Differential
Equations
& Applications
Volume 4, Number 1 (2012), 11–25

RECTIFIABLE OSCILLATIONS OF RADIALLY SYMMETRIC SOLUTIONS OF p–LAPLACE DIFFERENTIAL EQUATIONS

YŪKI NAITO, MERVAN PAŠIĆ, AND HIROYUKI USAMI

Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

(*Communicated by J.-M. Rakotoson*)

Abstract. Let $\Omega = \{x \in \mathbb{R}^N : r_0 \leqslant |x| < 1\}$ with $N \geqslant 2$ and $r_0 \in (0,1)$. We study a kind of geometric oscillatory and asymptotic behaviour near $|x| = 1$ of all radially symmetric solutions $u = u(x)$ of the p-Laplace partial differential equation (P): $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(|x|)|u|^{p-2}u$ in Ω, $u = 0$ on $|x| = 1$ for $p > 1$. Necessary and sufficient conditions on the coefficient $f(|x|)$ are given such that $u(x)$ oscillates near $|x| = 1$ and the surface area of graph $\Gamma(u) \subseteq \mathbb{R}^{N+1}$ of $u(x)$ is finite-rectifiable oscillations, and infinite-nonrectifiable oscillations. The L^1-integrability and L^p-nonintegrability of $|\nabla u|$ on Ω for $p > 1$ are also considered.

1. Introduction and statement of the main results

Let $\Omega = \{x \in \mathbb{R}^N : r_0 \leqslant |x| < 1\}$ with $N \geqslant 2$ and $r_0 \in (0,1)$. Let $u = u(x)$ be a solution of the p-Laplace partial differential equation,

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(|x|)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } |x| = 1, \\ u \in C^1(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad |\nabla u|^{p-2}\nabla u \in C^1(\Omega), \end{cases} \tag{1.1}$$

where $p > 1$. The coefficient $f = f(r)$ is supposed to be a real function satisfying

$$f \in C^2([r_0,1)), \quad f(r) > 0 \quad \text{on } (r_0,1) \quad \text{and} \quad \lim_{r\to 1} f(r) = \infty. \tag{1.2}$$

For $p = 2$, equation (1.1) becomes the appropriate linear Laplace equation $-\Delta u = f(|x|)u$ in Ω.

DEFINITION 1.1. A function $u \in C(\overline{\Omega})$ is said to be *oscillatory near* $|x| = 1$, if there is a sequence $x_n \in \Omega$ such that $u(x_n) = 0$ for all $n \in \mathbb{N}$ and corresponding sequence of real numbers $|x_n|$ is increasing and $|x_n| \to 1$ as $n \to \infty$.

Mathematics subject classification (2010): 26B15, 26B35, 34A26, 34C10, 34D05, 35B05, 35J25.
Keywords and phrases: p-Laplacian equation, radially symmetric solutions, oscillations, surface area, rectifiability.

Paper DEA-04-03

DEFINITION 1.2. A function $u = u(x)$, $u \in C^1(\Omega) \cap C(\overline{\Omega})$, is said to be *rectifiable oscillatory near* $|x| = 1$, if $u(x)$ oscillates near $|x| = 1$ and the surface area of graph $\Gamma(u)$ of u, denoted by $|\Gamma(u)|_S$, is finite, where:

$$\Gamma(u) = \{(x,y) \in \mathbb{R}^N \times \mathbb{R} : x \in \Omega,\ y = u(x)\} \quad \text{and} \quad |\Gamma(u)|_S = \int_{\Gamma(u)} dS.$$

However, if $|\Gamma(u)|_S$ is infinite, then $u = u(x)$ is said to be *nonrectifiable oscillatory near* $|x| = 1$.

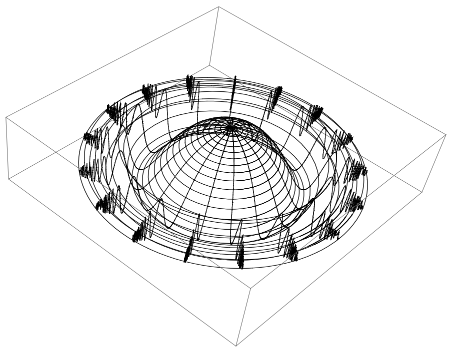

Figure 1: $u(x)$ is a radially symmetric function which oscillates near $|x| = 1$.

In the paper, we study the rectifiable oscillations of all radially symmetric solutions $u(x)$ of equation (1.1), where $u(x) = y(r)$ and $r = |x|$. We see that $y = y(r)$ is a solution of the following one-dimensional singular problem:

$$\begin{cases} \left(r^{N-1}|y'|^{p-2}y'\right)' + r^{N-1}f(r)|y|^{p-2}y = 0, & r \in (r_0, 1), \\ y(1) = 0. \end{cases} \tag{1.3}$$

In order to simplify the notation, we adopt the following definition.

DEFINITION 1.3. Equation (1.1) is said to be *rectifiable* (resp. *nonrectifiable*) *oscillatory near* $|x| = 1$ if all radially symmetric solutions $u(x)$ of (1.1) are rectifiable (resp. nonrectifiable) oscillatory near $|x| = 1$.

On the rectification and rectifiability of plane curves we refer reader to [2] and [16]. The rectifiable and unrectifiable oscillations near $x = 0$ for the first time were introduced and studied in the case of Euler type equation: $y'' + \lambda x^{-\sigma} y = 0$ in $(0, t_0]$, where $\lambda > 0$ and $\sigma \geqslant 2$, see [8] and [18]. Also, this kind of geometric oscillations have been studied in the case of Riemann-Weber version of Euler type equation, see [10]. Preceding results are generalized to the case of general linear differential equations $y'' + f(x)y = 0$, where $f(x)$ satisfies the so-called Hartman-Wintner type condition, see [4]. It is enlarged to the case of two-point oscillations on the interval $[0,1]$, see [13]. The most general results on the rectifiable and unrectifiable oscillations of linear differential equations have been obtained in the case of self-adjoint linear differential equations, see [11]. The more general approach to the rectifiable and unrectifiable oscillations is given in the notion of so-called fractal oscillations introduced in [9] and continued to study in [4], [13], [14] and [12]. Recently, rectifiable oscillation of second order half-linear differential equation

$$\left(|y'|^{p-2}y'\right)' + f(r)|y|^{p-2}y = 0, \quad r \in (r_0, 1), \tag{1.4}$$

was studied in [14]. However, since $r^{N-1} \not\equiv 1$ for $N \geqslant 2$, we are not able to apply directly on (1.3) the known results for (1.4). Hence, (1.3) is transformed into an equivalent equation of the type as (1.4). According to this procedure which is presented in details in Section 2, we are able to state and prove the main result of the paper by which the rectifiable oscillations of all radially symmetric solutions of equation (1.1) are characterized.

THEOREM 1.1. *Let coefficient $f(r)$ besides the structural conditions given in* (1.2) *satisfy the following Hartman-Wintner type condition*

$$f^{-\theta}\left(f^{-\eta}\right)'' \in L^1(r_0, 1), \tag{1.5}$$

where η and θ are arbitrary positive constants satisfying $\eta + \theta = 1/p$. Then equation (1.1) *is rectifiable oscillatory near $|x| = 1$ if and only if*

$$\lim_{\varepsilon \to 0} \int_{r_0}^{1-\varepsilon} [f(r)]^{1/p^2} dr < \infty. \tag{1.6}$$

REMARK 1.1. (i) By Lemma 2.5, condition (1.5) can be represented for example with

$$f^{-\frac{p-1}{p^2}}\left(f^{-\frac{1}{p^2}}\right)'' \in L^1(r_0, 1) \quad \text{or} \quad f^{-\frac{1}{2p}}\left(f^{-\frac{1}{2p}}\right)'' \in L^1(r_0, 1).$$

(ii) It should be mentioned that the conditions in Theorem 1.1 are independent of the space dimension N. It should be also mentioned that, by Pašić and Wong [14], our Theorem 1.1 holds true for the one-dimensional problem (1.4) with $y(1) = 0$. (See Lemma 2.2 below.) It is an open question whether rectifiable oscillation of all radially symmetric solutions of (1.1) are independent of N without the Hartman-Wintner type condition (1.5).

By an intuitive description, if we take, among of all, that equation (1.1) is the reduced suitable wave equation, then equation (1.1) describes the fundamental shapes of stretched membrane with finite or infinite surface area which depends on the integrability of function $[f(r)]^{1/p^2}$.

Applications of Theorem 1.1 to some linear elliptic partial differential equations are discussed in the next examples.

EXAMPLE 1.1. We consider the linear elliptic PDE:

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2 \ln^4 |x|} u & \text{in } \Omega \text{ with } N = 2, \lambda > 0, \\ u = 0 \quad \text{on } |x| = 1, & u \in C^2(\Omega) \cap C(\overline{\Omega}). \end{cases} \tag{1.7}$$

It is clear that $p = N = 2 > 1$ and the coefficient $f(r) = \lambda/(r^2 \ln^4 r)$ satisfies all required conditions from (1.2). Next, since $p = 2$ such $f(r)$ also satisfies:

$$f^{-\frac{1}{2p}}\left(f^{-\frac{1}{2p}}\right)'' = -\frac{1}{4\sqrt{\lambda}}\frac{\ln^2 r}{r} \in L^1(r_0,1),$$

$$\lim_{\varepsilon\to 0}\int_{r_0}^{1-\varepsilon}[f(r)]^{1/p^2}dr = -\lim_{\varepsilon\to 0}\int_{r_0}^{1-\varepsilon}\frac{\sqrt[4]{\lambda}}{\sqrt{r}\ln r}dr \geqslant -\lim_{\varepsilon\to 0}\sqrt[4]{\lambda r_0^2}\int_{r_0}^{1-\varepsilon}\frac{1}{r\ln r}dr = \infty.$$

Hence by Theorem 1.1 for $p = 2$, equation (1.7) is nonrectifiable oscillatory near $|x| = 1$.

More complicated way to verify nonrectifiable oscillations near $|x| = 1$ of equation (1.7) is to use the explicit solution's formula for all radially symmetric solutions $u(x)$ of equation (1.7): $u(x) = \left(\ln x\right)\left[c_1\cos\left(\sqrt{\lambda}/\ln x\right) + c_2\sin\left(\sqrt{\lambda}/\ln x\right)\right]$, $x \in \Omega$, where $\lambda > 0$ and $c_1, c_2 \in \mathbb{R}$.

EXAMPLE 1.2. Let $\lambda > 0$ and $\sigma > 2$. By using the same calculation as in the previous example, one can show that the function $f(|x|) = \lambda|x|^{-2}(-\ln|x|)^{-\sigma}$, satisfies all required assumptions of Theorem 1.1. Moreover, (1.6) is satisfied provided $2 < \sigma < 4$.

Now, we give an example for the coefficient $f(|x|)$ which does not satisfy all assumptions of Theorem 1.1.

EXAMPLE 1.3. We consider the linear elliptic PDE:

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2\ln^2|x|}u & \text{in } \Omega \text{ with } N = 2,\ \lambda > 1/4, \\ u = 0 \quad \text{on } |x| = 1, & u \in C^2(\Omega)\cap C(\overline{\Omega}). \end{cases} \tag{1.8}$$

Unlike equation (1.7), the coefficient $f(r) = \lambda/(r^2\ln^2 r)$ of equation (1.8) does not satisfy the Hartman-Wintner type condition (1.5) for $p = 2$, since for $r \in (r_0, 1)$ we have:

$$f^{-\frac{1}{2p}}\left(f^{-\frac{1}{2p}}\right)'' = -\frac{1}{4\sqrt{\lambda}}\frac{\ln^2 r + 1}{r\ln r} \notin L^1(r_0,1).$$

Therefore, we are not able to apply Theorem 1.1 to equation (1.8). However, the rectifiable oscillations near $|x| = 1$ of equation (1.8) can be immediately verified by using the following explicit formula for all radially symmetric solutions $u(x)$ of equation (1.8): $u(x) = \sqrt{\ln(1/x)}\left[c_1\cos(\rho\ln\ln(1/x)) + c_2\sin(\rho\ln\ln(1/x))\right], x \in \Omega$, $\rho = \sqrt{\lambda - 1/4}$ and $c_1, c_2 \in \mathbb{R}$.

Summarizing results from three previous examples, we state the following important consequence.

COROLLARY 1.1. *Let* $N = 2$, $\sigma \geqslant 2$ *and* $\lambda > 0$ *if* $\sigma > 2$ *and* $\lambda > 1/4$ *if* $\sigma = 2$. *Then the equation*

$$\begin{cases} -\Delta u = \frac{\lambda}{|x|^2(-\ln|x|)^\sigma} u & \text{in } \Omega \subseteq \mathbb{R}^2,\ \sigma \geqslant 2, \\ u = 0 \quad \text{on } |x| = 1, & u \in C^2(\Omega) \cap C(\overline{\Omega}), \end{cases} \tag{1.9}$$

is rectifiable oscillatory near $|x| = 1$ *provided* $2 \leqslant \sigma < 4$ *and nonrectifiable oscillatory near* $|x| = 1$ *provided* $\sigma \geqslant 4$.

Let us remark that equation (1.9) allows explicit form of their radially symmetric solutions only for $\sigma = 2$ and $\sigma = 4$.

We consider the case where the coefficient $f(|x|)$ admits a precise asymptotic behaviour near $|x| = 1$. We say that $f(r) \sim g(r)$ as $r \to 1$ if there exists two positive constants C_1, C_2 such that $C_1 g(r) \leqslant f(r) \leqslant C_2 g(r)$ near $r = 1$.

THEOREM 1.2. *Let coefficient* $f(r)$ *satisfy* (1.2) *and let there be a* $\sigma \in \mathbb{R}$ *such that* $\sigma > p$ *and*

$$f''(r) \sim (1-r)^{-\sigma-2} \quad \text{as } r \to 1. \tag{1.10}$$

Then equation (1.1) *is rectifiable oscillatory near* $|x| = 1$ *if and only if* $\sigma < p^2$.

The proof of previous theorem is given at the end of Section 2. An application of Theorem 1.2 to a linear elliptic PDE is given in the next example.

EXAMPLE 1.4. Let $N \geqslant 2$. We consider equation (1.1) with the case

$$f(r) = g(r)(1-r)^{-\sigma},$$

where $\sigma > p$ and g satisfies $g \in C^2([r_0, 1])$ and $g(r) > 0$ on $(r_0, 1]$. With the help of Theorem 1.2, one can conclude that, in this case, equation (1.1) is rectifiable oscillatory near $|x| = 1$ provided $\sigma < p^2$ and nonrectifiable oscillatory near $|x| = 1$ provided $\sigma \geqslant p^2$. In fact, the coefficient $f(r) = g(r)(1-r)^{-\sigma}$ obviously satisfies condition (1.2) and $f''(r) \sim (1-r)^{-\sigma-2}$ as $r \to 1$. Thus, we may apply Theorem 1.2 to the equation, which shows the statement of this example.

OPEN QUESTION 1.1. Let $u(x)$ be a radially symmetric solution of equation (1.1) which oscillates near $|x| = 1$ and let $x_n \in \Omega$ be the sequence of zero points of $u(x)$ determined as in Definition 1.2. Let $\Omega_n = \{x \in \mathbb{R}^N : |x_n| < |x| < |x_{n+1}|\}$, $n \in \mathbb{N}$. It is known that the Riccati type substitution (see for instance [3], [5], [6]),

$$\vec{\omega}_n(x) = \frac{|\nabla u|^{p-2}\nabla u}{|u|^{p-2}u}, \quad x \in \Omega_n,$$

transforms equation (1.1) into the following vector equation

$$\operatorname{div}\vec{\omega}_n + (p-1)|\vec{\omega}_n|^q + f(|x|) = 0, \quad x \in \Omega_n,$$

where $1/q + 1/p = 1$. Is it possible to derive some qualitative properties of the function $\vec{\omega}_n(x)$ that are related to the rectifiable and nonrectifiable oscillations of $u(x)$ near $|x| = 1$?

In the sequel, we consider the problems of L^1-integrability and L^p-nonintegrability of $|\nabla u|$ on Ω for all radially symmetric solutions $u(x)$ of equation (1.1). The L^1-integrability of $|\nabla u|$ on Ω is a direct consequence of Theorem 1.1 and Lemma 2.1 as follows.

COROLLARY 1.2. *Let $f(r)$ satisfy* (1.2) *and* (1.10) *with $\sigma > p$. For every radially symmetric solution $u(x)$ of equation* (1.1)*, we have $|\nabla u| \in L^1(\Omega)$ if and only if $\sigma < p^2$.*

For the L^p-nonintegrability of $|\nabla u|$ on Ω, some asymptotic properties of radially symmetric solutions of equation (1.1) are required as follows.

THEOREM 1.3. *Assume that conditions* (1.2) *and* (1.5) *hold and $f(r)$ is increasing near $r = 1$. Let $u(x) = y(|x|)$ be a radially symmetric solution of equation* (1.1) *such that there exists an increasing sequence $r_n \in (r_0, 1)$ of consecutive zeros of $y(r)$ satisfying $r_n \to 1$ and $r_{n+1} - r_n \sim r_{n+2} - r_{n+1}$ as $n \to \infty$. Then $|\nabla u| \notin L^p(\Omega)$.*

If the coefficient $f(|x|)$ admits a precise asymptotic behaviour near $|x| = 1$, then besides Corollary 1.2 we have the following result.

COROLLARY 1.3. *Let $f(r)$ satisfy* (1.2) *and* (1.10) *with $\sigma > p$. For every radially symmetric solution $u(x)$ of equation* (1.1)*, we have $|\nabla u| \notin L^p(\Omega)$.*

About the L^p integrability of solutions of quasilinear ellitpic equations, in most general setting, see in [15].

2. Proofs of Theorems 1.1 and 1.2

We firstly state the following lemma.

LEMMA 2.1. *Let $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be a radially symmetric function and let $y = y(r) = u(|x|)$, $r = |x|$. Then $u(x)$ is rectifiable oscillatory near $|x| = 1$ if and only if the next two conditions are satisfied:*

(i) *$y(r)$ oscillates near $r = 1$, that is, there is an increasing sequence $r_n \in (r_0, 1)$ such that $y(r_n) = 0$ and $r_n \to 1$ as $n \to \infty$;*

(ii) *the graph $\Gamma(y) = \{(r,t) \in \mathbb{R} \times \mathbb{R} : r \in [r_0, 1),\ t = y(r)\}$ of y is a rectifiable curve in $\mathbb{R}^2$, that is, $\sqrt{1 + y'^2} \in L^1(r_0, 1)$.*

Furthermore, $u(x)$ is nonrectifiable oscillatory near $|x| = 1$ if and only if the statement (i) *is fulfilled and $\Gamma(y)$ is nonrectifiable curve in $\mathbb{R}^2$, that is, $\sqrt{1 + y'^2} \notin L^1(r_0, 1)$.*

Proof. Since $u(x) = y(r)$, $r = |x|$, we have $|\nabla u| = |y'|$, and it follows that

$$|\Gamma(u)|_S = \int_{\Gamma(u)} dS = \omega_N \int_{r_0}^1 r^{N-1} \sqrt{1 + y'(r)^2} dr,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. Since

$$r_0^{N-1}\int_{r_0}^1 \sqrt{1+y'(r)^2}dr \leqslant \int_{r_0}^1 r^{N-1}\sqrt{1+y'(r)^2}dr \leqslant \int_{r_0}^1 \sqrt{1+y'(r)^2}dr,$$

$u(x)$ is rectifiable oscillatory at $|x|=1$ if and only if $\sqrt{1+y'(r)^2}\in L^1(r_0,1)$, which proves this lemma. □

Let u be a radially symmetric solution of (1.1) and let $y(r)=u(x)$, $r=|x|$. Then $y(r)$ satisfies the one-dimensional equation (1.3). By the change of variables

$$z(t)=y(r) \quad \text{with} \quad \begin{cases} r=e^t & \text{if } p=N, \\ r=t^{\frac{p-1}{p-N}} & \text{if } p\neq N, \end{cases} \tag{2.1}$$

equation (1.3) is transformed into the equivalent one (see for instance [17] and [7]):

$$(|z'|^{p-2}z')'+F(t)|z|^{p-2}z=0, \;\; t\in I_N, \tag{2.2}$$

where

$$F(t)=\begin{cases} e^{Nt}f(e^t) & \text{if } p=N, \\ \left|\frac{p-1}{p-N}\right|^p t^{\frac{pN-p}{p-N}} f(t^{\frac{p-1}{p-N}}) & \text{if } p\neq N, \end{cases} \tag{2.3}$$

and

$$I_N=\begin{cases} (t_0,0) \;\text{ for some } t_0\in(-\infty,0) & \text{if } p=N, \\ (t_0,1) \;\text{ for some } t_0\in(0,1) & \text{if } p>N, \\ (1,t_0) \;\text{ for some } t_0\in(1,\infty) & \text{if } p<N. \end{cases} \tag{2.4}$$

Since

$$\int_{r_0}^1 \left|\frac{dy}{dr}\right| dr = \int_{I_N}\left|\frac{dz}{dt}\right| dt, \tag{2.5}$$

we find that $dy/dr\in L^1([r_0,1])$ if and only if $dz/dt\in L^1(I_N)$. It is easy to see that the structural conditions from (1.2), that is, $f\in C^2([r_0,1))$, $f(r)>0$ on $(r_0,1)$ and $f(1-)=\infty$, are equivalent to:

$$F\in C^2(I_N),\; F(t)>0 \text{ on } I_N \;\text{ and }\; \begin{cases} F(0-)=\infty & \text{if } p=N, \\ F(1\pm)=\infty & \text{if } p\neq N. \end{cases} \tag{2.6}$$

Let us recall the results by [14] for the rectifiable oscillations of equation (2.2).

LEMMA 2.2. (See [14, Theorem 2].) *Let η and θ be arbitrary positive constants such that $\eta+\theta=1/p$. Let F satisfy (2.6) and*

$$F^{-\theta}\frac{d^2}{dt^2}F^{-\eta}\in L^1(I_N). \tag{2.7}$$

Then problem (2.2) with $z(1)=0$ is rectifiable oscillatory near $t=1$ if and only if

$$F^{1/p^2}\in L^1(I_N). \tag{2.8}$$

Define $Q = Q(r;\eta)$ by

$$Q(r;\eta) = r^{-\frac{p(N-1)}{p-1}\eta}[f(r)]^{-\eta}, \ \ r \in (r_0,1). \tag{2.9}$$

First we show the following lemma.

LEMMA 2.3. (i) *Let* η *and* θ *be arbitrary positive constants such that* $\eta+\theta = 1/p$. *Condition* (2.7) *is equivalent to*

$$Q^{\theta/\eta}\frac{d}{dr}\Big(r^{\frac{N-1}{p-1}}\frac{dQ}{dr}\Big) \in L^1(r_0,1). \tag{2.10}$$

(ii) *Condition* (2.8) *is equivalent to*

$$r^{-\frac{N-1}{p-1}}[Q(r;1/p^2)]^{-1} \in L^1(r_0,1). \tag{2.11}$$

Proof. (i) Let firstly consider the case $p = N$. Since $F(t) = e^{Nt}f(e^t)$, then we have

$$\int_{t_0}^0 \frac{1}{[F(t)]^\theta}\frac{d^2}{dt^2}\Big(\frac{1}{[F(t)]^\eta}\Big)dt = \int_{t_0}^0 [Q(e^t;\eta)]^{\theta/\eta}\frac{d^2}{dt^2}\big(Q(e^t;\eta)\big)dt = I_1.$$

Since

$$\frac{d^2}{dt^2}\big(Q(e^t;\eta)\big) = r\frac{d}{dr}\Big(r\frac{dQ}{dr}\Big),$$

we obtain

$$I_1 = \int_{t_0}^0 [Q(e^t;\eta)]^{\theta/\eta}\frac{d}{dr}\Big(r\frac{dQ}{dr}\Big)r\,dt = \int_{r_0}^1 [Q(r)]^{\theta/\eta}\frac{d}{dr}\Big(r\frac{dQ}{dr}\Big)dr.$$

Next we consider the case $p \neq N$. In this case, since

$$F(t) = C_{p,N}t^{\frac{pN-p}{p-N}}f(t^{\frac{p-1}{p-N}}), \quad \text{where } C_{p,N} = \Big|\frac{p-1}{p-N}\Big|,$$

we have

$$\int_a^b \frac{1}{[F(t)]^\theta}\frac{d^2}{dt^2}\Big(\frac{1}{[F(t)]^\eta}\Big)dt = C\int_a^b \big[Q(t^{\frac{p-1}{p-N}};\eta)\big]^{\theta/\eta}\frac{d^2}{dt^2}\Big(Q(t^{\frac{p-1}{p-N}};\eta)\Big)dt = I_2.$$

Here the interval $[a,b]$ denotes $[t_0,1]$ or $[1,t_0]$ respectively for $p > N$ or $p < N$. Since

$$\frac{d^2}{dt^2}\Big(Q(t^{\frac{p-1}{p-N}};\eta)\Big) = \Big(\frac{p-1}{p-N}\Big)^2 r^{\frac{N-1}{p-1}}\frac{d}{dr}\Big(r^{\frac{N-1}{p-1}}\frac{dQ}{dr}\Big),$$

we obtain

$$I_2 = C'\int_a^b [Q(t^{\frac{p-1}{p-N}};\eta)]^{\theta/\eta} r^{\frac{N-1}{p-1}}\frac{d}{dr}\Big(r^{\frac{N-1}{p-1}}\frac{dQ}{dr}\Big)dt$$

$$= C' \int_{r_0}^{1} [Q(r)]^{\theta/\eta} \frac{d}{dr}\Big(r^{\frac{N-1}{p-1}} \frac{dQ}{dr}\Big) dr,$$

which proves that (2.10) is equivalent to (2.7).

(ii) First let us consider the case $p = N$. Since

$$F(t) = e^{Nt} f(e^t) \quad \text{and} \quad Q(e^t; 1/p^2) = e^{-t/p}[f(e^t)]^{-1/p^2}, \ \ r \in (r_0, 1),$$

we have

$$\int_{-\varepsilon}^{0} [F(t)]^{1/p^2} dt = \int_{-\varepsilon}^{0} e^{t/N} [f(e^t)]^{1/p^2} dt = \int_{-\varepsilon}^{0} \frac{1}{Q(e^t; 1/p^2)} dt = \int_{e^{-\varepsilon}}^{1} \frac{1}{rQ(r; 1/p^2)} dr.$$

Next let us consider the case $p \neq N$. Since

$$F(t) = C_{p,N} t^{\frac{pN-p}{p-N}} f(t^{\frac{p-1}{p-N}}), \quad C_{p,N} = \left|\frac{p-1}{p-N}\right|^p,$$

then we have

$$\begin{aligned}
\int_{1}^{1+\varepsilon} [F(t)]^{1/p^2} dt &= C_{p,N}^{1/p^2} \int_{1}^{1+\varepsilon} t^{\frac{N-1}{p^2-pN}} [f(t^{\frac{p-1}{p-N}})]^{1/p^2} dt \\
&= C_{p,N}^{1/p^2} \int_{1}^{1+\varepsilon} \frac{1}{Q(t^{\frac{p-1}{p-N}}; 1/p^2)} dt \\
&= C'_{p,N} \int_{1-\tilde{\varepsilon}}^{1} \frac{1}{r^{\frac{N-1}{p-1}} Q(r; 1/p^2)} dr,
\end{aligned}$$

where $C'_{p,N} = \left|\frac{p-1}{p-N}\right|^{(p-1)/p}$ and ε and $\tilde{\varepsilon}$ have a relation $1+\varepsilon = (1-\tilde{\varepsilon})^{(N-1)/(p-1)}$. In the above, we have used (2.9), that is,

$$Q(t^{(p-1)/(p-N)}; 1/p^2) = \frac{1}{t^{(N-1)/(p^2-pN)} [f(t^{(N-1)/(p-N)})]^{1/p^2}}.$$

Thus, we have proved that (2.11) is equivalent to (2.8). □

LEMMA 2.4. *Let η and θ be arbitrary positive constants such that $\eta + \theta = 1/p$. Then condition* (2.10) *is equivalent to* (1.5).

In order to prove Lemma 2.4, we need the following result by [14].

LEMMA 2.5. (See [14, Lemma 1].) *Let f satisfy* (1.2)*. Let η_1, θ_1, η_2, and θ_2 be positive constants such that $\eta_1 + \theta_1 = \eta_2 + \theta_2$. Then $f^{-\theta_1}\big(f^{-\eta_1}\big)'' \in L^1(I)$ if and only if $f^{-\theta_2}\big(f^{-\eta_2}\big)'' \in L^1(I)$.*

PROOF OF LEMMA 2.4. Applying Lemma 2.5 with $f = F$, we may assume that (2.7) holds with $\theta = \eta = 1/(2p)$. Then, by Lemma 2.3 (i), we obtain (2.10) with $\theta = \eta = 1/(2p)$.

In order to simplify notation, let $q = (N-1)/(p-1)$. Then

$$Q(r;\eta) = r^{-pq\eta} f(r)^{-\eta}$$

and

$$Q_r(r;\eta) = -pq\eta r^{-pq\eta-1}[f^{-\eta}] + r^{-pq\eta}[f^{-\eta}]_r.$$

By a direct calculation, we obtain

$$\begin{aligned} Q(r;\eta)^{\theta/\eta} &\left(r^{(N-1)/(p-1)} Q_r(r;\eta)\right)_r \\ &= Q(r;\eta)^{\theta/\eta} \left(r^q Q_r(r;\eta)\right)_r \\ &= \frac{pq\eta(pq\eta+1-q)}{r^2}[f^{-\theta}][f^{-\eta}] \\ &\qquad + \frac{q(1-2p\eta)}{r}[f^{-\theta}][f^{-\eta}]_r + [f^{-\theta}][f^{-\eta}]_{rr}. \end{aligned} \tag{2.12}$$

Putting $\theta = \eta = 1/(2p)$ in (2.12), we have

$$\begin{aligned} Q(r;\eta) &\left(r^{(N-1)/(p-1)} Q_r(r;1/(2p))\right)_r \\ &= \frac{pq\eta(pq\eta+1-q)}{r^2}\left[f^{-1/(2p)}\right]\left[f^{-1/(2p)}\right] + \left[f^{-1/(2p)}\right]\left[f^{-1/(2p)}\right]_{rr}. \end{aligned} \tag{2.13}$$

Since $f(r) \to \infty$ as $r \to 1-$, we see that the function $r^{-2}[f^{-1/(2p)}][f^{-1/(2p)}]$ is bounded on $[r_0, 1)$ and so,

$$r^{-2}[f^{-1/(2p)}][f^{-1/(2p)}] \in L^1([r_0,1)).$$

Then it follows that

$$f^{-\frac{1}{2p}}\left(f^{-\frac{1}{2p}}\right)'' \in L^1(r_0,1). \tag{2.14}$$

By applying Lemma 2.5 again, we obtain (1.5).

Conversely, we assume that (1.5) holds. By Lemma 2.5 we may assume that (2.14) holds. By (2.13), we obtain (2.10) with $\theta = \eta = 1/(2p)$. By applying Lemma 2.5 with $f = F$, we have (2.10). □

LEMMA 2.6. *The condition* (2.11) *is equivalent to* (1.6).

Proof. Observe that

$$\frac{1}{r^{(N-1)/(p-1)} Q(r;1/p^2)} = r^{(N-1)/p} f(r)^{1/p^2}.$$

Then it follows that

$$r_0^{(N-1)/p} \int_{r_0}^{1-\varepsilon} f(x)^{1/p^2} dx \leqslant \int_{r_0}^{1-\varepsilon} \frac{1}{r^{(N-1)/(p-1)} Q(r;1/p^2)} dr \leqslant \int_{r_0}^{1-\varepsilon} f(x)^{1/p^2} dx.$$

Thus (2.11) is equivalent to (1.6). □

PROOF OF THEOREM 1.1. Let $u=u(x)=y(|x|)$ be a solution of our main equation (1.1), where the coefficient $f=f(r)$ satisfies conditions (1.2) and (1.5). Then by Lemmas 2.3 (i) and 2.4, the coefficient $F=F(t)$ (defined in (2.3)) of equation (2.2) satisfies the required conditions (2.6) and (2.7). Hence, by Lemma 2.2, equation (2.2) is rectifiable oscillatory if and only if condition (2.8) is fullfiled. Together with (ii) of Lemma 2.3 and Lemma 2.6, this proves that (2.2) is rectifiable oscillatory if and only if condition (1.6) is valid. We know that the rectifiability of the graph $G(z)$ of a smooth function $z(t)$, defined on the interval I, is equivalent to $dz/dt \in L^1(I)$, see [8, Theorem 1]. By (2.5), it means that $dy/dr \in L^1(I)$ if and only if (1.6) is valid. Lemma 2.1 implies that (1.1) is rectifiable oscillatory near $|x|=1$ if and only if (1.6) is valid, which proves this theorem. □

PROOF OF THEOREM 1.2. By integrating (1.10) we obtain the corresponding asymptotic behaviour for $f(r)$ and $f'(r)$ and hence, we have:

$$f(r) \sim (1-r)^{-\sigma}, \;\; f'(r) \sim -(1-r)^{-\sigma-1} \text{ and } f''(r) \sim (1-r)^{-\sigma-2} \text{ as } r \to 1. \quad (2.15)$$

Next, we show that a function $f(r)$ which satisfies (1.2) and (2.15) also satisfies the Hartman-Wintner condition (1.5) provided $\sigma > p$. In fact,

$$f^{-\frac{1}{2p}}(f^{-\frac{1}{2p}})'' = \frac{1+2p}{4p^2} f^{-\frac{1+2p}{p}}(f')^2 - \frac{1}{2p} f^{-\frac{1+p}{p}} f'',$$

and then

$$\left| f^{-\frac{1}{2p}}\left(f^{-\frac{1}{2p}}\right)''\right| \sim (1-r)^{\frac{\sigma}{p}-2} \in L^1(r_0,1),$$

since $\sigma > p$. Moreover, from (2.15) easily follows: $f^{\frac{1}{p^2}} \in L^1(r_0,1)$ if and only if $\sigma < p^2$. Now, with the help of Theorem 1.1 we complete the proof of this theorem. □

3. Proofs of Theorem 1.3 and Corollary 1.3

In this section, we give the proofs of Theorem 1.3 and Corollary 1.3 which have been stated in Section 1.

Let $u(x) = y(|x|)$ be a radially symmetric solution of equation (1.1), where the coefficient $f(r)$ satisfies the required assumptions (1.2) and (1.5). By the change of variables given in (2.1), we know that the function $z(t) = y(r)$ satisfies the differential equation (2.2): $(|z'|^{p-2}z')' + F(t)z^{p-2}z = 0$, $t \in I_N$, where the coefficient $F(t)$ and the interval I_N are given respectively by (2.3) and (2.4).

We consider the case $I_N = (t_0, 1)$. The other cases from (2.4), that is $I_N = (t_0, 0)$ and $I_N = (1, t_0)$ can be analogously considered and they are left to the reader.

The generalized sine function $\sin_p$ is defined by the solution to the problem

$$(|S'|^{p-2}S')' + (p-1)|S|^{p-2}S = 0, \quad S(0) = 0 \text{ and } S'(0) = 1.$$

The function $\sin_p$ is defined on $\mathbb{R}$ and is periodic with period $2\pi_p$, where

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)}.$$

It is known that $\sin_p$ satisfies $|\sin_p t|^p + |\sin_p' t|^p = 1$ for all $t \in \mathbb{R}$,

$$\sin_p(k\pi_p) = 0 \quad \text{for all } k \in \mathbb{N},$$

$$\sin_p'(k\pi_p + \pi_p/2) = 0 \quad \text{and} \quad |\sin_p(k\pi_p + \pi_p/2)| = 1 \quad \text{for all } k \in \mathbb{N},$$

where $\sin_p'$ denotes the first derivative of $\sin_p$.

Similar to as in [14, Lemma 2, Propositions 1 and 2], we obtain the following lemma.

LEMMA 3.1. *Let η and θ be arbitrary positive constants such that $\eta + \theta = 1/p$. Let F satisfy* (2.6) *and* (2.7)*. Then all solutions $z = z(t)$ of equation* (2.2) *admit the following asymptotic formula:*

$$z(t) = (p-1)^{\frac{1}{pq}} F^{-\frac{1}{pq}}(t) V^{\frac{1}{p}}(t) \sin_p(\varphi(t)) \ \textit{near } t = 1, \tag{3.1}$$

where $1/p + 1/q = 1$ and the energy functional $V(t)$ and the phase $\varphi(t)$ satisfy:

$$\begin{cases} 0 < \lim_{t\to 1-} V(t) < \infty & \textit{and} \quad \lim_{t\to 1-} \varphi(t) = \infty, \\ \varphi'(t) > 0 \ \textit{on } (t_0, 1) & \textit{and} \quad \varphi'(t) \sim F^{\frac{1}{p}}(t) \ \textit{as } t \to 1. \end{cases} \tag{3.2}$$

PROOF OF THEOREM 1.3. From the assumption of this theorem, we know that there exists an increasing sequence $r_n \in (r_0, 1)$ of consecutive zeros of $y(r)$ such that $r_n \to 1$ and $r_{n+1} - r_n \sim r_{n+2} - r_{n+1}$ as $n \to \infty$. Hence by the change of variables given in (2.1) we get an increasing sequence $t_n \in (t_0, 1)$ of consecutive zeros of $z(t)$ such that

$$t_n \to 1 \quad \text{and} \quad t_{n+1} - t_n \sim t_{n+2} - t_{n+1} \quad \text{as } n \to \infty. \tag{3.3}$$

Since $z \in W_0^{1,p}(t_0, 1)$, there exists a $c > 0$ depending only on p such that (see [1])

$$\sup_{(t_n, t_{n+1})} |z(t)| \leqslant c(t_{n+1} - t_n)^{1-1/p} ||z'||_{L^p(t_n, t_{n+1})}.$$

It allows us to conclude that for some $n_0 \in \mathbb{N}$ and for all $m \geqslant n_0$,

$$||z'||^p_{L^p(t_0, t_{m+1})} \geqslant \sum_{n=n_0}^{m} ||z'||^p_{L^p(t_n, t_{n+1})} \geqslant c \sum_{n=n_0}^{m} \frac{\max_{(t_n, t_{n+1})} |z(t)|^p}{(t_{n+1} - t_n)^{p-1}}. \tag{3.4}$$

Moreover, from (3.3) and (3.4), we observe that

$$||z'||^p_{(t_0, t_{m+1})} \geqslant c \sum_{n=n_0}^{m} \frac{\max_{(t_n, t_{n+1})} |z(t)|^p}{(t_{n+2} - t_{n+1})^{p-1}}. \tag{3.5}$$

Since t_n is an increasing sequence of consecutive zeros of $z(t)$, from the asymptotic formula (3.1) and from (3.2), we derive that there exists an $n_* \in \mathbb{N}$ such that:

$$\varphi(t_n) = (n+n_*)\pi_p \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

We also get the existence of a sequence $s_n \in (t_n, t_{n+1})$ such that

$$\varphi(s_n) = (n+n_*+1/2)\pi_p \quad \text{and} \quad |\sin_p(\varphi(s_n))| = 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

By the mean value theorem, there exists $\sigma_{n+1} \in (t_{n+1}, t_{n+2})$ such that

$$\pi_p = \varphi(t_{n+2}) - \varphi(t_{n+1}) = \varphi'(\sigma_{n+1})(t_{n+2} - t_{n+1}).$$

From (3.2), we get the existence of a positive constant c_1 and of an $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$,

$$\frac{1}{(t_{n+2}-t_{n+1})^{p-1}} = \frac{1}{\pi_p^{p-1}}[\varphi'(\sigma_{n+1})]^{p-1} \geqslant c_1\left[F^{\frac{1}{p}}(\sigma_{n+1})\right]^{p-1}. \tag{3.8}$$

Note here that $F(t)$ is increasing near $t=1$, since $f(t)$ is increasing near $t=1$ and F is given by (2.3). From $t_n < s_n < t_{n+1} < \sigma_{n+1} < t_{n+2}$, we have

$$\frac{1}{(t_{n+2}-t_{n+1})^{p-1}} \geqslant c_1\left[F^{\frac{1}{p}}(s_n)\right]^{p-1} = c_1[F(s_n)]^{1-\frac{1}{p}}. \tag{3.9}$$

On the other hand, from (3.1), (3.2) and (3.7), we obtain the existence of a positive constant c_2 and of an $n_2 \in \mathbb{N}$ such that for all $n \geqslant n_2$,

$$\max_{(t_n,t_{n+1})} |z(t)|^p \geqslant |z(s_n)|^p \geqslant c_2\left[F^{-\frac{1}{pq}}(s_n)\right]^p = c_2[F(s_n)]^{\frac{1}{p}-1}, \tag{3.10}$$

since $1/p+1/q=1$.

Finally, by chosing $n_0 = \max\{n_1, n_2\}$, where n_1, n_2 are as in (3.8) and (3.10) and using (3.9) and (3.10) into (3.5), we obtain:

$$\begin{aligned} ||z'||^p_{(t_0,t_{m+1})} &\geqslant c\sum_{n=n_0}^{m} \frac{\max_{(t_n,t_{n+1})}|z(t)|^p}{(t_{n+2}-t_{n+1})^{p-1}} \geqslant c_3\sum_{n=n_0}^{m}[F(s_n)]^{1-\frac{1}{p}}[F(s_n)]^{\frac{1}{p}-1} \\ &= c_3\sum_{n=n_0}^{m} 1 = c_3(m-n_0). \end{aligned}$$

Taking $m \to \infty$ in previous inequality, we get $z' \notin L^p(t_0,1)$ which implies that $y' \notin L^p(r_0,1)$ and $|\nabla u| \notin L^p(\Omega)$, $p>1$. It proves this theorem. □

PROOF OF COROLLARY 1.3. In the proof of Theorem 1.2, it is shown that assumptions (1.2) and (1.10) ensure that the coefficient $f(r)$ also satisfies the Hartman-Wintner condition (1.5). Hence, $f(r)$ satisfies all required assumptions of Theorem 1.3. Thus, in order to be able to use Theorem 1.3, it remains to show that every radially symmetric solution $u(x) = y(|x|)$ admits an increasing sequence of zeros $r_n \in (r_0, 1)$ such that

$$r_n \to 1 \quad \text{and} \quad r_{n+1} - r_n \sim r_{n+2} - r_{n+1} \quad \text{as } n \to \infty. \tag{3.11}$$

In fact, by the reasons already presented at the begining of the proof of Theorem 1.3, we have all solutions $z = z(t)$ of equation (2.2) allowing the asymptotic formula (3.1) and (3.2) (let for instance $I_N = (t_0, 1)$, the other cases: $I_N = (t_0, 0)$ and $I_N = (1, t_0)$ can be analogously considered).

By the asymptotic assumption for $f(t)$ near $t = 1$ given in (2.15) we have $F(s) \sim (1-s)^{-\sigma}$ as $s \to 1$. By the asymptotic behaviour of $\varphi'(t)$ near $t = 1$, determined in (3.2): $\varphi'(t) \sim F^{1/p}(t)$ as $t \to 1$, we obtain:

$$\varphi(t) \sim \int_{t_0}^{t} F^{\frac{1}{p}}(s)ds \sim \int_{t_0}^{t} (1-s)^{-\frac{\sigma}{p}} ds \sim (1-t)^{-\frac{\sigma}{p}+1} \quad \text{as } t \to 1. \tag{3.12}$$

Next, with the help of (3.3) we know that sequence t_n of consecutive zeros of every solution $z(t)$ of (2.2) satisfies $\varphi(t_n) \sim n$ as $n \to \infty$ (see (3.6)), which together with (3.12) shows that $t_n \sim 1 - n^{-p/(\sigma-p)}$ as $n \to \infty$ and thus,

$$t_{n+1} - t_n \sim n^{-\frac{\sigma}{\sigma-p}} \sim (n+1)^{-\frac{\sigma}{\sigma-p}} \sim t_{n+2} - t_{n+1} \quad \text{as } n \to \infty.$$

It proves the desired statement (3.11) by using the change of variables given in (2.1). □

REFERENCES

[1] H. BREZIS, *Analyse fonctionelle. Théorie et applications*, Masson, Paris, 1983.

[2] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, New York, 1999.

[3] P. HARTMAN, *Ordinary Differential Equations*, Second edition, Birkhauser, Boston, Basel, Stuttgart, 1982.

[4] M. K. KWONG, M. PAŠIĆ, AND J. S. W. WONG, *Rectifiable oscillations in second order linear differential equations*, J. Differential Equations, **245** (2008), 2333–2351.

[5] R. MARIK, *Oscillation criteria for PDE with p - Laplacian via the Riccati technique*, J. Math. Anal. Appl., **248** (2000), 290–308.

[6] R. MARIK, *Ordinary differential equations in the oscillation theory of partial half-linear differential equation*, J. Math. Anal. Appl., **338** (2008), 194–208.

[7] Y. NAITO, *Uniqueness of positive solutions of quasilinear differential equations*, Differential Integral Equations, **8** (1995), 1813–1822.

[8] M. PAŠIĆ, *Rectifiable and unrectifiable oscillations for a class of second-order linear differential equations of Euler type*, J. Math. Anal. Appl., **335** (2007), 724–738.

[9] M. PAŠIĆ, *Fractal oscillations for a class of second-order linear differential equations of Euler type*, J. Math. Anal. Appl., **341** (2008), 211–223.

[10] M. PAŠIĆ, *Rectifiable and unrectifiable oscillations for a generalization of the Riemann - Weber version of Euler differential equations*, Georgian Math. J., **15** (2008), 759–774.

[11] M. PAŠIĆ AND S. TANAKA, *Rectifiable oscillations of self-adjoint and damped linear differential equations of second-order*, J. Math. Anal. Appl., **381** (2011), 27–42.

[12] M. PAŠIĆ AND S. TANAKA, *Fractal oscillations of self-adjoint and damped linear differential equations of second-order*, Appl. Math. Comp., **218** (2011), 2281–2293.

[13] M. PAŠIĆ AND J. S. W. WONG, *Two-point oscillations oscillations in second-order linear differential equations*, Differ. Equ. Appl., **1** (2009), 85–122.

[14] M. PAŠIĆ AND J. S. W. WONG, *Rectifiable oscillations in second-order half-linear differential equations*, Ann. Mat. Pura Appl. (4), **188** (2009), 517–541.
[15] J.-M. RAKOTOSON, *Equivalence between the growth of* $\int_{B(x,r)} |\nabla u|^p dy$ *and* T *in the equation* $P[u] = T$, J. Differential Equations, **86** (1990), 102–122.
[16] C. TRICOT, *Curves and Fractal Dimension*, Springer-Verlag, New York, 1995.
[17] J. WANG, *On second order quasilinear oscillations*, Funkcialaj Ekvacioj, **41** (1998), 25–54.
[18] J. S. W. WONG, *On rectifiable oscillation of Euler type second order linear differential equations*, E. J. Qualitative Theory of Diff. Equ., **20** (2007), 1–12.

(Received August 19, 2011)

Yūki Naito
Department of Mathematics
Ehime University
Matsuyama 790-8577
Japan
e-mail: ynaito@ehime-u.ac.jp

Mervan Pašić
Department of Mathematics, FER
University of Zagreb
10000 Zagreb
Croatia
e-mail: mervan.pasic@fer.hr

Hiroyuki Usami
Department of Applied Mathematics
Gifu University
Gifu 501-1193
Japan
e-mail: husami@gifu-u.ac.jp