-r,r]$

$$|f(t,x) - f(t,y)| \to 0 \text{ as } x \to y$$

uniformly for all $t \in [0,2\pi]$. Hence, for any $t,\tau \in [0,2\pi]$ and for any $x \in S$ one has

$$\begin{aligned}
|Ax(t) - Ax(\tau)| &= |f(t,x(t)) - f(\tau,x(\tau))| \\
&\leqslant |f(t,x(t)) - f(\tau,x(t))| + |f(\tau,x(t)) - f(\tau,x(\tau))| \\
&\to 0 \quad \text{as} \quad t \to \tau
\end{aligned}$$

uniformly for all $x \in S$. This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a,b]$.

Step II: Next we show that B is totally bounded operator on $[a,b]$. To finish, we shall show that $B(S)$ is uniformly bounded ad equi-continuous set in X for any subset S of $[a,b]$. Since the cone K in X is normal, the order interval $[a,b]$ is norm-bounded. Let $y \in B(S)$ be arbitrary. Then,

$$y(t) = \int_0^{2\pi} G_m(t,s) g_m(s,x(s))\, ds$$

for some $x \in S$. By hypothesis (B_2), one has

$$\begin{aligned}
|y(t)| &= \int_0^{2\pi} G_m(t,s)|g_m(s,x(s))|\, ds \\
&= \int_0^{2\pi} G_m(t,s)|g_m(s,b(s))|\, ds \\
&\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \int_0^{2\pi} h(s)\, ds \\
&\leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \|h\|_{L^1},
\end{aligned}$$

where h is given in hypothesis (B_4). Taking the supremum over t,

$$\|y\| \leqslant \left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right] \|h\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Again, for any $t_1, t_2 \in J$, one has

$$\begin{aligned}
&|Bx(t_1) - Bx(t_1)| \\
&\quad \leqslant \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\, |g_m(s,x(s))|\, ds \\
&\quad \leqslant \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\, h(s)\, ds \\
&\quad \leqslant \left(\int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|^2\, ds\right)^{1/2} \left(\int_0^{2\pi} |h(s)|^2\, ds\right)^{1/2}. \qquad (5.2)
\end{aligned}$$

Hence for all $t_1, t_2 \in J$,

$$|Bx(t_1) - Bx(t_1)| \to 0 \text{ as } t_1 \to t_2,$$

uniformly for all $x \in S$. This shows that $B(S)$ is a equi-continuous set of functions in $[a,b]$ for all $S \subset [a,b]$. Now $B(S)$ is a uniformly bounded and equi-continuous, so it is totally bounded by Arzelà-Ascoli theorem. Thus all the conditions of Theorem 3.3 are satisfied and hence an application of it yields that the PBVP (2.1) has a maximal and a minimal positive solution defined on J.

THEOREM 5.3. *Suppose that the assumptions* (A_0), (A_2), (A_4), (B_0)-(B_2) *and* (C_1)-(C_2) *hold. Further if*

$$L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1}<1,$$

where h *is given in Remark* 5.1 *and* $L=\max_{t\in J}\ell(t)$, *then the PBVP* (2.1) *has a minimal and a maximal positive solution defined on* J.

Proof. Now the PBVP (2.1) is equivalent to integral equation (4.6) defined on J. Let $X=C(J,\mathbb{R})$. Define two operators A and B on X by (4.9) and (4.10) respectively. Then integral equation (4.6) is transformed into an operator equation $Ax(t)\,Bx(t)=x(t)$ in a Banach algebra X. Notice that hypothesis (B_0) implies $A,B:[a,b]\to K$. Note also that the conditions (B_0) and (B_2) provides that $a\leqslant AaBa$ and $AbBb\leqslant b$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3.1 and Theorem 3.4 that the operator A is a Lipschitz with a Lipschitz constant $\alpha=L$ and B is totally bounded with bound

$$M=\|B([a,b])\|=\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1}.$$

Furthermore,

$$\alpha M=L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|h\|_{L^1}<1,$$

so the desired conclusion follows by an application of Theorem 3.4.

6. An Example

Let $a,b\in\mathbb{R}$, $a>0$, $b>0$. Define a function $f:J\times\mathbb{R}\to\mathbb{R}$ by

$$f(t,x)=\begin{cases}a+bx & \text{if } x\geqslant 0,\\ a & \text{if } x<0.\end{cases}$$

Given a closed and bounded interval $J=[0,2\pi]$ of the real line $\mathbb{R}$, consider the PBVP of ordinary second order differential equation,

$$\left.\begin{aligned}-\frac{d^2}{dt^2}\left[\frac{x(t)}{f(t,x(t))}\right]=-\left[\frac{x(t)}{f(t,x(t))}\right]+\frac{t\,x^2(t)}{\pi^2[1+x^2(t)]}\ \text{a.e. } t\in J\\ x(0)=x(2\pi),\ x'(0)=x'(2\pi).\end{aligned}\right\}\tag{6.1}$$

Here,

$$g(t,x)=-\left[\frac{x}{f(t,x(t))}\right]+\frac{t\,x^2}{\pi^2[1+x^2]}$$

for all $t \in J$ and $x \in \mathbb{R}_+$. Taking $m = 1$, we obtain

$$g_m(t,x) = g_1(t,x) = \frac{tx^2}{\pi^2[1+x^2]} > 0$$

for all $t \in J$ and $x \in \mathbb{R}_+$.

Next, the function f defines a mapping $f : J \times \mathbb{R} \to \mathbb{R}_+$. It is easy to verify that f is continuous and satisfies the hypotheses (A_0) and (A_4) in view of Remark 4.1. Further, f is Lipschitz on $J \times \mathbb{R}$ with the Lipschitz function $\ell(t) = b$ for all $t \in J$ and so, $L = b$.

Now,

$$|g_m(t,x)| = |g_1(t,x)| = \left| \frac{tx^2}{\pi^2[1+x^2]} \right| = \frac{t}{\pi^2} = \gamma(t)\psi(|x|)$$

for all $t \in J$ and $x \in \mathbb{R}_+$, where $\gamma(t) = \frac{t}{\pi^2}$ and $\psi(r) = 1$ for all $r > 0$. Hence,

$$\|\gamma\|_{L^1} = \frac{1}{\pi^2}\int_0^{2\pi} t\,dt = 2.$$

Again,

$$L\left[\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\right]\|\gamma\|_{L^1}\psi(r) = b\frac{e^{2\pi}+1}{(e^{2\pi}-1)}.$$

Thus,

$$\text{if } \; b < \frac{e^{2\pi}-1}{(e^{2\pi}+1)}, \quad \text{then } \; b\frac{e^{2\pi}+1}{(e^{2\pi}-1)} < 1,$$

and consequently, by Theorem 4.1, the PBVP (6.1) has a positive solution defined on J in a closed ball $\overline{\mathscr{B}_r(0)}$, where the real number $r > 0$ satisfies the inequality

$$r > \frac{a(e^{2\pi}+1)}{(e^{2\pi}-1) - b(e^{2\pi}+1)}.$$

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(Received April 25, 2010)

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