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ON SECOND–ORDER FUNCTIONAL DIFFERENTIAL INCLUSIONS IN HILBERT SPACES

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Abstract. We prove the existence result of monotone solutions, in Hilbert space, for the differential inclusion $\ddot{x}(t) \in f(t,T(t)x,\dot{x}(t)) + F(T(t)x,\dot{x}(t))$, where f is a Carathéodory single-valued mapping and F is an upper semicontinuous set-valued mapping with compact values contained in the Clarke subdifferential $\partial_c V(x)$ of a uniformly regular function V.

1. Introduction

Let H be a separable Hilbert space with the norm $\|.\|$ and the scalar product $\langle .,. \rangle$. For any segment I in $\mathbb{R}$, we denote by $\mathscr{C}(I,H)$ the Banach space of continuous functions from I to H equipped with the norm $\|x(.)\|_\infty := \sup\{\|x(t)\|; t \in I\}$. For all positive number a, we put $\mathscr{C}_a := \mathscr{C}([-a,0],H)$ and for any $t \in [0,T]$, $T > 0$, we define the operator $T(t)$ from $\mathscr{C}([-a,T],H)$ to $\mathscr{C}_a$ by $(T(t)x)(s) = x(t+s)$. For a given nonempty subset K of H, we introduce the set $\mathscr{K}_0 := \left\{\varphi \in \mathscr{C}_a; \varphi(0) \in K\right\}$.

This paper is devoted to prove the existence of solutions to the following Cauchy problem:

$$\begin{cases} \ddot{x}(t) \in f(t,T(t)x,\dot{x}(t)) + F(T(t)x,\dot{x}(t)) & \text{a.e. on } [0,\tau], \\ x(s) = \varphi(s), \ \forall s \in [-a,0], \\ x(s) \in P(x(t)), \ \forall t \in [0,\tau], \ \forall s \in [t,\tau], \end{cases} \tag{1.1}$$

where F is an upper semicontinuous multifunction with compact values, f is a Carathéodory function and P is a lower semicontinuous multifunction.

Existence of solutions of second-order differential inclusions has been studied by many authors. For instance see [1, 2, 4, 6, 14, 15] and the references therein.

Existence of viability result for functional differential inclusions was first suggested by Haddad [10, 11], when the right-hand side is upper semicontinuous with convex and compact values, in finite dimensional vector space. For review of other results on functional differential inclusions, we refer the reader to the papers by Gavioli and Malaguti [9], Syam [17] and the references therein.

The viability result for second order differential inclusions (1.1) was given by Lupulescu [13] in the case in which $f \equiv 0$ and $P(x) = K$. Ibrahim and Al-Adsani

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[12] proved the existence of monotone solutions for (1.1) without perturbation $(f \equiv 0)$. Note that in [12, 13], the right hand-side is contained in the subdifferential of a proper convex function. Cernia [6] considered the same situation but the right hand-side is contained in the Fréchet subdifferential of a ϕ-convex function of order two. Here it is necessary to mention that the works [6, 12, 13] has been studied in the finite dimensional space.

This work extends results which are presented in [6, 12, 13] to the infinite dimensional case. Furthermore, we assume that F is contained in the Clarke subdifferential $\partial_c V$, where V belongs to the class of uniformly regular functions which contains strictly the class of convex functions and the class of lower-C^2 functions. As is known, viability problems need tangential conditions. For the problems (1.1), we shall use a tangency condition which is weaker than that used in [6, 12, 13].

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, in Section 3, we give some preliminary results, while in Section 4, we prove the existence of solutions for (1.1).

2. Preliminaries and statement of the main result

For $x \in H$ and $r > 0$ let $B(x,r) := \{y \in H; \|y-x\| < r\}$ be the open ball centered at x with radius r, $\overline{B}(x,r)$ be its closure and let $B = B(0,1)$. For $\varphi \in \mathscr{C}_a$ let $B_a(\varphi,r) := \{\psi \in \mathscr{C}_a; \|\varphi - \psi\|_\infty < r\}$ and $\overline{B}_a(\varphi,r)$ be its closure. For $x \in H$ and for a set $A \subset H$ we denote by $d_A(x)$ the distance from x to A given by $d_A(x) := \inf\{\|y-x\| : y \in A\}$.

We shortly review the definitions of the various extensions of derivatives used in this paper (see [7, 8, 16] as general references).

Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and x be any point where V is finite. The generalized Rockafellar directional derivative $V^\uparrow(x,.)$ is

$$V^\uparrow(x,v) := \limsup_{x' \to x, V(x') \to V(x), t \to 0^+} \inf_{v' \to v} \frac{V(x'+tv') - V(x')}{t}.$$

The Clarke subdifferential of V at x is defined by

$$\partial_c V(x) := \left\{ y \in H : \langle y, v \rangle \leqslant V^\uparrow(x,v), \text{ for all } v \in H \right\},$$

and that the proximal subdifferential $\partial_p V(x)$ of V at x is the set of all $y \in H$ for which there exist $\delta,\ \sigma > 0$ such that for all $x' \in x + \delta \overline{B}$,

$$\langle y, x'-x \rangle \leqslant V(x') - V(x) + \sigma \|x'-x\|^2.$$

Note that $\partial_c V(x)$ is convex and closed and $\partial_p V(x)$ is convex, but not necessarily closed. On the other hand, one always has $\partial_p V(x) \subset \partial_c V(x)$.

In the following proposition we summarize some useful properties of Clarke generalized directional derivatives.

PROPOSITION 2.1. [7, 8] *Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be locally Lipschitz. Then the following conditions hold:*

(i) $\partial_c V(x) = \{p \in H : V^o(x,v) \geqslant \langle p,v\rangle,\ \forall v \in H\} = \{p \in H : V_o(x,v) \leqslant \langle p,v\rangle, \forall v \in H\}$;

(ii) $V^o(x,v) = \max\{\langle p,v\rangle, p \in \partial_c V(x)\}$ *and* $V_o(x,v) = \min\{\langle p,v\rangle, p \in \partial_c V(x)\} = -V^o(x,-v)$.

Let us recall the definition of the concept of regularity that will be used in the sequel.

DEFINITION 2.2. [5] Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $U \subset \mathrm{Dom}(V)$ be a nonempty open subset. We will say that V is uniformly regular over U if there exists a positive number β such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has

$$\langle \xi, x' - x\rangle \leqslant V(x') - V(x) + \beta \|x' - x\|^2 \quad \text{for all } x' \in U.$$

We say that V is uniformly regular over a closed set S if there exists an open set U containing S such that V is uniformly regular over U.

The class of functions that are uniformly regular over sets is so large. Any l.s.c. proper convex function V is uniformly regular over any nonempty subset of its domain with $\beta = 0$. For more details to the concept of regularity, we refer the reader to [5].

The following proposition summarizes some important properties for uniformly regular locally Lipschitz functions over sets needed in this paper.

PROPOSITION 2.3. [5] *Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and S a nonempty closed set. If V is uniformly regular over S, then the following conditions hold:*

(a) *the proximal subdifferential of V is closed as a multifunction over S, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ weakly with $\xi_n \in \partial_p V(x_n)$ one has $\xi \in \partial_p V(x)$;*

(b) *the proximal subdifferential of V coincides with the Clarke subdifferential of V for any point x;*

(c) *the proximal subdifferential of V is upper semicontinuous over S, that is, the support function $x \mapsto \sigma(v, \partial_p V(x))$ is u.s.c. over S for every $v \in H$.*

Assume that the following hypotheses hold:

(H1) (a) K is a nonempty locally compact subset in H and Ω is a nonempty open subset in H such that $\overline{\Omega}$ is compact and $K \times \overline{\Omega} \subset \mathrm{Graph}(T_K)$, where $T_K(x)$ is the Bouligand's contingent cone of K at x,

(b) $P : H \to 2^K$ is a lower semicontinuous set-valued map satisfying:

(i) for all $x \in K$, $x \in P(x)$,

(ii) for all $x \in K$ and all $y \in P(x)$ we have $P(y) \subseteq P(x)$;

(H2) $F : \mathscr{K}_0 \times \Omega \to 2^H$ is an upper semicontinuous multifunction with compact values satisfying $F(\varphi, y) \subset \partial_c V(y) \cap W$ for all $(\varphi, y) \in \mathscr{K}_0 \times \Omega$, where W is a compact subset of H and $V : H \to \mathbb{R}$ is a locally Lipschitz function and uniformly regular over Ω;

(H3) $f : \mathbb{R} \times \mathscr{C}_a \times H \to H$ is a function with the following properties:

(i) for all $(\varphi, y) \in \mathscr{C}_a \times H$, $t \mapsto f(t, \varphi, y)$ is measurable,

(ii) for all $t \in \mathbb{R}$, $(\varphi, y) \mapsto f(t, \varphi, y)$ is continuous,

(iii) for all bounded subset S of $\mathscr{C}_a \times H$, there exists a nonempty compact subset C of H such that $f(t, \varphi, y) \in C$ for all $(t, \varphi, y) \in \mathbb{R} \times S$;

(H4) (*Tangential condition*) $\forall (t, \varphi, y) \in [0,1] \times \mathscr{K}_0 \times \overline{\Omega}$, $\exists v \in F(\varphi, y)$ such that

$$\liminf_{h \mapsto 0^+} \frac{2}{h^2} d_{P(\varphi(0))}\left(\varphi(0) + hy + \frac{h^2}{2} v + \int_t^{t+h} (t+h-s) f(s, \varphi, y) ds\right) = 0.$$

We shall prove the following result:

THEOREM 2.4. *If assumptions* (H1)-(H4) *are satisfied, then there exist* $T > 0$ *and an absolutely continuous function* $x(.) : [-a, T] \to H$, *for which* $\dot{x}(.) : [0, T] \to H$ *is also absolutely continuous such that* $x(.)$ *is a solution of* (1.1).

In all the paper, we suppose that the assumptions (H1)-(H4) are satisfied, we fix $(\varphi, y_0) \in \mathscr{K}_0 \times \Omega$ and we choose $r > 0$ such that $K_0 = K \cap \overline{B}(\varphi(0), r)$ is compact, $\Omega_0 = \overline{B}(y_0, r) \subset B(y_0, 2r) \subset \Omega$ and V is Lipschitz continuous on $B(y_0, 2r)$ with Lipschitz constant $\lambda > 0$. Then $\partial_c V(y) \subset \lambda \overline{B}$ for every $y \in \Omega_0$. Let $\sigma > 0$ such that $\overline{\Omega} \subset B(0, \sigma)$ and let C be a compact subset of H such that

$$f(t, \psi, y) \in C, \quad \forall (t, \psi, y) \in \mathbb{R} \times \left(\mathscr{K}_0 \cap \overline{B}_a(\varphi, 2r)\right) \times B(0, \sigma).$$

Let $M > 0$ such that $C \subset B(0, M)$. For $\varepsilon > 0$ set

$$\eta(\varepsilon) := \sup\left\{\rho \in]0, \varepsilon] : \|\varphi(t_1) - \varphi(t_2)\| < \varepsilon \text{ if } |t_1 - t_2| \leqslant \rho\right\}. \tag{2.1}$$

REMARK 2.5. If $K \cap \overline{B}(\varphi(0), r)$ is closed in H, then $\mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$ is closed in $\mathscr{C}_a$. Indeed, let $(\psi_n)_{n \in \mathbb{N}}$ a sequence in $\mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$ which converges to ψ. We have $\psi_n \in \overline{B}_a(\varphi, r)$ for all $n \in \mathbb{N}$, so $\psi \in \overline{B}_a(\varphi, r)$. On the other hand since $\psi_n \in \mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$ for all $n \in \mathbb{N}$, one has $\psi_n(0) \in K \cap \overline{B}(\varphi(0), r)$ for all $n \in \mathbb{N}$, so by the closedness of $K \cap \overline{B}(\varphi(0), r)$ we get $\psi(0) \in K$. Hence $\psi \in \mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$.

3. Preliminary results

In this section, we shall prove some auxiliary results needed in the next section. Consider first the following hypotheses which we shall use throughout this section.

(A1) $G : K \times \Omega \to 2^H$ is an upper semicontinuous multifunction with compact values satisfying $G(x,y) \subset \partial_c V(y) \cap W$ for all $(x,y) \in K \times \Omega$;

(A2) $g : \mathbb{R} \times H \times H \to H$ is a function with the following properties:

(i) for all $(x,y) \in H \times H$, $t \mapsto g(t,x,y)$ is measurable,

(ii) for all $t \in \mathbb{R}$, $(x,y) \mapsto g(t,x,y)$ is continuous,

(iii) $g(t,x,y) \in C$ for all $(t,x,y) \in \mathbb{R} \times \overline{B}(\varphi(0),2r) \times B(0,\sigma)$;

(A3) (*Tangential condition*) $\forall (t,x,y) \in [0,1] \times K \times \overline{\Omega}$, $\exists v \in G(x,y)$ such that

$$\liminf_{h \mapsto 0^+} \frac{2}{h^2} d_{P(x)} \left(x + hy + \frac{h^2}{2} v + \int_t^{t+h} (t+h-s) g(s,x,y) ds \right) = 0.$$

In the sequel, we will use the following important Lemma. It will play a crucial role in the proof of Proposition 3.3.

LEMMA 3.1. *If assumptions* (A1)-(A3) *are satisfied, then for all* $\varepsilon > 0$, *there exists* $\eta > 0$ $(\eta < \varepsilon)$ *such that* $\forall (t,x,y) \in [0,1] \times K_0 \times \Omega_0$, *there exist* $h_{t,x,y} \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ *and* $u \in G(x,y) + \varepsilon B$ *such that*

$$\left(x + h_{t,x,y} y + \frac{h_{t,x,y}^2}{2} u + \int_t^{t+h_{t,x,y}} (t + h_{t,x,y} - s) g(s,x,y) ds \right) \in P(x).$$

Proof. Let $\varepsilon > 0$ and $(t,x,y) \in [0,1] \times K_0 \times \Omega_0$ be fixed. Since G is upper semicontinuous on (x,y), there exists $\delta_{x,y} > 0$ such that $G(\bar{x},\bar{y}) \subset G(x,y) + \frac{\varepsilon}{2} B$ for all $(\bar{x},\bar{y}) \in B((x,y),\delta_{x,y})$. Let $(s,\bar{x},\bar{y}) \in [0,1] \times K_0 \times \overline{\Omega}$. By the tangential condition, there exist $v \in G(\bar{x},\bar{y})$ and $h_{s,\bar{x},\bar{y}} \in]0, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ such that

$$d_{P(\bar{x})} \left(\bar{x} + h_{s,\bar{x},\bar{y}} \bar{y} + \frac{h_{s,\bar{x},\bar{y}}^2}{2} v + \int_s^{s+h_{s,\bar{x},\bar{y}}} (s + h_{s,\bar{x},\bar{y}} - \tau) g(\tau,\bar{x},\bar{y}) d\tau \right) < \frac{h_{s,\bar{x},\bar{y}} \varepsilon}{8}.$$

Consider the subset $N(s,\bar{x},\bar{y})$ of all $(\tilde{s},\tilde{x},\tilde{y})$ in $\mathbb{R} \times B(\varphi(0),2r) \times B(0,\sigma)$ such that

$$d_{P(\tilde{x})} \left(\tilde{x} + h_{s,\bar{x},\bar{y}} \tilde{y} + \frac{h_{s,\bar{x},\bar{y}}^2}{2} v + \int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x},\bar{y}}} (\tilde{s} + h_{s,\bar{x},\bar{y}} - \tau) g(\tau,\tilde{x},\tilde{y}) d\tau \right) < \frac{h_{s,\bar{x},\bar{y}}^2 \varepsilon}{8}.$$

Moreover, by hypothesis (A2), the dominated convergence theorem applied to the sequence of functions

$$\left(\chi_{[\tilde{s},\tilde{s}+h_{s,\bar{x},\bar{y}}]}(.) \varphi_{\tilde{s}}(.,\tilde{x},\tilde{y}) \right)_{\tilde{s}}, \text{ where } \varphi_{\tilde{s}}(\tau,\bar{x},\bar{y}) = (\tilde{s} + h_{s,\bar{x},\bar{y}} - \tau) g(\tau,\tilde{x},\tilde{y}),$$

shows that the function

$$(\tilde{s},\tilde{x},\tilde{y}) \mapsto \tilde{x} + h_{s,\bar{x},\bar{y}} \tilde{y} + \frac{h_{s,\bar{x},\bar{y}}^2}{2} v + \int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x},\bar{y}}} (\tilde{s} + h_{s,\bar{x},\bar{y}} - \tau) g(\tau,\tilde{x},\tilde{y}) d\tau$$

is continuous. Since P is lower semicontinuous, by Corollary 1.2.1 in [3], the function

$$(\tilde{s},\tilde{x},\tilde{y}) \mapsto d_{P(\tilde{x})}\left(\tilde{x}+h_{s,\bar{x},\bar{y}}\tilde{y}+\frac{h^2_{s,\bar{x},\bar{y}}}{2}v+\int_{\tilde{s}}^{\tilde{s}+h_{s,\bar{x},\bar{y}}}(\tilde{s}+h_{s,\bar{x},\bar{y}}-\tau)g(\tau,\tilde{x},\tilde{y})d\tau\right)$$

is upper semicontinuous. So $N(s,\bar{x},\bar{y})$ is open. Furthermore, since $(s,\bar{x},\bar{y})$ belongs to $N(s,\bar{x},\bar{y})$, there exists $0<\eta_{s,\bar{x},\bar{y}}<\delta_{x,y}$ such that $B((s,\bar{x},\bar{y}),\eta_{s,\bar{x},\bar{y}})$ is contained in $N(s,\bar{x},\bar{y})$, therefore, the compact subset $[0,1]\times K_0\times\overline{\Omega}$ can be covered by q such balls $B((s_i,\bar{x}_i,\bar{y}_i),\eta_{s_i,\bar{x}_i,\bar{y}_i})$. For simplicity, set $h_i := h_{s_i,\bar{x}_i,\bar{y}_i}$ and $\eta_i := \eta_{s_i,\bar{x}_i,\bar{y}_i}$, $i = 1,...,q$. Put $\eta = \min\{h_i/1 \leqslant i \leqslant q\}$. There exists $i\in\{1,...,q\}$ such that $(t,x,y)\in B((s_i,\bar{x}_i,\bar{y}_i),\eta_{s_i,\bar{x}_i,\bar{y}_i})$, hence $(t,x,y)\in N(s_i,\bar{x}_i,\bar{y}_i)$. Then there exists $v_i\in G(\bar{x}_i,\bar{y}_i)$ such that

$$d_{P(x)}\left(x+h_iy+\frac{h_i^2}{2}v_i+\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right)<\frac{h_i^2\varepsilon}{8}.$$

Let $x_i\in P(x)$ such that

$$\frac{2}{h_i^2}\left\|x_i-\left(x+h_iy+\frac{h_i^2}{2}v_i+\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right)\right\|$$
$$\leqslant \frac{2}{h_i^2}d_{P(x)}\left(x+h_iy+\frac{h_i^2}{2}v_i+\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right)+\frac{\varepsilon}{4},$$

hence

$$\left\|\frac{2}{h_i^2}\left(x_i-x-h_iy-\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right)-v_i\right\|<\frac{\varepsilon}{2}.$$

Set

$$u=\frac{2}{h_i^2}\left(x_i-x-h_iy-\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right),$$

then $u\in G(\bar{x}_i,\bar{y}_i)+\frac{\varepsilon}{2}B$ and

$$x_i=\left(x+h_iy+\frac{h_i^2}{2}u+\int_t^{t+h_i}(t+h_i-\tau)g(\tau,x,y)d\tau\right)\in P(x).$$

Since $\|(x,y)-(\bar{x}_i,\bar{y}_i)\|\leqslant\delta_{x,y}$ we have $u\in G(x,y)+\varepsilon B$. □

In the sequel, we need the following Lemma.

LEMMA 3.2. *For all* $0<\varepsilon<a$ *there exists* $0<\alpha<\varepsilon$ *such that for all* $z\in B(y_0,r)$ *and* $x\in B(\varphi(0),r)$, *there exist* $\rho\in]0,1]$ *and* $b\in[\alpha,\inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}),1\}]$ *satisfying* $B(z,\rho)\subset B(y_0,r)$, $B(x,\rho)\subset B(\varphi(0),r)$ *and* $b(\|z\|+a+\rho+\lambda+2M)\leqslant\rho/2$.

Proof. Let $0<\varepsilon<a$, $z\in B(y_0,r)$ and $x\in B(\varphi(0),r)$ be fixed. Consider $0<\rho\leqslant 1$ such that $B(z,\rho)\subset B(y_0,r)$ and $B(x,\rho)\subset B(\varphi(0),r)$. Let $(\bar{\rho},\bar{z})\in[0,1]\times\overline{\Omega}$. There exists $b_{\bar{\rho},\bar{z}} \in]0, \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$ such that $b_{\bar{\rho},\bar{z}}(\|\bar{z}\| + a + \bar{\rho} + \lambda + 2M) < \bar{\rho}/4$. Consider the open subset

$$N(\bar{\rho}, \bar{z}) = \Big\{(\mu, v) \in \mathbb{R} \times H : b_{\bar{\rho},\bar{z}}(\|v\| + a + \mu + \lambda + 2M) < \frac{\mu}{4}\Big\}.$$

Since $(\bar{\rho}, \bar{z}) \in N(\bar{\rho}, \bar{z})$ there exists $\tau > 0$ such that $B((\bar{\rho}, \bar{z}), \tau) \subset N(\bar{\rho}, \bar{z})$. The compact subset $[0,1] \times \overline{\Omega}$ can be covered by q such balls $B((\bar{\rho}_i, \bar{z}_i), \tau_i)$. Set $b_i = b_{\bar{\rho}_i,\bar{z}_i}$ and $\alpha = \inf\{b_i, 0 \leqslant i \leqslant q\}$. Let $i \in \{1, \ldots, q\}$ such that $(\rho, z) \in B((\bar{\rho}_i, \bar{z}_i), \tau_i)$. Hence $(\rho, z) \in N(\bar{\rho}_i, \bar{z}_i)$. So

$$b_i(\|z\| + a + \rho + \lambda + 2M) < \rho/4,$$

where $b_i \in [\alpha, \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$. □

In all the paper, for $\varepsilon > 0$ we denote $\alpha(\varepsilon)$ the number α given by Lemma 3.2. In the next section, we need the following Proposition.

PROPOSITION 3.3. *If assumptions* (A1)-(A3) *are satisfied, then for all* $\varepsilon \in]0, a[$, $t_0 \in [0,1]$, $x_0 \in K \cap B(\varphi(0), r)$ *and* $z_0 \in B(y_0, r)$ *there exist* $b_0 \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$, *continuous functions* $x(.), y(.) : [t_0, +\infty[\to H$, *a function* $v(.) : [t_0, +\infty[\to H$ *and step functions* $\theta(.), \bar{\theta}(.) : [t_0, +\infty[\to [t_0, +\infty[$ *such that*

(*i*) $x(t_0) = x_0$, $x(t_0 + b_0) \in K \cap B(\varphi(0), r)$, $x(\bar{\theta}(t)) \in P(x(\theta(t)))$ *and* $x(\theta(t)) \in K \cap B(\varphi(0), r)$ *for all* $t \in [t_0, t_0 + b_0]$;

(*ii*) $\ddot{x}(t) - g(t, x(\theta(t)), y(\theta(t))) \in G(x(\theta(t)), y(\theta(t))) + \varepsilon B$ *a.e. on* $[t_0, t_0 + b_0]$;

(*iii*) $y(t_0) = z_0$, $y(t_0 + b_0) \in B(y_0, r)$, $y(\theta(t)) \in B(y_0, r)$ *for all* $t \in [t_0, t_0 + b_0]$, $\|\dot{x}(t) - y(t)\| \leqslant \varepsilon$, $\|\ddot{x}(t) - \dot{y}(t)\| \leqslant \varepsilon$ *and* $\|\dot{x}(t)\| \leqslant \|y_0\| + r + \lambda + a + M$ *for almost all* $t \in [t_0, t_0 + b_0]$;

(*iv*) $y(t) = y(t_0) + \int_{t_0}^{t} \Big(g(s, x(\theta(s)), y(\theta(s))) + v(s)\Big) ds$ *for all* $t \in [t_0, t_0 + b_0]$;

(*v*) $0 \leqslant t - \theta(t) \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leqslant \bar{\theta}(t) - t \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$ *and* $v(t) \in G(x(\theta(t)), y(\theta(t)))$ *for all* $t \in [t_0, t_0 + b_0]$.

Proof. Let $0 < \varepsilon < a$, $t_0 \in [0,1]$, $x_0 \in K \cap B(\varphi(0), r)$ and $z_0 \in B(y_0, r)$. By Lemma 3.2 there exist $\rho \in]0,1]$ and $b_0 \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}), 1\}]$ such that $B(x_0, \rho) \subset B(\varphi(0), r)$, $B(z_0, \rho) \subset B(y_0, r)$ and

$$b_0(\|z_0\| + a + \rho + \lambda + 2M) \leqslant \frac{\rho}{2}. \tag{3.1}$$

By Lemma 3.1, there exist $\eta > 0$, $h_0 \in [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$ and $u_0 \in G(x_0, z_0) + \varepsilon B$ such that

$$x_1 = \left(x_0 + h_0 z_0 + \frac{h_0^2}{2} u_0 + \int_{t_0}^{t_0 + h_0} (t_0 + h_0 - s) g(s, x_0, z_0) ds\right) \in P(x_0).$$

Set

$$t_1 = t_0 + h_0 \quad \text{and} \quad z_1 = z_0 + h_0 v_0 + \int_{t_0}^{t_0+h_0} g(s, x_0, z_0)ds,$$

where $v_0 \in G(x_0, z_0)$ such that $\|u_0 - v_0\| \leqslant \varepsilon$. If $h_0 \leqslant b_0$, by (A1), (A2) and (3.1), we have

$$\begin{aligned}\|x_1 - x_0\| &= \left\| h_0 z_0 + \frac{h_0^2}{2} u_0 + \int_{t_0}^{t_0+h_0} (t_0 + h_0 - s) g(s, x_0, z_0) ds \right\| \\ &\leqslant (\|z_0\| + \lambda + a + 2M) b_0 \\ &< \rho\end{aligned}$$

and

$$\begin{aligned}\|z_1 - z_0\| &= \left\| h_0 v_0 + \int_{t_0}^{t_0+h_0} g(s, x_0, z_0) ds \right\| \\ &\leqslant (\lambda + M) b_0 \\ &< \rho.\end{aligned}$$

Thus $x_1 \in K \cap B(\varphi(0), r)$ and $z_1 \in B(y_0, r)$. Set $h_{-1} = 0$. We reiterate this process for constructing sequences $(h_p)_{p \geqslant 0} \subset [\eta, \frac{1}{4}\eta(\frac{\varepsilon}{4})]$, $(t_p)_{p \geqslant 0}$, $(x_p)_{p \geqslant 0}$, $(z_p)_{p \geqslant 0}$, $(u_p)_{p \geqslant 0}$, $(v_p)_{p \geqslant 0}$ such that

(a) $t_p = t_0 + \sum_{i=0}^{p-1} h_i$ and $x_p \in P(x_{p-1})$;

(b) $x_p = x_{p-1} + h_{p-1} z_{p-1} + \frac{h_{p-1}^2}{2} u_{p-1} + \int_{t_{p-1}}^{t_p} (t_p - s) g(s, x_{p-1}, z_{p-1}) ds$;

(c) $z_p = z_{p-1} + h_{p-1} v_{p-1} + \int_{t_{p-1}}^{t_p} g(s, x_{p-1}, z_{p-1}) ds$;

(d) $x_p \in K \cap B(\varphi(0), r)$ and $z_p \in B(y_0, r)$ if $\sum_{i=0}^{p-1} h_i \leqslant b_0$;

(e) $u_{p-1} \in G(x_{p-1}, z_{p-1}) + \varepsilon B$, $v_{p-1} \in G(x_{p-1}, z_{p-1})$ and $\|u_{p-1} - v_{p-1}\| \leqslant \varepsilon$.

It is easy to see that for $p = 1$ the assertions (a)-(e) are fulfilled. Let now $p \geqslant 1$. Assume that (a)-(e) are satisfied for any $p = 1, \ldots, q$. If $t_0 + b_0 \leqslant t_q$, then we stop this process of iterations and we get (a)-(e) satisfied with $t_{q-1} < t_0 + b_0 \leqslant t_q$. In the other case, we can apply for (t_q, x_q, z_q) the same technique applied for (t_0, x_0, z_0) at the beginning of this proof, and we get (a), (b), (c) and (e) satisfied for $p = q + 1$. It remains to prove (e). By induction, we have

$$x_{q+1} = x_0 + \sum_{i=0}^{q} h_i z_i + \sum_{i=0}^{q} \frac{h_i^2}{2} u_i + \sum_{j=0}^{q} \int_{t_0 + \sum_{i=0}^{j} h_{i-1}}^{t_0 + \sum_{i=0}^{j} h_i} (t_0 + \sum_{i=0}^{j} h_i - s) g(s, x_j, z_j) ds$$

and

$$z_{q+1} = z_0 + \sum_{i=0}^{q} h_i v_i + \sum_{j=0}^{q} \int_{t_0+\sum_{i=0}^{j} h_{i-1}}^{t_0+\sum_{i=0}^{j} h_i} g(s, x_j, z_j) ds.$$

Then, if $\sum_{i=0}^{q} h_i \leqslant b_0$, by (A1), (A2), and (3.1) we have

$$\begin{aligned} \|z_{q+1} - z_0\| &\leqslant \sum_{i=0}^{q} h_i \lambda + \sum_{i=0}^{q} h_i M \\ &\leqslant b_0(\lambda + M) \\ &< \rho \end{aligned}$$

and

$$\begin{aligned} \|x_{q+1} - x_0\| &\leqslant \sum_{i=0}^{q} h_i(\|z_0\| + \rho) + \sum_{i=0}^{q} h_i(\lambda + a) + \sum_{i=0}^{q} h_i(2M) \\ &\leqslant b_0(\|z_0\| + \rho + \lambda + a + 2M) \\ &< \rho. \end{aligned}$$

Hence $z_{q+1} \in B(y_0, r)$ and $x_{q+1} \in K \cap B(\varphi(0), r)$. Since $h_p \geqslant \eta > 0$ there exists an integer s such that

$$t_s = t_0 + \sum_{i=0}^{s-1} h_i < t_0 + b_0 \leqslant t_{s+1} = t_0 + \sum_{i=0}^{s} h_i.$$

Define on $[t_0, +\infty[$ the functions $x(.)$, $y(.)$, $v(.)$, $\theta(.)$ and $\bar{\theta}(.)$ as follows:

$$x(t) = x_{q-1} + (t - t_{q-1}) z_{q-1} + \frac{(t - t_{q-1})^2}{2} u_{q-1} + \int_{t_{q-1}}^{t} (t - s) g(s, x_{q-1}, z_{q-1}) ds$$

$$y(t) = z_{q-1} + (t - t_{q-1}) v_{q-1} + \int_{t_{q-1}}^{t} g(s, x_{q-1}, z_{q-1}) ds \quad \text{for all } t \in [t_{q-1}, t_q];$$

$$\theta(t) = t_{q-1},\ v(t) = v_{q-1} \quad \text{and} \quad \bar{\theta}(t) = t_q \quad \text{for all } t \in [t_{q-1}, t_q[.$$

Finally, the above definitions will enable us to derive the assertions (i)-(v). □

4. Proof of the Theorem 2.4

Set $\varphi(0) = x_0$ and let

$$T = \inf\left\{1, \frac{1}{4}\eta\left(\frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}\right)\right\}.$$

We shall show the following Proposition. It will be used in order to obtain a sequence of approximated solutions.

PROPOSITION 4.1. *For all $0<\varepsilon<a$ there exist continuous maps:*

$$x(.):[-a,+\infty[\to H,\ \ y(.):[0,+\infty[\to H,\ \ \Gamma(.):[0,+\infty[\to \mathscr{C}_a,$$

a function $v(.):[0,+\infty[\to H$ and step functions $\theta(.),\bar{\theta}(.),\tilde{\theta}(.):[0,+\infty[\to[0,+\infty[$ such that

(i) $x(\theta(t))\in K\cap B(\varphi(0),r)$ and $x(\bar{\theta}(t))\in P(x(\theta(t)))$, for all $t\in[0,T]$ and $x\equiv\varphi$ on $[-a,0]$;

(ii) $\ddot{x}(t)-f(t,\Gamma(t),y(\theta(t)))\in F(\Gamma(t),y(\theta(t)))+\varepsilon B$ for almost all $t\in[0,T]$;

(iii) $0\leqslant t-\theta(t)\leqslant\frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0\leqslant t-\tilde{\theta}(t)\leqslant\frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $0\leqslant\bar{\theta}(t)-t\leqslant\frac{1}{4}\eta(\frac{\varepsilon}{4})$ for all $t\in[0,T]$;

(iv) $y(\theta(t))\in B(y_0,r)$ for all $t\in[0,T]$, $\|\dot{x}(t)-y(t)\|\leqslant\varepsilon$, $\|\ddot{x}(t)-\dot{y}(t)\|\leqslant\varepsilon$ and $\|\dot{x}(t)\|\leqslant\|y_0\|+r+\lambda+a+M$ for almost all $t\in[0,T]$;

(v) $y(t)=y(0)+\int_0^t\Big(f(s,\Gamma(s),y(\theta(s)))+v(s)\Big)ds$ and $v(t)\in F(\Gamma(t),y(\theta(t)))$ for all $t\in[0,T]$;

(vi) For all $t\in[0,T]$

$$\Gamma(t)(s)=\begin{cases}x(\tilde{\theta}(t)+\frac{1}{4}\eta(\frac{\varepsilon}{4})+s), & -a\leqslant s\leqslant-\frac{1}{4}\eta(\frac{\varepsilon}{4}),\\[2ex] -\frac{4s}{\eta(\frac{\varepsilon}{4})}x(\tilde{\theta}(t))+\big(1+\frac{4s}{\eta(\frac{\varepsilon}{4})}\big)x(\theta(t)), & -\frac{1}{4}\eta(\frac{\varepsilon}{4})\leqslant s\leqslant 0.\end{cases}$$

Proof. Let $0<\varepsilon<a$ be fixed. Set $t_0=0$ and put $x(t)=\varphi(t)$ for all $t\in[-a,0]$. Consider the function $\Gamma_0:H\to\mathscr{C}_a$ defined as follows: for all $x\in H$

$$\Gamma_0(x)(s)=\begin{cases}x(t_0+\frac{1}{4}\eta(\frac{\varepsilon}{4})+s), & -a\leqslant s\leqslant-\frac{1}{4}\eta(\frac{\varepsilon}{4}),\\[2ex] -\frac{4s}{\eta(\frac{\varepsilon}{4})}x(t_0)+\big(1+\frac{4s}{\eta(\frac{\varepsilon}{4})}\big)x, & -\frac{1}{4}\eta(\frac{\varepsilon}{4})\leqslant s\leqslant 0.\end{cases}$$

The set-valued maps

$$G_0:K\times\Omega\to 2^H\quad\text{and}\quad g_0:\mathbb{R}\times H\times H\to H$$

defined by $G_0(x,y)=F(\Gamma_0(x),y)$ and $g_0(t,x,y)=f(t,\Gamma_0(x),y)$ satisfy all assumptions (A1)-(A3). By Proposition 3.3, there exist $b_0\in[\alpha(\varepsilon),\inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}),1\}]$, continuous maps $x_0(.),y_0(.):[t_0,+\infty[\to H$, a function $v_0(.):[t_0,+\infty[\to H$ and step functions $\theta_0(.),\bar{\theta}_0(.):[t_0,+\infty[\to[t_0,+\infty[$ such that:

(i) $x_0(t_0)=x_0$, $x(t_0+b_0)\in B(\varphi(0),r)$, $x_0(\bar{\theta}_0(t))\in P(x_0(\theta_0(t)))$ and $x_0(\theta_0(t))\in K\cap B(\varphi(0),r)$ for all t in $[t_0,t_0+b_0]$;

(ii) $\ddot{x}_0(t) - f(t,\Gamma_0(x_0(\theta_0(t))),y_0(\theta_0(t))) \in F(\Gamma_0(x_0(\theta_0(t))),y_0(\theta_0(t))) + \varepsilon B$ for almost all $t \in [t_0,t_0+b_0]$;

(iii) $\|\dot{x}_0(t) - y_0(t)\| \leqslant \varepsilon$, $\|\ddot{x}_0(t) - \dot{y}_0(t)\| \leqslant \varepsilon$ and $\|\dot{x}_0(t)\| \leqslant \|y_0\| + r + \lambda + a + M$ for almost all $t \in [t_0,t_0+b_0]$, $y_0(t_0) = y_0$, $y_0(t_0+b_0) \in B(y_0,r)$ and $y_0(\theta_0(t)) \in B(y_0,r)$ for all $t \in [t_0,t_0+b_0]$;

(iv) $y_0(t) = y_0(t_0) + \int\limits_{t_0}^{t} \Big(f(s,\Gamma_0(x_0(\theta_0(s))),y_0(\theta_0(s))) + v_0(s)\Big)ds$ for all t in $[t_0,t_0+b_0]$;

(v) For all $t \in [t_0,t_0+b_0]$, $0 \leqslant t - \theta_0(t) \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leqslant \bar{\theta}_0(t) - t \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $v_0(t) \in F(\Gamma_0(x_0(\theta_0(t))),y_0(\theta_0(t)))$.

Set $t_1 = t_0 + b_0$, $x(t) = x_0(t)$ and $y(t) = y_0(t)$ for all $t \in [t_0,t_1]$.

We reiterate this process for constructing sequences $b_i \in [\alpha(\varepsilon), \inf\{\frac{1}{4}\eta(\frac{\varepsilon}{4}),1\}]$, $x_i(.)$, $y_i(.), v_i(.) : [t_i,+\infty[\to H$, $\theta_i(.), \bar{\theta}_i(.) : [t_i,+\infty[\to [t_i,+\infty[$, $\Gamma_i : H \to \mathscr{C}_a$ and continuous functions $x(.) : [-a,t_{i+1}] \to H$ and $y(.) : [0,t_{i+1}] \to H$ satisfying the following assertions for $i \geqslant 0$:

(a) $t_{i+1} = t_i + b_i$, $x_i(t_i+b_i) \in B(\varphi(0),r)$, $x(t) = x_i(t)$, $x_i(\bar{\theta}_i(t)) \in P(x_i(\theta_i(t)))$ and $x_i(\theta_i(t)) \in K \cap B(\varphi(0),r)$ for all $t \in [t_i,t_{i+1}]$;

(b) $\ddot{x}_i(t) - f(t,\Gamma_i(x_i(\theta_i(t))),y_i(\theta_i(t))) \in F(\Gamma_i(x_i(\theta_i(t))),y_i(\theta_i(t))) + \varepsilon B$ for almost all $t \in [t_i,t_{i+1}]$;

(c) $y_i(t_i+b_i) \in B(y_0,r)$, $y(t) = y_i(t)$ and $y_i(\theta_i(t)) \in B(y_0,r)$ for all $t \in [t_i,t_{i+1}]$, and $\|\dot{x}_i(t) - y_i(t)\| \leqslant \varepsilon$, $\|\ddot{x}_i(t) - \dot{y}_i(t)\| \leqslant \varepsilon$ and $\|\dot{x}_i(t)\| \leqslant \|y_0\| + r + \lambda + a + M$ for almost all $t \in [t_i,t_{i+1}]$;

(d) $y_i(t) = y_i(t_i) + \int\limits_{t_i}^{t} \Big(f(s,\Gamma_i(x_i(\theta_i(s))),y_i(\theta_i(s))) + v_i(s)\Big)ds$ on $[t_i,t_{i+1}]$;

(e) For all $t \in [t_i,t_{i+1}]$, $0 \leqslant t - \theta_i(t) \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$, $0 \leqslant \bar{\theta}_i(t) - t \leqslant \frac{1}{4}\eta(\frac{\varepsilon}{4})$ and $v_i(t) \in F(\Gamma_i(x_i(\theta_i(t))),y_i(\theta_i(t)))$;

(f) For all $x \in H$

$$\Gamma_i(x)(s) = \begin{cases} x(t_i + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s), & -a \leqslant s \leqslant -\frac{1}{4}\eta(\frac{\varepsilon}{4}), \\ \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}x(t_i) + \big(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\big)x, & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leqslant s \leqslant 0. \end{cases}$$

The assertions (a)-(f) are fulfilled for $i = 0$. Let now $i \geqslant 1$. Assume that (a)-(f) are satisfied for any $i = 1,...,q$. If $T \leqslant t_{q+1}$, then we stop this process of iterations and we get (a)-(f) satisfied with $t_q < T \leqslant t_{q+1}$. In the other case: $t_{q+1} < T$, consider the function $\Gamma_{q+1} : H \to \mathscr{C}_a$ defined as follows: for all $x \in H$,

$$\Gamma_{q+1}(x)(s) = \begin{cases} x(t_{q+1} + \frac{1}{4}\eta(\frac{\varepsilon}{4}) + s), & -a \leqslant s \leqslant -\frac{1}{4}\eta(\frac{\varepsilon}{4}), \\ \\ -\frac{4s}{\eta(\frac{\varepsilon}{4})}x(t_{q+1}) + \big(1 + \frac{4s}{\eta(\frac{\varepsilon}{4})}\big)x, & -\frac{1}{4}\eta(\frac{\varepsilon}{4}) \leqslant s \leqslant 0. \end{cases}$$

The set-valued map $G_{q+1} : K \times \Omega \to 2^H$ and the map $g_{q+1} : \mathbb{R} \times H \times H \to H$, defined by

$$G_{q+1}(x,y) = F(\Gamma_{q+1}(x),y) \quad \text{and} \quad g_{q+1}(t,x,y) = f(t,\Gamma_{q+1}(x),y),$$

satisfy all assumptions (A1)-(A3). In view of Proposition 3.3, there exist $b_{q+1} \in [\alpha(\varepsilon), \inf\{\eta(\varepsilon/4)/4, 1\}]$, continuous functions $x_{q+1}(.), y_{q+1}(.)$, a function $v_{q+1}(.)$ and step functions $\theta_{q+1}(.)$ and $\bar{\theta}_{q+1}(.)$, defined on $[t_{q+1}, +\infty[$, satisfying (a)-(f) for $i = q+1$. Set

$$t_{q+2} = t_{q+1} + b_{q+1}, \; x(t) = x_{q+1}(t) \quad \text{and} \quad y(t) = y_{q+1}(t),$$

for all $t \in [t_{q+1}, t_{q+2}]$. Thus the conditions (a)-(f) are satisfied for $i = q+1$. Since $t_{i+1} - t_i = b_i \geqslant \alpha(\varepsilon)$, there exists an integer s such that $t_s < T \leqslant t_{s+1}$. Further on, we define the functions $\theta(.), \bar{\theta}(.), \tilde{\theta}(.) : [0,+\infty[\to [0,+\infty[$, $\Gamma : [0,+\infty[\to \mathscr{C}_a$ and $v(.) : [0,+\infty[\to H$ as follows: for all $t \in [t_q, t_{q+1}[$, $\theta(t) = \theta_q(t)$, $\bar{\theta}(t) = \bar{\theta}_q(t)$, $\tilde{\theta}(t) = t_q$, $v(t) = v_q(t)$ and $\Gamma(t) = \Gamma_q(x_q(\theta_q(t)))$. Hence the proof of Proposition 4.1 is complete. □

Now we are prepared to prove our Theorem 2.4. Let $k \in \mathbb{N}^*$ such that

$$\frac{1}{k} < \inf\left\{a, \frac{r}{1 + \|y_0\| + r + \lambda + a + M}\right\}.$$

By Proposition 4.1, we can define sequences $s_k \in \mathbb{N}^*$, $(t_q^k)_{0 \leqslant q \leqslant s_k+1}$, $x_k(.) : [-a,+\infty[\to H$, $y_k(.), v_k(.) : [0,+\infty[\to H$, $\theta_k(.), \bar{\theta}_k(.), \tilde{\theta}_k(.) : [0,+\infty[\to [0,+\infty[$ and $\Gamma_k(.) : [0,+\infty[\to \mathscr{C}_a$ such that:

(1) $x_k(\theta_k(t)) \in B(\varphi(0), r)$ and $x_k(\bar{\theta}_k(t)) \in P(x_k(\theta_k(t)))$, for all $t \in [0,T]$ and $x_k \equiv \varphi$ on $[-a, 0]$;

(2) $\ddot{x}_k(t) - f(t, \Gamma_k(t), y_k(\theta_k(t))) \in F(\Gamma_k(t), y_k(\theta_k(t))) + \frac{1}{k}B$ for almost all t in $[0,T]$;

(3) $0 \leqslant t - \theta_k(t) \leqslant \frac{1}{4}\eta(\frac{1}{4k})$, $0 \leqslant t - \tilde{\theta}_k(t) \leqslant \frac{1}{4}\eta(\frac{1}{4k})$ and $0 \leqslant \bar{\theta}_k(t) - t \leqslant \frac{1}{4}\eta(\frac{1}{4k})$ for all $t \in [0,T]$;

(4) for almost all $t \in [0,T]$, $\|\dot{x}_k(t) - y_k(t)\| \leqslant 1/k$, $\|\ddot{x}_k(t) - \dot{y}_k(t)\| \leqslant 1/k$ and $\|\dot{x}(t)\| \leqslant \|y_0\| + r + \lambda + a + M$ and for all $t \in [0,T]$, $y_k(\theta_k(t)) \in B(y_0, r)$;

(5) for all $t \in [0,T]$, $y_k(t) = y_k(0) + \int_0^t \Big(f(s, \Gamma_k(s), y_k(\theta_k(s))) + v_k(s)\Big) ds$ and $v_k(t) \in F(\Gamma_k(t), y_k(\theta_k(t)))$;

(6) for all $t \in [0,T]$,

$$\Gamma_k(t)(s) = \begin{cases} x_k(\tilde{\theta}(t) + \frac{1}{4}\eta(\frac{1}{4k}) + s), & -a \leqslant s \leqslant -\frac{1}{4}\eta(\frac{1}{4k}), \\ -\frac{4s}{\eta(\frac{1}{4k})} x_k(\tilde{\theta}_k(t)) + \big(1 + \frac{4s}{\eta(\frac{1}{4k})}\big) x_k(\theta_k(t)), & -\frac{1}{4}\eta(\frac{1}{4k}) \leqslant s \leqslant 0. \end{cases}$$

CLAIM 4.2. *$\Gamma_k(t) \in \mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$ and $\|T(t)x_k - \Gamma_k(t)\|_{+\infty} \leqslant \frac{1}{2k}(1 + \|y_0\| + r + a + M + \lambda)$ for all $t \in [0,T]$.*

Proof. First, remark that for all $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leqslant \eta(\rho)$ we have $\|x_k(t) - x_k(\bar{t})\| \leqslant \rho(1 + \|y_0\| + r + \lambda + a + M)$. Indeed, let $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leqslant \eta(\rho)$. If $t, \bar{t} \in [0, T]$ and $\bar{t} \leqslant t$, by (4) and (5) we have

$$\begin{aligned}
\|x_k(t) - x_k(\bar{t})\| &\leqslant \int_{\bar{t}}^{t} \|\dot{x}_k(s)\| ds \\
&\leqslant (t - \bar{t})(\|y_0\| + r + \lambda + a + M) \\
&\leqslant \rho(\|y_0\| + r + \lambda + a + M).
\end{aligned}$$

If $t, \bar{t} \in [-a, 0]$, by construction, $\|x_k(t) - x_k(\bar{t})\| = \|\varphi(t) - \varphi(\bar{t})\| \leqslant \rho$. If $t \in [0, T]$ and $\bar{t} \in [-a, 0]$, one has $|t| \leqslant \eta(\rho)$ and $|\bar{t}| \leqslant \eta(\rho)$. Then

$$\begin{aligned}
\|x_k(t) - x_k(\bar{t})\| &\leqslant \|x_k(t) - x_k(0)\| + \|\varphi(\bar{t}) - \varphi(0)\| \\
&\leqslant \rho(\|y_0\| + r + \lambda + a + M) + \rho \\
&= \rho(1 + \|y_0\| + r + \lambda + a + M).
\end{aligned}$$

Hence we conclude that for all $t, \bar{t} \in [-a, T]$ such that $|t - \bar{t}| \leqslant \eta(\rho)$, we have

$$\|x_k(t) - x_k(\bar{t})\| \leqslant \rho(1 + \|y_0\| + r + \lambda + a + M).$$

Now, let $t \in [0, T]$, if $-a \leqslant s \leqslant -\frac{1}{4}\eta(\frac{1}{4k})$ we have

$$\begin{aligned}
&\left|\tilde{\theta}_k(t) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) + s - s\right| \\
&\quad \leqslant \tilde{\theta}_k(t) - t + t + \frac{1}{4}\eta\left(\frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}\right) \\
&\quad \leqslant \frac{1}{4}\eta\left(\frac{1}{4k}\right) + \frac{1}{4}\eta\left(\frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}\right) \\
&\qquad + \frac{1}{4}\eta\left(\frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}\right) \\
&\quad \leqslant \eta\left(\frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\|\Gamma_k(t)(s) - \varphi(s)\| &= \left\| x_k\left(\tilde{\theta}_k(t) + \frac{1}{4k}\eta\left(\frac{1}{4k}\right) + s\right) - x_k(s) \right\| \\
&\leqslant \frac{r}{4(1 + \|y_0\| + r + \lambda + a + M)}(1 + \|y_0\| + r + \lambda + a + M) \\
&\leqslant r.
\end{aligned}$$

If $-\frac{1}{4}\eta(\frac{1}{4k}) \leqslant s \leqslant 0$ we get

$$\begin{aligned}
&|\theta_k(t) - \tilde{\theta}_k(t)| \\
&\quad \leqslant |\theta_k(t) - t| + |\tilde{\theta}_k(t) - t| \\
&\quad \leqslant \frac{1}{4}\eta\left(\frac{r}{4(1+\|y_0\|+r+\lambda+a+M)}\right) + \frac{1}{4}\eta\left(\frac{r}{4(1+\|y_0\|+r+\lambda+a+M)}\right) \\
&\quad \leqslant \eta\left(\frac{r}{4(1+\|y_0\|+r+\lambda+a+M)}\right)
\end{aligned}$$

and

$$\begin{aligned}
|\theta_k(t) - s| &\leqslant \frac{1}{4}\eta\left(\frac{r}{4(1+\|y_0\|+r+\lambda+a+M)}\right) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) \\
&\leqslant \eta\left(\frac{r}{4(1+\|y_0\|+r+\lambda+a+M)}\right).
\end{aligned}$$

So

$$\begin{aligned}
\|\Gamma_k(t)(s) - \varphi(s)\| &= \left\| x_k(s) + \frac{4s}{\eta(\frac{1}{4k})} x_k(\tilde{\theta}_k(t)) - \left(1 + \frac{4s}{\eta(\frac{1}{4k})}\right) x_k(\theta_k(t)) \right\| \\
&\leqslant \|x_k(s) - x_k(\theta_k(t))\| + \|x_k(\tilde{\theta}_k(t)) - x_k(\theta_k(t))\| \\
&\leqslant \frac{2r}{4(1+\|y_0\|+r+\lambda+a+M)}(1+\|y_0\|+r+\lambda+a+M) \\
&\leqslant r.
\end{aligned}$$

Thus we conclude that $\Gamma_k(t) \in \overline{B}_a(\varphi, r)$. Since $\Gamma_k(t)(0) = x_k(\theta_k(t)) \in K$, we have $\Gamma_k(t) \in \mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$.

For the second assertion, let $t \in [0,T]$, if $-a \leqslant s \leqslant -\frac{1}{4}\eta(\frac{1}{4k})$ we have

$$\begin{aligned}
\left|\tilde{\theta}_k(t) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) + s - t - s\right| &\leqslant |\tilde{\theta}_k(t) - t| + \frac{1}{4}\eta\left(\frac{1}{4k}\right) \\
&\leqslant \frac{1}{4}\eta\left(\frac{1}{4k}\right) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) \\
&\leqslant \eta\left(\frac{1}{4k}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\|T(t)x_k(s) - \Gamma_k(t)(s)\| &= \left\| x_k(t+s) - x_k\left(\tilde{\theta}_k(t) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) + s\right) \right\| \\
&\leqslant \frac{1}{4k}(1+\|y_0\|+r+\lambda+a+M).
\end{aligned}$$

If $-\frac{1}{4}\eta(\frac{1}{4k}) \leqslant s \leqslant 0$ we get

$$|\theta_k(t) - t - s| \leqslant |\theta_k(t) - t| + |s| \leqslant \frac{1}{4}\eta\left(\frac{1}{4k}\right) + \frac{1}{4}\eta\left(\frac{1}{4k}\right) \leqslant \eta\left(\frac{1}{4k}\right)$$

and

$$|\theta_k(t)-\tilde{\theta}_k(t)| \leqslant |\theta_k(t)-t|+|\tilde{\theta}_k(t)-t| \leqslant \frac{1}{4}\eta\left(\frac{1}{4k}\right)+\frac{1}{4}\eta\left(\frac{1}{4k}\right) \leqslant \eta\left(\frac{1}{4k}\right).$$

So

$$\begin{aligned}
&\|T(t)x_k(s)-\Gamma_k(t)(s)\| \\
&= \left\| x_k(t+s)+\frac{4s}{\eta(\frac{1}{4k})}x_k(\tilde{\theta}_k(t))-\left(1+\frac{4s}{\eta(\frac{1}{4k})}\right)x_k(\theta_k(t))\right\| \\
&\leqslant \|x_k(t+s)-x_k(\theta_k(t))\|+\|x_k(\tilde{\theta}_k(t))-x_k(\theta_k(t))\| \\
&\leqslant \frac{1}{4k}(1+\|y_0\|+r+\lambda+a+M)+\frac{1}{4k}(1+\|y_0\|+r+\lambda+a+M) \\
&\leqslant \frac{1}{2k}(1+\|y_0\|+r+\lambda+a+M).
\end{aligned}$$

Thus

$$\|T(t)x_k-\Gamma_k(t)\|_{+\infty} \leqslant \frac{1}{2k}(1+\|y_0\|+r+\lambda+a+M) \quad \text{for all } t\in[0,T]. \ \square$$

Now, from (5), we deduce that $\|\dot{y}_k(t)\| \leqslant M+\lambda$ for almost every $t\in[0,T]$ and for all $t\in[0,T]$, $y_k(t)\in y_0+[0,T](C+W)$, which is compact. Therefore, by Arzelà-Ascoli's theorem (see [3]), we can select a subsequence, again denoted by $(y_k(.))_k$, which converges uniformly to an absolutely continuous function $y(.)$ on $[0,T]$, moreover $\dot{y}_k(.)$ converges weakly to $\dot{y}(.)$ in $L^2([0,T],H)$. In addition, since $y_k(\theta_k(t))\in B(y_0,r)\subset\Omega_0$ for all $t\in[0,T]$, one has $y(t)\in\Omega_0$ for all $t\in[0,T]$. Now, consider the function $x(.):[-a,T]\to H$ defined by $x(t)=\varphi(t)$ for all $t\in[-a,0]$ and $\dot{x}(t)=y(t)$, $\forall t\in[0,T]$. Remark that, for almost all $t\in[0,T]$, by (4), we have

$$\begin{aligned}
\|\dot{x}_k(t)-\dot{x}(t)\| &\leqslant \|\dot{x}_k(t)-y_k(t)\|+\|y_k(t)-\dot{x}(t)\| \\
&\leqslant \frac{1}{k}+\|y_k(t)-\dot{x}(t)\|.
\end{aligned}$$

The last term of the above inequality converges to 0, then $\dot{x}_k(.)$ converges uniformly to $\dot{x}(.)$ almost everywhere on $[0,T]$. Since

$$\|x_k(t)-x(t)\| \leqslant \int_0^t \|\dot{x}_k(s)-\dot{x}(s)\|ds,$$

we conclude that $x_k(.)$ converges uniformly to $x(.)$ on $[-a,T]$. Now, by (1), (2) and (3), for all $t\in[0,T]$, we have

$$\lim_{k\to+\infty}\|x(t)-x_k(\overline{\theta}_k(t))\|=0$$

and $x_k(\overline{\theta}_k(t))\in P(x_k(\theta_k(t)))\cap B(\varphi(0),r)\subset K_0$, then $x(t)\in K$ for all $t\in[0,T]$. By (H1), we conclude that $x(t)\in P(x(t))$ for all $t\in[0,T]$. It remains to prove that if $t'<t$

then $x(t) \in P(x(t'))$. Let $t', t \in [0,T]$ be such that $t' < t$. Then for k large enough we can find $p, q \in \{0, ..., s_k\}$ such that $q = p + i$ where $0 \leqslant i \leqslant s_k$, $t' \in [t_p^k, t_{p+1}^k]$, $t \in [t_q^k, t_{q+1}^k]$, $\lim_{k\to+\infty} t_q^k = t$ and $\lim_{k\to+\infty} t_p^k = t'$. Note that, by construction, one has $x_k(t_{q-1}^k) \in P(x_k(t_{q-2}^k))$, which together with (H1) gives

$$P(x_k(t_{q-1}^k)) \subseteq P(x_k(t_{q-2}^k)).$$

Similarly, $P(x_k(t_{q-2}^k)) \subseteq P(x_k(t_{q-3}^k))$. If we continue for $i-1$ steps, we obtain $P(x_k(t_{q-1}^k)) \subseteq P(x_k(t_p^k)))$. By the fact that $x_k(t_q^k) \in P(x_k(t_{q-1}^k))$, we conclude that $x_k(t_q^k) \in P(x_k(t_p^k))$. By letting $k \to +\infty$, we get $x(t) \in P(x(t'))$. Remark that, from Claim 4.2, we deduce that

$$\Gamma_k(t) \text{ converges to } T(t)x \tag{4.1}$$

in $\mathscr{C}_a$ and $T(t)x \in \mathscr{K}_0 \cap \overline{B}_a(\varphi, r)$.

PROPOSITION 4.3. $\dot{y}(t) - f(t, T(t)x, y(t)) \in \partial_c V(y(t))$ *for almost all* $t \in [0,T]$.

Proof. The weak convergence of $\dot{y}_k(.)$ to $\dot{y}(.)$ in $L^2([0,T], H)$ and the Mazur's Lemma entail

$$\dot{y}(t) \in \bigcap_k \overline{\text{co}}\{\dot{y}_m(t) : m \geqslant k\}, \text{ for a.e. on } [0,T].$$

Fix any $t \in [0,T]$ such that $t \neq t_q^k$ for all $k > 1$ and $0 \leqslant q \leqslant s+1$. Then for all $z \in H$,

$$\langle z, \dot{y}(t) \rangle \leqslant \inf_m \sup_{k \geqslant m} \langle z, \dot{y}_k(t) \rangle.$$

By (5) and (H2) one has $\dot{y}_k(t) \in \partial_c V(y_k(\theta_k(t))) + f(t, \Gamma_k(t), y_k(\theta_k(t)))$. Thus for all m,

$$\langle z, \dot{y}(t) \rangle \leqslant \sup_{k \geqslant m} \sigma\big(z, \partial_c V(y_k(\theta_k(t))) + f(t, \Gamma_k(t), y_k(\theta_k(t)))\big),$$

from which we deduce that

$$\langle z, \dot{y}(t) \rangle \leqslant \limsup_{k\to+\infty} \sigma\big(z, \partial_c V(y_k(\theta_k(t))) + f(t, \Gamma_k(t), y_k(\theta_k(t)))\big).$$

By Proposition 2.3, the function $x \mapsto \sigma\big(z, \partial_c V(x)\big)$ is $u.s.c$ and hence we get

$$\langle z, \dot{y}(t) \rangle \leqslant \sigma\big(z, \partial_c V(y(t)) + f(t, T(t)x, y(t))\big).$$

So, the convexity and the closedness of the set $\partial_c V(y(t))$ (see [7, 8]) ensure

$$\dot{y}(t) - f(t, T(t)x, y(t)) \in \partial_c V(y(t)). \square$$

PROPOSITION 4.4. *The set* $\big\{\langle p, \dot{y}(t) \rangle, p \in \partial_c V(y(t))\big\}$ *is reduced to the singleton* $\big\{\frac{d}{dt} V(y(t))\big\}$ *for almost every* $t \in [0,T]$.

Proof. Since $y(.)$ is absolutely continuous function and V is locally Lipschitz continuous. The function $V \circ y(.)$ is absolutely continuous and then for almost all t there exists $\frac{d}{dt}V(y(t))$. Let $t \in [0,T]$ be such that there exist both $\dot{y}(t)$ and $\frac{d}{dt}V(y(t))$. There is $\delta > 0$ such that for every $|h| < \delta$,

$$y(t+h) \in B(y_0, 2r),\ (y(t)+h\dot{y}(t)) \in B(y_0, 2r),\ (y(t)-h\dot{y}(t)) \in B(y_0, 2r)$$
$$y(t+h) - y(t) - h\dot{y}(t) = r(h),\ \text{where } \lim_{h\to 0} \|r(h)\|/h = 0.$$

Since V is Lipschitz continuous on $B(y_0, 2r)$ with Lipschitz constant $\lambda > 0$, we have

$$|V(y(t+h)) - V(y(t)+h\dot{y}(t))| \leqslant \lambda \|r(h)\|,$$

whenever $|h| < \delta$. Consequently, the function $h \to V(y(t)+h\dot{y}(t))$ is differentiable at $h = 0$, and its derivative is the same as the derivative of $h \to V(y(t+h))$ at $h = 0$. Hence

$$\frac{d}{dt}V(y(t)) = \lim_{h\to 0} \frac{V(y(t)+h\dot{y}(t)) - V(y(t))}{h}. \tag{4.2}$$

Since V is uniformly regular over Ω, there exists $\beta \geqslant 0$ such that for all $x \in \Omega$ and for all $\xi \in \partial_p V(x)$ one has

$$\langle \xi, x' - x \rangle \leqslant V(x') - V(x) + \beta \|x' - x\|^2 \quad \text{for all } x' \in \Omega. \tag{4.3}$$

Let $0 < h < \delta$. Applying the inequality (4.3) with $x = y(t)$ and $x' = y(t) + h\dot{y}(t)$, we have

$$\langle \xi, h\dot{y}(t) \rangle \leqslant V(y(t)+h\dot{y}(t)) - V(y(t)) + \beta \|h\dot{y}(t)\|^2.$$

Then

$$\langle \xi, \dot{y}(t) \rangle \leqslant \frac{V(y(t)+h\dot{y}(t)) - V(y(t))}{h} + \beta h \|\dot{y}(t)\|^2.$$

By passing to the limit, we get

$$\langle \xi, \dot{y}(t) \rangle \leqslant \lim_{h\to 0^+} \frac{V(y(t)+h\dot{y}(t)) - V(y(t))}{h}.$$

By (4.2), it follows that

$$\max\left\{\langle \xi, \dot{y}(t) \rangle, \xi \in \partial_p V(y(t))\right\} \leqslant \frac{d}{dt}V(y(t)).$$

In view of Proposition 2.3 and Proposition 2.1, one has

$$V^o(y(t), \dot{y}(t)) \leqslant \frac{d}{dt}V(y(t)). \tag{4.4}$$

If we Apply the inequality (4.3) with $x' = y(t) + h(-\dot{y}(t))$ and $x = y(t)$, we obtain by the same argument

$$\langle \xi, -\dot{y}(t) \rangle \leqslant \lim_{h\to 0^+} \frac{V(y(t)+(-h)\dot{y}(t)) - V(y(t))}{h}.$$

Thus

$$\langle \xi, -\dot{y}(t)\rangle \leqslant -\lim_{h\to 0^-} \frac{V(y(t)+h\dot{y}(t))-V(y(t))}{h}.$$

Consequently

$$V^o(y(t),-\dot{y}(t)) \leqslant -\frac{d}{dt}V(y(t)).$$

Since $V^o(y(t),-\dot{y}(t)) = -V_o(y(t),\dot{y}(t))$, we have

$$-V_o(y(t),\dot{y}(t)) \leqslant -\frac{d}{dt}V(y(t)).$$

In other words

$$\frac{d}{dt}V(y(t)) \leqslant V_o(y(t),\dot{y}(t)). \tag{4.5}$$

By (4.4) and (4.5), we deduce that

$$V^o(y(t),\dot{y}(t)) \leqslant \frac{d}{dt}V(y(t)) \leqslant V_o(y(t),\dot{y}(t)),$$

which implies that

$$V^o(y(t),\dot{y}(t)) = \frac{d}{dt}V(y(t)) = V_o(y(t),\dot{y}(t)).$$

This means that for almost all t the set $\left\{\langle p,\dot{y}(t)\rangle, p\in\partial_c V(y(t))\right\}$ reduces to the singleton $\left\{\frac{d}{dt}V(y(t))\right\}$. □

PROPOSITION 4.5. *$\ddot{x}(t)\in f(t,T(t)x,\dot{x}(t))+F(T(t)x,\dot{x}(t))$ for almost all $t\in[0,T]$.*

Proof. By Proposition 4.3 and Proposition 4.4, we obtain

$$\frac{d}{dt}V(y(t)) = \langle \dot{y}(t), \dot{y}(t)-f(t,T(t)x,y(t))\rangle \text{ a.e. on } [0,T],$$

therefore,

$$V(y(T))-V(y_0) = \int_0^T \|\dot{y}(s)\|^2 ds - \int_0^T \langle \dot{y}(s), f(s,T(s)x,y(s))\rangle ds. \tag{4.6}$$

For simplicity, in the rest of the paper, we take $t^k_{s_k+1}=T$. On the other hand, by construction, for all $q=1,\dots,s_k+1$, one has

$$\dot{y}_k(t)-f(t,\Gamma_k(t),y_k(t^k_{q-1}))\in\partial_c V(y_k(t^k_{q-1})).$$

Since V is uniformly regular over Ω, there exists $\beta\geqslant 0$ and

$$\begin{aligned} &V(y_k(t^k_q))-V(y_k(t^k_{q-1}))\\ &\quad\geqslant \left\langle y_k(t^k_q)-y_k(t^k_{q-1}), \dot{y}_k(t)-f(t,\Gamma_k(t),y_k(t^k_{q-1}))\right\rangle - \beta\|y_k(t^k_q)-y_k(t^k_{q-1})\|^2 \end{aligned}$$

$$= \left\langle \int_{t_{q-1}^k}^{t_q^k} \dot{y}_k(s)ds, \dot{y}_k(t) - f(t,\Gamma_k(t), y_k(t_{q-1}^k)) \right\rangle - \beta \|y_k(t_q^k) - y_k(t_{q-1}^k)\|^2$$

$$= \int_{t_{q-1}^k}^{t_q^k} \langle \dot{y}_k(s), \dot{y}_k(s)\rangle ds - \int_{t_{q-1}^k}^{t_q^k} \langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle ds$$
$$- \beta \|y_k(t_q^k) - y_k(t_{q-1}^k)\|^2.$$

By adding, we obtain

$$V(y_k(T))) - V(y_0) \geqslant \int_0^T \|\dot{y}_k(s)\|^2 ds - \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle ds$$
$$- \sum_{q=1}^{s_k+1} \beta \|y_k(t_q^k) - y_k(t_{q-1}^k)\|^2. \tag{4.7}$$

CLAIM 4.6.

$$\lim_{k\to+\infty} \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle ds = \int_0^T \langle \dot{y}(s), f(s,T(s)x, y(s))\rangle ds.$$

Proof. We have

$$\left| \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle ds - \int_0^T \langle \dot{y}(s), f(s,T(s)x, y(s))\rangle ds \right|$$
$$= \left| \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \Big(\langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle - \langle \dot{y}(s), f(s,T(s)x, y(s))\rangle \Big) ds \right|$$
$$\leqslant \left| \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \Big(\langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle - \langle \dot{y}_k(s), f(s,T(s)x, y(s))\rangle \Big) ds \right|$$
$$+ \left| \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \Big(\langle \dot{y}_k(s), f(s,T(s)x, y(s))\rangle - \langle \dot{y}(s), f(s,T(s)x, y(s))\rangle \Big) ds \right|$$
$$\leqslant \sum_{q=1}^{s_k+1} \int_{t_{q-1}^k}^{t_q^k} \Big| \langle \dot{y}_k(s), f(s,\Gamma_k(s), y_k(t_{q-1}^k))\rangle - \langle \dot{y}_k(s), f(s,T(s)x, y(s))\rangle \Big| ds$$
$$+ \left| \int_0^T \Big(\langle \dot{y}_k(s), f(s,T(s)x, y(s))\rangle - \langle \dot{y}(s), f(s,T(s)x, y(s))\rangle \Big) ds \right|.$$

Since

$$\|\dot{y}_k(t)\| \leqslant \lambda + M, \quad \lim_{k\to+\infty} f(s,\Gamma_k(s), y_k(t_{q-1}^k)) = f(s,T(s)x, y(s))$$

and $\dot{y}_k(.)$ converges weakly to $\dot{y}(.)$, the last term converges to 0. This completes the proof of the Claim. □

CLAIM 4.7.

$$\lim_{k\to+\infty}\sum_{q=1}^{s_k+1}\beta\|y_k(t_q^k)-y_k(t_{q-1}^k)\|^2=0.$$

Proof. By construction we have

$$\begin{aligned}\|y_k(t_q^k)-y_k(t_{q-1}^k)\| &= \left\|\int_{t_{q-1}^k}^{t_q^k}\Big(f(s,\Gamma_k(s),y_k(t_{q-1}^k))+v_k(s)\Big)ds\right\|\\ &\leqslant (t_q^k-t_{q-1}^k)\big(M+\lambda\big).\end{aligned}$$

Hence

$$\begin{aligned}\|y_k(t_q^k)-y_k(t_{q-1}^k)\|^2 &\leqslant (t_q^k-t_{q-1}^k)^2(M+\lambda)^2\\ &\leqslant (t_q^k-t_{q-1}^k)\frac{1}{k}(M+\lambda)^2.\end{aligned}$$

Then

$$\sum_{q=1}^{s_k+1}\beta\|y_k(t_q^k)-y_k(t_{q-1}^k)\|^2\leqslant\frac{\beta T(\lambda+M)^2}{k},$$

so

$$\lim_{k\to+\infty}\sum_{q=1}^{s_k+1}\beta\|y_k(t_q^k)-y_k(t_{q-1}^k)\|^2=0.\ \square$$

By passing to the limit for $k\to\infty$ in (4.7) and using the continuity of the function V on the ball $B(y_0,2r)$, we obtain

$$V(y(T))-V(y_0)\geqslant\limsup_{k\to+\infty}\int_0^T\|\dot y_k(s)\|^2ds-\int_0^T\langle\dot y(s),f(s,T(s)x,y(s))\rangle ds.$$

Moreover, by (4.6),

$$\|\dot y(.)\|_2^2\geqslant\limsup_{k\to+\infty}\|\dot y_k(.)\|_2^2$$

and by the weak $l.s.c$ of the norm ensures

$$\|\dot y(.)\|_2^2\leqslant\liminf_{k\to+\infty}\|\dot y_k(.)\|_2^2.$$

Hence we get

$$\|\dot y(.)\|_2^2=\lim_{k\to+\infty}\|\dot y_k(.)\|_2^2.$$

Finally, there exists a subsequence of $(\dot y_k(.))_k$ (still denoted $(\dot y_k(.))_k$) converges point-wisely to $\dot y(.)$. In addition, by (4), for almost all $t\in[0,T]$

$$\begin{aligned}\|\ddot x_k(t)-\ddot x(t)\| &\leqslant \|\ddot x_k(t)-\dot y_k(t)\|+\|\dot y_k(t)-\dot y(t)\|\\ &\leqslant \frac{1}{k}+\|\dot y_k(t)-\dot y(t)\|.\end{aligned}$$

The last term of the above inequality converges to 0, then $\ddot{x}_k(.)$ converges point-wisely to $\ddot{x}(.)$ almost everywhere on $[0,T]$. Now, by (2), for almost all $t \in [0,T]$,

$$\ddot{x}_k(t) - f(t,\Gamma_k(t), y_k(\theta_k(t))) \in F(\Gamma_k(t), y_k(\theta_k(t))) + \frac{1}{k}B,$$

then

$$\begin{aligned} &d_{\mathrm{Graph}(F)}\Big(\big((\Gamma_k(t),\dot{x}_k(t)),\ddot{x}_k(t) - f(t,\Gamma_k(t),y_k(\theta_k(t)))\big)\Big) \\ &\leqslant \|\dot{x}_k(t) - y_k(\theta_k(t))\| + \frac{1}{k} \\ &\leqslant \|\dot{x}_k(t) - y_k(t)\| + \|y_k(t) - y(t)\| + \|y(t) - y_k(\theta_k(t))\| + \frac{1}{k} \\ &\leqslant \frac{1}{k} + \|y_k(t) - y(t)\| + \|y(t) - y_k(\theta_k(t))\| + \frac{1}{k} \end{aligned}$$

hence

$$\lim_{k\to+\infty} d_{\mathrm{Graph}(F)}\Big(\big((\Gamma_k(t),\dot{x}_k(t)),\ddot{x}_k(t) - f(t,\Gamma_k(t),y_k(\theta_k(t)))\big)\Big) = 0,$$

from which we conclude that

$$d_{\mathrm{Graph}(F)}\Big(\big((T(t)x,\dot{x}(t)),\ddot{x}(t) - f(t,T(t)x,\dot{x}(t))\big)\Big) = 0$$

and as F has a closed graph, we obtain

$$\ddot{x}(t) \in f(t,T(t)x,\dot{x}(t)) + F(T(t)x,\dot{x}(t)) \text{ a.e. on } [0,T].$$

The proof is complete. □

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