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TRAJECTORY ATTRACTORS OF ENERGY BALANCE CLIMATE MODELS WITH BIO–FEEDBACK

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Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

(*Communicated by J.-M. Rakotoson*)

Abstract. Motivated by coupling an energy balance climate model and a two-species competition model for the bio-sphere, one is led to the study of functional reaction-diffusion equations with memory and a nonlocal Volterra operator. The existence of a trajectory attractor is established. The work is motivated by similar studies in [12] for a energy balance model with latent heat flux and uses techniques developed in [11] and [12]. It is a continuation of [18], where an abstract global existence and boundedness result was established.

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1. Introduction

This paper establishes the existence of a trajectory attractor for the nonnegative solutions of a family of reaction-diffusion equation with degenerate diffusion. Both reaction terms depend on a nonlocal Volterra operator, and one of them additionally is set-valued and depends on a memory term. Such problems arise in the context of energy balance climate models.

Energy balance climate models describe the evolution of a, say, ten-year mean of temperature u in Kelvin by employing the balance equation for the heat fluxes involved and modeling the ten-year mean of the horizontal heat flux as a diffusion operator. A bio-feedback is introduced in terms of a Volterra operator $V = V(u)$, which is in a paradigmatic daisy world scenario, e.g., the solution of an initial value problem of two-species diffusion competition system with u as a parameter (cf. [18] and section 3 for brief outlines). The resulting reaction-diffusion problem is

$$\begin{cases} c(x)\partial_t u - \nabla\cdot[k(x)\,|\nabla u|^{p-2}\nabla u] + g(u,V(u)) \\ \qquad\qquad \in F(t,x,u,\overline{u},V(u)(t)),\ t>0,\ x\in M, \\ \overline{u}(t,x) := \int_{-T}^{0}\beta(s,x)u(t+s,x)\,ds,\ t>0,\ x\in M, \\ u(s,x) = u_0(s,x),\ -T\leqslant s\leqslant 0,\ x\in M. \end{cases} \tag{1.1}$$

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Here M stands for the earth's surface, e.g. the unit sphere S^2, u is the climate indicator, a ten-year mean of temperature, the degenerate diffusion operator ($p > 2$) approximates the horizontal energy flux, $c(x)u$ stands for the sensible heat flux, g for the emitted radiation flux, and F for the absorbed radiation flux. The absorbed radiation flux is given as incoming solar radiation flux times co-albedo (1-relative reflexivity) with the co-albedo much lower over ice- and snow-covered regions than elsewhere. The continental ice sheets (e.g. in Greenland) cannot be modeled by u, but by a long-term average $\overline{u}$ of u, which explains the dependence of F on $\overline{u}$, the memory term of the system. Moreover, the co-albedo as a function of u is discontinuous at the snow-line, which leads to a set-valued F.

J. Ildefonso Díaz has made many fundamental contributions to the mathematical analysis of climate models, and the reader is referred to [1], [3], [9], [11], [12], [13], [14], [15], and the references therein. The present paper is motivated by my collaboration with him on climate issues and our collaboration with Lourdes Tello, cf. [11] and [12]. The techniques developed in these two papers are employed here. Both papers are devoted to establishing the mathematical foundations for an energy balance model that accounts for the latent energy flux, which dramatically changes on very long time scales (accumulation of continental ice during an ice-age). The mathematical questions studied in this paper arise from incorporating a bio-feedback into an energy balance model. Such an effect is relevant on a much shorter time-scale, therefore variations of the latent energy flux are neglected in (1.1) as in most energy balance models.

Global existence of nonnegative solutions and a priori bounds for (1.1) have been recently established in [18] in an abstract setting which applies immediately to the case $c \equiv 1$ under hypotheses stated later. The somewhat more general and climatologically relevant case of a spatially varying c (land-water distribution) requires certain straightforward modifications which can be verified by inspecting the proofs of Theorems 3.1 and 4.1 in [18]. The approach in dealing with the set-valued right-hand side goes back to [11] where we used an regularization technique based on the approximate selection theorem for upper semi-continuous set-valued mappings. The presence of the nonlocal operator V requires significant modifications as the proofs in [18] show. It should be noted that, as for all Budyko-type models, one cannot expect unique solvability for (1.1). In fact, the ten-year average of snow-cover is determined by sub-scale processes not resolved in an energy balance model, and cannot be forecasted based on a ten-year mean of temperature. Therefore the compact interval-valued function F accounts for all possible values of the co-albedo for given data $(t, x, \mathbf{y}, V(u))$.

The lack of uniqueness suggests to study the existence of a trajectory attractor or alternatively a pullback attractor. Following [18] and [12], we deal here with the trajectory attractor, a concept which goes back to work by Sell [23] and V.V. Chepyzhov, M.I. Vishik [4], [5], cf. also [24] and [6]. More precisely, write F in (1.1) as $Q(t,x)F_0(x,u,\overline{u},V(u))$ with Q the incoming solar radiation flux and F_0 the co-albedo. Q depends on t due to seasons and variation of the earth's orbit (Milankovitch theory), and is quasi- or more general almost periodic. Hence considering just the nonnegative solutions of (1.1) does not yield an invariant set, rather a suitable invariant set contains the union of the sets of nonnegative solutions of (1.1) with Q in F replaced by functions $\tilde{Q}$ and $\tilde{Q}$ varying in the hull of Q. Our main result establishes the existence of a trajectory attractor in this setting.

The rest of the paper is organized as follows. Section 2 briefly describes some preliminary material, section 3 states the hypotheses and discusses an existence result following [18], and section 4 establishes the main result, the existence of a trajectory attractor for (1.1).

2. Preliminaries

We very briefly outline some of the basic concepts used in this paper and refer to [25] for details.

Let X be a real Banach space. The *upper semi-inner product*

$$(x,y)_+ : X \times X \to \mathbb{R}$$

is defined by

$$(x,y)_+ := \lim_{h\downarrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}, \quad x,\ y \in X.$$

In particular, the upper semi-inner product on $C(M)$ is given by

$$(\phi,\theta)_+ = \begin{cases} \|\phi\|_\infty \max\{\theta(x)\operatorname{sgn}(\phi(x)) : x \in M, |\phi(x)| = \|\phi\|_\infty\}, & \|\phi\|_\infty \neq 0, \\ 0, & \|\phi\|_\infty = 0 \end{cases} \tag{2.1}$$

for $\theta,\phi \in C(M)$.

An operator $A : X \supseteq D \to X$ is called *accretive*, iff $(x_1 - x_2, Ax_1 - Ax_2)_+ \geqslant 0$ for all $x_1, x_2 \in D$. An accretive operator A is *called m-accretive*, iff $A + \lambda\mathrm{Id}$ is onto for all $\lambda \in (0,\infty)$.

Consider

$$\dot{U}(t) + A\,U(t) = Z(t), \ \ t \in \operatorname{int}(I) \tag{2.2}$$

for $I := [a,b]$, $a < b$, $A : X \supseteq D \to X$ m-accretive, and $Z \in L^1(I,X)$. Let $\varepsilon > 0$. The expression

$$((t_0,t_1,\ldots,t_n),(Z_1,\ldots,Z_n)) \in [a,b]^{n+1} \times X^n$$

is called *an* ε*-discretization of* (2.2) iff

$$a \leqslant t_0 < t_1 < \ldots < t_n \leqslant b, \quad t_0 - a \leqslant \varepsilon,$$
$$t_j - t_{j-1} \leqslant \varepsilon \quad \text{for} \quad 1 \leqslant j \leqslant n, \quad b - t_n \leqslant \varepsilon,$$
$$\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\|Z(t) - Z_j\right\| \, dt < \varepsilon.$$

Let $((t_0,t_1,\ldots,t_n),(Z_1,\ldots,Z_n)) \in [a,b]^{n+1} \times X^n$ be an ε-discretization of (2.2). The step function $\sigma : [t_0,t_n] \to D$ ($\sigma|(t_{j-1},t_j]$ constant) is called a *solution of the backward*

Euler scheme associated with the ε*-discretization of* (2.2), iff

$$\frac{\sigma(t_j)-\sigma(t_{j-1})}{t_j-t_{j-1}}+A\sigma(t_j)=Z_j,\quad 1\leqslant j\leqslant n.$$

Note, since A is m-accretive, the backward Euler scheme is uniquely solvable. $U\in C([a,b],X)$ is called a *mild solution of* (2.2), iff for every $\varepsilon>0$ there exists an ε-discretization and a solution σ of the associated Euler scheme such that $\|u(t)-\sigma(t)\|\leqslant\varepsilon$ for all $t\in[t_0,t_n]$.

An integral solution of (2.2) on $[a,b]$ is a function $U\in C([a,b],X)$ with $u(t)\in\mathrm{cl}(D)$ for $t\in[a,b]$ which satisfies

$$\|U(t)-x\|\leqslant\|U(s)-x\|+\int_s^t\big(U(\tau)-x,Z(\tau)-Ax\big)_+\,d\tau,$$

for all $a\leqslant s<t\leqslant b$, $x\in D$.

If $U_0\in\mathrm{cl}(D)$, then the initial value problem (2.2), $U(0)=U_0$, has a unique mild solution which is also the unique integral solution (cf. [22] for a proof).

Let Y,Z be topological spaces. $G:Y\to 2^Z$ is called *upper semi-continuous*, iff $\{y\in Y: G(y)\cap A\neq\emptyset\}$ is closed for each closed $A\subseteq Z$.

Let Z be a Banach space, $S\subset Z$, and $r>0$. We set

$$B(S,r):=\Big\{z\in Z:\inf_{\zeta\in S}\|z-\zeta\|<r\Big\}.$$

The Approximate Selection Theorem states (cf. [2], section 9.2 and [8], section 2.4)

THEOREM 2.1. *Let* Y *be a metric space,* Z *be a Banach space, and* $F:Y\to 2^Z$ *be upper semi-continuous with* $F(y)$ *non-empty and convex for* $y\in Y$*. Then for every* $\varepsilon>0$*, there exists a Lipschitz continuous function* $f_\varepsilon:X\to Z$ *with* $\mathrm{graph}(f_\varepsilon)\subseteq B(\mathrm{graph}(F),\varepsilon)$.

3. Mild solutions of (1.1)

The reaction-diffusion problem (1.1) will be considered under the following hypotheses:

$(H1)$ M two-dimensional, compact, oriented Riemannian C^∞-manifold without boundary; $\mathrm{meas}(M)>0$, where meas denotes the measure induced by the Riemannian metric on M; $m\in\mathbb{N}$, $p>2$; $c,k\in C^2(M)$ positive.

$(H2)$ $T>0$, $\beta\in C^1([-T,0]\times M,\mathbb{R}_+)$, $\beta(-T,\cdot)\equiv 0$, $\beta(s,x)>0$ for $s\in(-T,0]$ and $x\in M$, $\int_{-T}^0\beta(s,x)\,ds=1$ for $x\in M$.

$(H3)$ $g\in C^2(\mathbb{R}_+\times\mathbb{R}_+^m,\mathbb{R}_+)$, $g(0,\cdot)\equiv 0$, $\kappa\in[1,\infty)$, $\lim\limits_{y\to\infty}\frac{g(y,z)}{y^\kappa}=\infty$, uniformly for $z\in\mathbb{R}_+^m$.

$(H4)$ $F(t,x,y,\overline{y},z) = Q(t,x)F_0(x,,y,\overline{y},z)$ for $(t,x,y,\overline{y},z) \in \mathbb{R} \times M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^m$ with: $Q \in C^2(\mathbb{R} \times M, (0,\infty))$ bounded and uniformly continuous, $Q(\cdot,x)$ almost periodic for $x \in M$, $F_0 : M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^m \to 2^{(0,\infty)}$ bounded, upper semi-continuous, $F_0(x,y,\overline{y},z)$ a nonempty, compact interval for every $(x,y,\overline{y},z) \in M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^m$.

$(H5)$ $V : C_b([-T,\infty),C(M,\mathbb{R}_+)) \to C_b([0,\infty),C(M,\mathbb{R}_+^m))$ continuous in the topologies of uniform convergence on compacta on $C_b([-T,\infty),C(M))$ and $C_b([0,\infty),C(M,\mathbb{R}^m))$, respectively, V satisfies:

(i) $\{V(\vartheta)(t) : 0 \leqslant t \leqslant \overline{t}, \vartheta \in \mathfrak{B}\}$ bounded for all $\overline{t} > 0$ and uniformly bounded subsets $\mathfrak{B}$ of $C_b([-T,\infty),C(M,\mathbb{R}_+))$;

(ii) V has the Volterra property, i.e.,

$$V(\vartheta_1)|_{[0,t]} = V(\vartheta_2)|_{[0,t]} \text{ for all } \vartheta_1, \vartheta_2 \in C_b([-T,\infty),C(M,\mathbb{R}_+))$$

with $\vartheta_1|_{[-T,t]} = \vartheta_2|_{[-T,t]}$ [;

(iii) let $\overline{t} > 0$, then there exist r, $\mu > 0$ with

$$\sup_{0 \leqslant t \leqslant \overline{t}} \|V(u_1)(t) - V(u_2)(t)\|_\infty \leqslant \mu \sup_{-T \leqslant t \leqslant \overline{t}} \|u_1(t) - u_2(t)\|_\infty$$

for all u_1, $u_2 \in C_b([-T,\infty),C(M,\mathbb{R}_+))$ with $\sup_{-T \leqslant t \leqslant \overline{t}} \|u_j(t)\| \leqslant r$, $j = 1, 2$;

(iv) for $\tilde{t} > 0$ and $\vartheta_1, \vartheta_2 \in C_b([-T,\infty),C(M,\mathbb{R}_+))$ with $\vartheta_1(t) = \vartheta_2(t)$ for $t \in [-T,\tilde{t}]$, there exist $\tau > 0$ and $C > 0$ with

$$\|\big(V(\vartheta_1) - V(\vartheta_2)\big)|_{[\tilde{t},\tilde{t}+\tau]}\|_\infty \leqslant C\|\big(\vartheta_1 - \vartheta_2\big)|_{[\tilde{t},\tilde{t}+\tau]}\|_\infty.$$

The following comments address the climatological motivations for some of the assumptions.

REMARKS. 1. T in $(H2)$ is the memory span of the system (thousands of years due to the continental ice-sheets). β is a weight accounting for the land-water distribution and the fading memory for example.

2. The growth condition in $(H3)$ reflect the Stefan-Boltzmann law. The radiation flux of a black body at temperature u in Kelvin is given by σu^4, σ the Stefan-Boltzman constant. In the case of the earth, σ is a function of temperature (greenhouse feedback, e.g.). The vegetation is affecting the CO_2 in the atmosphere, hence $\sigma = \sigma(u,V(u))$, when accounting for the bio-feedback. The fact that σ is positive and bounded, and has a positive infimum is reflected in $(H3)$. □

In order to motivate $(H5)$, consider a daisy world model, i.e. a planet M covered by a vegetation consisting of black daisies (population density v_1) and white daises (population density v_2). On the one hand, their fitness (growth rate, carrying capacity, ...) varies with the changing climate and on the other hand, they affect the climate (black daisies absorb more solar radiation than white ones, both extract species-specific amounts of CO_2 from the atmosphere). Employing a standard competition-diffusion model one is led to

$$\begin{aligned} \frac{\partial v_1}{\partial t} - k_1 \Delta v_1 &= v_1 f_1(x,u,v_1,v_2), \quad x \in M,\ t > 0, \\ \frac{\partial v_2}{\partial t} - k_2 \Delta v_2 &= v_2 f_2(x,u,v_1,v_2), \quad x \in M,\ t > 0 \end{aligned} \tag{3.1}$$

under the following hypotheses:

- $f_j \in C^2\left(M \times \left(\mathbb{R}^+\right)^3, \mathbb{R}\right)$, $f_j(x,0,0,0) > 0$ for $x \in M$, $j = 1,2$;
- $\partial_{v_k} f_j < 0$ for $j,k = 1,2$;
- $\limsup_{v_1 \to \infty} f_1(x,u,v_1,v_2) < 0$ uniformly for all $x \in M, u, v_2 \geqslant 0$;
- $\limsup_{v_2 \to \infty} f_2(x,u,v_1,v_2) < 0$ uniformly for all $x \in M, u, v_1 \geqslant 0$.

Fixing nonnegative initial conditions $v_0 := (v_1^0, v_2^0)$, one finds for every

$$u \in C_b([-T,\infty), C(M))$$

a unique solution $V(u) = (v_1, v_2)$ of (3.1) which belongs to $C_b([0,\infty), C(M,\mathbb{R}^2))$ and satisfies $V(u)(0) = v_0$. We refer to [18] for an outline of how to derive the assumptions stated in $(H5)$.

Alternatively, consider nonlocal dispersal. Then one is led to the integro-differential system

$$\begin{aligned} \frac{\partial v_1}{\partial t} - k_1\left(\mathscr{K} v_1 - v_1\right) &= v_1 f_1(x,u,v_1,v_2), \quad x \in M,\ t > 0, \\ \frac{\partial v_2}{\partial t} - k_2\left(\mathscr{K} v_2 - v_2\right) &= v_2 f_2(x,u,v_1,v_2), \quad x \in M,\ t > 0, \end{aligned} \tag{3.2}$$

where $\mathscr{K}\phi(x) = \int_M k(x,y)\phi(y)\, dy$ for $x,y \in M$ and $k \in C^1(M \times M, \mathbb{R}^+)$ satisfies:

(i) $\int_M k(x,y)\, dy \leqslant 1$ for $x \in M$;

(ii) $x \mapsto \int_M k(x,y) dy \not\equiv 1$;

(iii) there is $\delta_0 > 0$ such that for each $x \in M$, $k(x,y) > 0$ for $y \in M$ and $\mathrm{dist}_M(x,y) < \delta_0$, where dist_M denotes the distance function on M induces by the Riemannian metric.

Clearly, (3.2) can be rewritten as an abstract ordinary differential equation of the form $\dot{w} + Lw = N(t,w)$ in $C(M,\mathbb{R}_+^2)$, where L is a compact perturbation of the identity and N is locally Lipschitz. The hypothesis $f_j(x,0,0,0) > 0$ for $x \in M$ and $j = 1,2$ guarantees that the standard existence and uniqueness theory for equations on closed sets applies (cf. [7]), and fixing v_0, again, yields the Volterra operator V. It is not to hard to derive the properties stated under $(H5)$.

The reader is referred to [19], [21], [20] and the references therein for the significance of nonlocal dispersal in two-species competition models. Though these papers address questions of asymptotic behavior in case of autonomous reaction terms and do not cover climate issues, they indicate the potential of also utilizing nonlocal dispersal when modeling complex planetary vegetation. Clearly, $(H5)$ allows for many species vegetation with various interactions, e.g. some species competing with each other and in a symbiotic relationship with others.

MILD SOLUTION OF (1.1). We follow the approach of [11] in dealing with the technicality that c is not constant. Set

$$H := L^2(M),$$
$$<\varphi,\psi>_H := \int_M \varphi\psi c \quad \text{for } \varphi,\psi \in H,$$
$$\|\varphi\|_H := \sqrt{<\varphi,\varphi>_H} \quad \text{for } \varphi \in H.$$

Define A_H to be the subdifferential of

$$J : \varphi \mapsto \begin{cases} \frac{1}{p}\int_M k|\nabla\varphi|^p, & \varphi \in W^{1,p}(M), \\ \infty, & \varphi \in H \setminus W^{1,p}(M), \end{cases}$$

then A_H is an m-accretive operator in H. Let I be a non-degenerate interval and $z \in L^1(I,H)$. A mild solution u of $\dot{u}+A_Hu = z$ is a mild solution of the nonhomogeneous degenerate diffusion equation

$$c(x)\partial_t u(t,x) - \nabla\cdot[k|\nabla u|^{p-2}\nabla u](t,x) = c(x)z(t,x) \ \text{ on } I\times M.$$

Therefore

$$u \in C([a-T,b),C(M,\mathbb{R}_+)) \ \ (0 \leqslant a < b)$$

is called an L^2*-mild solution of* (1.1) *on* $[a-T,b)$, iff there exists a $z \in L^1([a,b),H)$ with

$$\dot{u}+A_Hu = z \quad \text{and} \quad c(x)z(t,x)+g(u(t,x),V(u)(t,x)) \in F(t,x,u,\overline{u},V(u))$$

for $(t,x) \in (a,b)\times M$ a.e.

On the other hand, using the standard norm on $L^2(M)$, one obtains an m-accretive operator $\hat{A}$ as subdifferential of J and defines the m-accretive operator A_C in $C(M)$ by $\mathrm{dom}(A_C) := \{\varphi \in \mathrm{dom}(\hat{A}) : \hat{A}\varphi \in C(M)\}$ and $A_C(\varphi) = \frac{1}{c}\hat{A}(\varphi)$ for $\varphi \in \mathrm{dom}(A_C)$. Note that $\mathrm{dom}(A_C)$ is dense in $C(M)$ under the maximum norm, since M is a manifold without boundary. Now consider (1.1) with F replaced by a single-valued continuous function $f : [0,\infty)\times M\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$, i.e.

$$\begin{cases} c(x)\partial_t u - \nabla\cdot[k(x)\,|\nabla u|^{p-2}\,\nabla u] + g(u,V(u)) \\ \qquad\qquad\qquad\qquad = f(t,x,u,\overline{u},V(u)(t)),\ t>0,\ x\in M, \\ \overline{u}(t,x) := \int_{-T}^{0}\beta(s,x)u(t+s,x)\,ds,\ t>0,\ x\in M, \\ u(s,x) = u_0(s,x),\ -T \leqslant s \leqslant 0, x\in M. \end{cases} \tag{3.3}$$

Let $0 \leqslant a < b$. A $C(M)$-mild solution $u \in C([a-T,b),C(M))$ of $\dot{u}+A_C u = \frac{z}{c}$ on $[a-T,b)$ is called a *mild solution of* (3.3) *on* $[a-T,b)$, iff

$$z(t) = f(t,x,u,\overline{u},V(u)) - g(u,V(u)) \text{ for } t \in (a,b) \text{ and } x \in M.$$

THEOREM 3.1. *Let* $(H1)$-$(H5)$ *be satisfied and* $u_0 \in C([-T,0],C(M,\mathbb{R}_+))$. *Then* (1.1) *has at least one* L^2*-mild solution* $u \in C_b([-T,\infty),C(M,\mathbb{R}_+))$. *Moreover, there exists an a priori bound* $M > 0$ *such that* $\|u\|_\infty \leqslant \max\{M,\|u_0\|\}$ *for all* L^2*-mild solution* $u \in C_b([-T,\infty),C(M,\mathbb{R}_+))$.

In the case that $c \equiv 1$, the existence part of Theorem 3.1 follows by applying [18, Theorem 4.1], see section 5 in [18] for a quite similar example. The general case requires to repeat the proofs of Theorems 3.1 and 4.1 of [18], since A_C is not a restriction of A_H as assumed in (H1) of [18]. Employing Theorem 2.1, one finds a sequence of Lipschitz-continuous functions (f_j) which approximate F in the sense of graphs as stated in the theorem. One then follows the reasoning of the proof of [18, Theorem 3.1] in order to establish existence of $C(M)$-mild solutions (note f in that proof corresponds to $f_j - g$ here) and the a priori bound for the sequence. Next, one notes that all solutions are indeed L_2-mild solution in light of the discussion in [11, Section 2.1]. Therefore quite the same reasoning as in the proof of [18, Theorem 4.1.] yields the existence of L^2-mild solutions and the a priori bounds for the so obtained solutions as claimed in Theorem 3.1. Finally, it is not too hard to conclude that every L^2-mild solution u of (1.1) which belongs to $C_b([-T,\infty),C(M,\mathbb{R}_+))$ satisfies the stated a priori estimate, since such a solution is the uniform limit on compacta of solutions of regularized problems, hence one can again employ the argument of the proof of [18, Theorem 4.1] as outlined before.

4. Trajectory attractor

Though global solutions of (1.1) belong to $C_b([-T,\infty),C(M))$, it is well-known that the maximum norm is not suitable for the concept of a trajectory attractor. Rather one employs the Fréchet space $C\big(\mathbb{R}_+,C([-T,0],C(M))\big)$ under the metric

$$d(U_1,U_2) := \sup\{\|U_1(t)-U_2(t)\|_\infty : t \in [0,1]\} + \sum_{l=2}^{\infty} \frac{1}{2^l} \frac{\sup\{\|U_1(t)-U_2(t)\|_\infty : t \in [0,l]\}}{1+\sup\{\|U_1(t)-U_2(t)\|_\infty : t \in [0,l]\}} \quad (4.1)$$

for $U_1,\ U_2 \in C(\mathbb{R}_+,C([-T,0],C(M)))$. To this end, one identifies functions

$$u \in C_b([-T,\infty],C(M))$$

with functions in $C(\mathbb{R}_+,C([-T,0],C(M)))$ via the natural linear homeomorphism

$$\mathscr{I} : C([-T,\infty],C(M))) \to C(\mathbb{R}_+,C([-T,0],C(M)))$$

defined by $\mathscr{I}u(t)(s) = u(t+s)$ for $t \in \mathbb{R}_+$, $u \in C([-T,\infty),C(M))$, and $s \in [-T,0]$. We frequently write U for $\mathscr{I}u$.

The convergence in the sense of d is uniform convergence on compact sets. The choice of the nontraditional term

$$\sup\{\|U_1(t)-U_2(t)\|_\infty : t\in[0,1]\}$$

in d guarantees that $\{u|_{[-T,0]} : u\in B\}$ is uniformly bounded, if for the set B which is the subset of $C([-T,\infty),C(M))$, holds that $\mathscr{I}(B)$ is d-bounded.

In this context, the *shift semigroup* replaces the evolution (solution) family which one considers in case of unique solvability. Recall the definition.

DEFINITION 4.1. $\hat{S}:\mathbb{R}\times C(\mathbb{R}_+,C([-T,0],C(M)))\to C(\mathbb{R}_+,C([-T,0],C(M)))$ *defined by* $\hat{S}(t)(U)(s):=U(t+s)$ *for* $t\in\mathbb{R}_+$, $U\in C(\mathbb{R}_+,C([-T,0],C(M)))$, *and* $s\in[-T,0]$, *is called the shift semigroup on* $C(\mathbb{R}_+,C([-T,0],C(M)))$.

Clearly, $\hat{S}$ is in fact a semigroup. Since Q is time dependent, one cannot expect that $\hat{S}$ leaves the set

$$\{\mathscr{I}u : u\in C([-T,0],C(M,\mathbb{R}_+))\ L^2\text{-mild solution of (1.1)}\}$$

invariant. Therefore one introduces the following definition.

DEFINITION 4.2. *The closure* $\mathscr{Q}$ *of* $\{(s,x)\mapsto Q(t+s,x) : t\in\mathbb{R}, x\in M\}$ *in the compact open topology is called the* hull *of* Q.

Note that $\mathscr{Q}$ is compact, that all elements in $\mathscr{Q}$ satisfy the hypotheses stated for Q in (H4), and that $\|\tilde{Q}\|_\infty=\|Q\|_\infty$ for all $\tilde{Q}\in\mathscr{Q}$, hence Theorem 3.1 applies to

$$\begin{cases} c(x)\partial_t u-\nabla\cdot[k(x)\,|\nabla u|^{p-2}\nabla u]+g(u,V(u)) \\ \qquad\qquad \in F_{\tilde{Q}}(t,x,u,\overline{u},V(u)(t)),\ \ t>0,\ x\in M, \\ \overline{u}(t,x):=\int_{-T}^0\beta(s,x)u(t+s,x)\,ds,\ \ t>0,\ x\in M, \end{cases} \tag{4.2}$$

where $F_{\tilde{Q}}(t,x,y,\overline{y},z)=\tilde{Q}(t,x)F_0(x,y,\overline{y},z)$ for $(t,x,y,\overline{y},z)\in\mathbb{R}\times M\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m$ and $\tilde{Q}\in\mathscr{Q}$.

Note that $\hat{S}(t)\mathscr{I}u(s)=u(t+s)$ for $s\in[-T,0]$ shows that $\hat{S}(t)$ is the time-t-map, if unique solvability holds. This suggest to introduces the *united trajectory space* as follows.

DEFINITION 4.3. *One calls*

$$\mathscr{X}:=\Big\{\mathscr{I}u : u\in C_b([-T,\infty),C(M,\mathbb{R}^+)) \\ L^2\text{-mild solution of (4.2) for some } \tilde{Q}\in\mathscr{Q}\Big\} \tag{4.3}$$

the united trajectory space associated with (4.2).

Clearly, $\mathscr{X}$ is invariant under $\hat{S}$, and we denote by S the restriction of $\hat{S}$ to $\mathbb{R}_+\times\mathscr{X}$. The main result of this paper is

THEOREM 4.1. *Let $(H1)$-$(H5)$ be satisfied. Then S has a compact global attractor $\mathscr{A}$, the so-called trajectory attractor of* (4.2).

The proof employs the following well-known result for abstract dissipative systems (cf. [16, Theorem 3.4.8], e.g.).

THEOREM 4.2. *Let Y be a complete metric space, $\underline{t} \in (0,\infty)$, and $\mathscr{T} : \mathbb{R}_+ \times Y \to Y$ be a continuous semigroup which is completely continuous for $t > \underline{t}$ and point dissipative. Then $\mathscr{T}$ has a compact attractor.*

The following three lemmas establish the hypotheses of Theorem 4.2 and thus prove Theorem 4.1.

LEMMA 4.3. *Assume that $(H1)$-$(H5)$ are satisfied. Then $\mathscr{X}$ is a closed subset of $C(\mathbb{R}_+, C([-T,0], C(M,\mathbb{R}^+)))$, hence $(\mathscr{X}, d|_{\mathscr{X}\times\mathscr{X}})$ a complete metric space.*

Proof. Let $(U_j) \in \mathscr{X}^{\mathbb{N}}$ be a d-convergent sequence with limit U_∞. Set

$$u_j := \mathscr{I}^{-1} U_j \quad \text{for } j \in \mathbb{N} \cup \{\infty\},$$

then (u_j) is uniformly bounded in view of $(u_j|_{[-T,0]})$ uniformly bounded and the a priori bounds mentioned in Theorem 3.1. Hence u_∞ belongs to $C_b([-T,\infty), C(M))$. Since u_j is a mild solution of (4.2) for $j \in \mathbb{N}$ one finds $(\gamma_j) \in L^1_{\text{loc}}(\mathbb{R}_+, L^2(M))$ with

$$c(\cdot)\gamma_j(t,\cdot) + g(u_j(t,\cdot), V(u_j)(t,\cdot)) \in F_{Q_j}(t,\cdot,u_j(t,\cdot),\overline{u_j}(t,\cdot),V(u_j)(t,\cdot)).$$

Since $\mathscr{Q}$ is compact, we can assume by passing to a subsequence of (U_j), if necessary, that (Q_j) converges uniformly on compact subsets of $\mathbb{R} \times M$ to some $Q_\infty \in \mathscr{Q}$. As in the proof of [18, Theorem 4.1], the fact that the sequence (γ_j) is uniformly bounded, the Dunford-Pettis, and Cantor's diagonal argument yield that, by passing once more to a subsequence of (U_j), if necessary, one can assume the existence of an $\gamma_\infty \in L^1_{\text{loc}}(\mathbb{R}_+, L^2(M))$ with $\gamma|_{[0,n]} \rightharpoonup \gamma_\infty|_{[0,n]}$ for $n \in \mathbb{N}$. Again, as in [18, Theorem 4.1] one concludes by means of a theorem of Baras that u_∞ is a mild solution of $\dot{u} + A_H u = \gamma_\infty$. Thus, it remains to establish that

$$c(\cdot)\gamma_\infty(t,\cdot) + g(u_\infty(t,\cdot), V(u_\infty)(t,\cdot)) \in F_{Q_\infty}(t,\cdot,u_\infty(t,\cdot),\overline{u_\infty}(t,\cdot),V(u_\infty)(t,\cdot))$$

for $t \in \mathbb{R}_+$ a.e.

Since F_0 is upper semi-continuous and compact interval-valued, there exist a lower semi-continuous function $\underline{f} : M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^m_+ \to (0,\infty)$ and an upper semi-continuous $\overline{f} : M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^m_+ \to (0,\infty)$ with

$$F_{\tilde{Q}}(t,x,y,\overline{y},z) = [\tilde{Q}(t,x)\underline{f}(x,y,\overline{y},z), \tilde{Q}(t,x)\overline{f}(x,y,\overline{y},z)]$$

for every $(t,x,y,\overline{y},z,\tilde{Q}) \in \mathbb{R} \times M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathscr{Q}$ (cf. [2], [8]). Since

$$\|Q_j(t,\cdot) - Q_\infty(t,\cdot)\|_\infty \to 0,$$

$$\|\overline{u_j}(t,\cdot)-\overline{u_\infty}(t,\cdot)\|_\infty \to 0,$$
$$\|V(u_j)(t,\cdot)-V(u_\infty)(t,\cdot)\|_\infty \to 0,$$
$$\|g(u_j(t,\cdot),V(u_j)(t,\cdot))-g_\infty(u_\infty(t,\cdot),V(u_\infty)(t,\cdot))\|_\infty \to 0,$$

all uniformly on compact subsets of $\mathbb{R}_+$, one has to show that

$$Q_\infty \underline{f}(\cdot,u_\infty,\overline{u_\infty},V(u_\infty)) \leqslant c\gamma_\infty + g_\infty(u_\infty,V(u_\infty) \leqslant Q_\infty \overline{f}(\cdot,u_\infty,\overline{u_\infty},V(u_\infty))$$

a.e. on $\mathbb{R}_+\times M$. Suppose that this is not true, then there exists an $n\in\mathbb{N}$ with either $\overline{S}_n := \{t\in[0,n] : \operatorname{meas}(\Gamma_t)>0\}$ has positive measure, where

$$\Gamma_t := \big\{x\in M : c(x)\gamma_\infty(t,x) > Q_\infty(t,x)\overline{f}(x,u_\infty(t,x),\overline{u_\infty}(t,x),V(u_\infty)(t,x)) \\ - g_\infty(u_\infty(t,x),V(u_\infty)(t,x))\big\},$$

or $\underline{S}_n := \{t\in[0,n] : \operatorname{meas}(\hat{\Gamma}_t)>0\}$ has positive measure with

$$\hat{\Gamma}_t := \{x\in M : c(x)\gamma_\infty(t,x) < Q_\infty(t,x)\underline{f}(x,u_\infty(t,x),\overline{u_\infty}(t,x),V(u_\infty)(t,x)) \\ - g_\infty(u_\infty(t,x),V(u_\infty)(t,x))\}. \quad (4.4)$$

Consider the first alternative. Let

$$\Phi(w) := \int_{\overline{S}_n}\Big(\int_{\Gamma_t} c(x)w(t,x)\,dx\Big)\,dt \ \text{ for } \ w\in L^1([0,n],L^2(M)).$$

Since $\Phi\in L^1([0,n],L^2(M))^*$, one has $\Phi(\gamma_j|_{[0,n]}) \to \Phi(\gamma_\infty|_{[0,n]})$.

On the other hand, the upper semi-continuity of $\overline{f}$ yields

$$\begin{aligned}\limsup_{j\to\infty} c(x)\gamma_j(t,x) &\leqslant \limsup_{j\to\infty} Q_j(t,x)\overline{f}(x,u_j(t,x),\overline{u_j}(t,x),V(u_j)(t,x)) \\ &\qquad - g(u_\infty(t,x),V(u_\infty)(t,x)) \\ &\leqslant Q_\infty(t,x)\overline{f}(x,u_\infty(t,x),\overline{u_\infty}(t,x),V(u_\infty)(t,x)) \\ &\qquad - g(u_\infty(t,x),V(u_\infty)(t,x)).\end{aligned}$$

Thus,

$$c(x)\gamma_\infty(t,x) - \limsup_{j\to\infty} c(x)\gamma_j(t,x) > 0 \ \text{ for } t\in[0,n] \text{ and } x\in\Gamma_t,$$

which contradicts $\Phi(\gamma_j|_{[0,n]}) \to \Phi(\gamma_\infty|_{[0,n]})$. □

The proof of the complete continuity property of S is similar to that of [12, Lemma 6.4]. We need the following regularity result (cf. [12, Theorem 5.5]).

THEOREM 4.4. *Let $(H1)$ be satisfied, $\sigma\in(0,1)$, $\bar{t}\in(0,\infty)$, $h\in L^\infty((0,\bar{t})\times M)$, $u_0\in C^\sigma(M)$, and $u:[0,\bar{t}]\times M\to\mathbb{R}$ be a bounded weak solution of*

$$c(x)u_t - \nabla\cdot[k(x)|\nabla u|^{p-2}\nabla u] = h(t,x).$$

Then there exist $\hat{\sigma} \in (0,\sigma)$ *and* $\gamma > 0$*, which depend on* σ *and* $\bar{t}$*, but not on* u *such that*

$$|u(t_1,x_1) - u(t_2,x_2)| \leqslant \gamma \|u\|_\infty \Big[\mathrm{dist}_M(x_1,x_2) + |t_1 - t_2|^{\frac{p-2}{p}}\Big]^{\hat{\sigma}}$$

for all $t_1,t_2 \in [0,\bar{t}]$ *and* $x_1,x_2 \in M$*.*

LEMMA 4.5. *Let* $(H1)$*-*$(H5)$ *be satisfied. Then* $S(t)$ *is completely continuous for* $t > T$*.*

Proof. It follows from Theorem 3.1 and $\|\tilde{Q}\|_\infty = \|Q\|_\infty$ for $\tilde{Q} \in \mathscr{Q}$ that there is an $M > 0$ such that

$$\|u\|_\infty \leqslant \max\{\|u|_{[-T,0]}\|_\infty, M\} \quad \text{for } \mathscr{I}u \in \mathscr{X}.$$

Fix $\underline{t} > T$ and recall that $S(t)U = U(\cdot + t)$ for $t \geqslant \underline{t}$ and $U \in \mathscr{X}$. Consider a d-bounded sequence $(U_j) = (\mathscr{I}u_j) \in \mathscr{X}^{\mathbb{N}}$. Then $(u_j|_{[-T,0]})$ is uniformly bounded, therefore (u_j) is uniformly bounded. Moreover, u_j is a weak solution of

$$c(x)u_t - \nabla \cdot [k(x)|\nabla u|^{p-2}\nabla u] = h(t,x)$$

with

$$h(t,x) + g(u(t,x), V(u)(t,x)) \in F_{\tilde{Q}}(t,x,u(t,x),\overline{u}(t,x),V(u)(t,x))$$

thanks to a theorem of Brezis (cf. [25, Theorem 1.9.3] and [12, Remark 5.4]. Consequently, given $b > 0$, Theorem 4.4 and $\underline{t} > T$ imply that $(u_j|_{[\underline{t}-T,\underline{t}+b]\times M})$ is bounded in some $C^\sigma([\underline{t}-T,\underline{t}+b]\times M,\mathbb{R})$. In particular, $\{u_j(t) : j \in M\}$ is relatively compact in $C(M)$ for $t \in [\underline{t}-T,\underline{t}+b]$ and $\{u_j|_{[\underline{t}-T,\underline{t}+b]} : j \in \mathbb{N}\}$ is equicontinuous as a mapping into $C(M)$. Thus, Ascoli's theorem implies that $(u_j|_{[\underline{t}-T,\underline{t}+b]})$ has a uniformly convergent subsequence. Since $b > 0$ is arbitrary, employing Cantor's diagonal procedure yields a subsequences of $(S(\underline{t})U_j)$ which converges uniformly on compacta. This is the convergence with respect to d, hence $S(\underline{t})$ is completely continuous. □

LEMMA 4.6. *Let* $(H1)$*-*$(H5)$ *be satisfied. Then* S *is point dissipative.*

Proof. Since Q and F_0 are uniformly bounded by (H4), one has that

$$\|F\|_\infty := \sup \bigcup \Big\{ F_{\tilde{Q}}(t,x,y,\overline{y},z) : (t,x,y,\overline{y},z,\tilde{Q}) \in \mathbb{R}\times M \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^m \times \mathscr{Q} \Big\} < \infty. \quad (4.5)$$

Hence $(H3)$ and $(H4)$ guarantee the existence of a $K > 0$ with

$$c(x)\|F\|_\infty - g(y,z) < -\min c \ \text{ for all } x \in M,\ y \geqslant K \text{ and } z \in \mathbb{R}_+.$$

Set $\mathscr{B} := \{U \in \mathscr{X} : \|U\|_\infty \leqslant 2K\}$. The goal is to show that for every $U \in \mathscr{X}$ there exists a $\underline{t}$ with $S(t)U \in \mathscr{B}$ for $t \geqslant \underline{t} + T$, which means $u(t) \leqslant 2K$ for all $t \geqslant \underline{t}$, where

$u = \mathscr{I}^{-1}U$. Let $u \in C_b([-t,\infty), C(M,\mathbb{R}_+))$ be a mild solution of (4.2). Then, following the existence proof for Theorem 3.1 (cf. [18, Theorem 4.1.]), one finds sequences

$$(u_j) \in C_b([-T,\infty), C(M,\mathbb{R}_+))^{\mathbb{N}} \quad \text{and} \quad (\gamma_j) \in C_b(\mathbb{R}_+, C(M,\mathbb{R}_+))^{\mathbb{N}}$$

such that

$$c(x)\gamma_j(t,x) \in \left[0, \max F(t,x,u_j(t,x),\overline{u_j}(t,x),V(u_j)(t,x)) + \frac{\min c}{2j}\right]$$

for all $(t,x) \in \mathbb{R}_+ \times M$, u_j is a mild solution of

$$\dot{u}_j + A_C u_j = \gamma_j - \frac{1}{c(x)} g(u_j, V(u_j)),$$

$u_j|_{[-T,0]} = u_j|_{[-T,0]}$, and $u_j \to u$, uniformly on compact subsets of $\mathbb{R}_+$. In particular, if $\|u(0)\|_\infty > 2K$, then there exists an $\bar{t} > 0$ with $\|u(t)\|_\infty > 2K$ for $t \in (0,\bar{t}]$. Moreover, one can find a $j_0 \in \mathbb{N}$ such that

$$\|u_j(t)\|_\infty \geqslant K \quad \text{for } j \geqslant j_0 \text{ and } t \in [0,\bar{t}],$$

hence

$$\begin{aligned} \gamma_j(t,x) - \frac{1}{c(x)} g(u_j(t,x),V(u_j)(t,x)) \\ &\leqslant \frac{1}{c(x)}\big(\|F\|_\infty - g(u_j(t,x),V(u_j)(t,x))\big) + \frac{1}{2j} \\ &\leqslant -1 + \frac{1}{2j} \leqslant -\frac{1}{2} \end{aligned}$$

for $j \geqslant j_0$ and all $x \in M$ with $u_j(t,x) = \|u_j(t)\|_\infty$. Since every mild solution of (4.2) is an integral solution (cf. [25]) and $A_C(0) = 0$, one has

$$\|u_j(t)\|_\infty^2 = \|u_j(0)\|_\infty^2 + 2\int_0^t \left(u_j(s), \gamma_j(s) - \frac{1}{c(\cdot)} g(u_j(s),V(u_j)(s))\right)_+ ds. \tag{4.6}$$

Observing that $u_j(s) \geqslant 0$ and therefore

$$\begin{aligned} (u_j(s),\gamma_j(s))_+ = \|u_j(s)\|_\infty \max\Big\{\gamma_j(s)(x) - \frac{1}{c(x)} g(u_j(s,x),V(u_j)(s,x)) \\ : x \in M,\ u_j(s)(x) = \|u_j(x)\|_\infty\Big\}, \end{aligned}$$

one obtains

$$\|u_j(t)\|_\infty^2 \leqslant \|u(0)\|_\infty^2 - Kt \quad \text{for } j \geqslant j_0 \text{ and } t \in (0,\bar{t}],$$

which implies $\bar{t} \leqslant \|u(0)\|_\infty^2 / K$. This shows that u enters the ball $B(0,2K)$ in $C(M)$ after a finite time. Using again (4.6), one argues likewise that no solution can leave that ball. Employing the isomorphism $\mathscr{I}$, one obtains that $S(t)U \in \mathscr{B}$ for every $U \in \mathscr{X}$ and t sufficiently large, i.e., S is point dissipative. □

REMARK. Theorem 4.1 also holds for the easier case of linear diffusion $p = 2$. Rather than using Theorem 4.4 in Lemma 4.5 for proving complete continuity, one employs the imbedding results for Sobolev spaces. Otherwise, one can follow the arguments given here for $p > 2$.

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