Differential
Equations
& Applications
Volume 4, Number 3 (2012), 459–484
doi:10.7153/dea-04-27

THE ADMB–KDV EQUATION IN ANISOTROPIC SOBOLEV SPACES

AMIN ESFAHANI

(*Communicated by Pavel I. Naumkin*)

Abstract. Considered herein is the anisotropic dissipation-modified Boussinesq-KdV equation which is a two-dimensional version of the KdV equation. It is established that the Cauchy problem associated to this equation is locally well-posed in anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$, for all $s_1 > -1/2$ and $s_2 \geqslant 0$. A global existence result for this equation will be obtained under suitable conditions.

1. Introduction

In this work, we consider the anisotropic dissipation-modified Boussinesq-KdV (ADMB-KdV) equation

$$u_t + u_{xxx} + 2uu_x + \mu\left(u_{xx} + u_{xxxx} + \gamma\partial_x^2(u^2) - u_{yy}\right) = 0, \tag{1.1}$$

where $\mu > 0$ and γ are real constants, and $u = u(x,y,t)$ is real-valued function. The ADMB-KdV equation (1.1) arises in modeling anisotropic systems such as the nonlinear waves generated by a long-wave instability in a viscous film flowing down an inclined rigid surface [1, 5]. In the case of an isotropic system (e.g., Béenard-Marangoni waves), the problem is governed by the dissipation-modified Kadomtsev-Petviashvili equation which was treated by the author in [15]. Equation (1.1) can be considered as a two-dimensional dissipated generalization of the KdV equation,

$$u_t + u_{xxx} + uu_x = 0. \tag{1.2}$$

The KdV equation arises in modeling for one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media [6, 16, 30], as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [29]. Equation (1.1) is also a natural two-dimensional version of the KdV-Kuramoto-Sivashinsky (KdV-KS) equation

$$u_t + u_{xxx} + uu_x + \mu(u_{xx} + u_{xxxx}) = 0, \tag{1.3}$$

which arises in interesting physical situations, for example as a model for long waves on a viscous fluid flowing down an inclined plane [28] and to derive drift waves in a plasma [12].

Mathematics subject classification (2010): 35Q53, 35M10, 35G25, 42B35.
Keywords and phrases: Cauchy Problem, anisotropic Sobolev spaces, Boussinesq-KdV equation.

Paper DEA-04-27

The aim of this paper is to establish local and global well-posedness to the initial value problem for (1.1). The notion of well-posedness will be the usual one in the context of nonlinear equations, that is, it includes existence, uniqueness, persistence property, and continuous dependence upon the data. In the last decade, Bourgain developed a new method to study the Cauchy problem for nonlinear dispersive equations which was successfully applied to several dispersive equations such as Schrödinger, KdV and KP-II equations [8, 9, 7]. See also [19, 20]. One of the particularity of this method is to use special Fourier transform restriction spaces strongly related to the symbol of the associated linear equation. Concerning the Cauchy problem for the KdV equation, Bourgain [8] showed the well-posedness in $H^s(\mathbb{R})$ for $s \geqslant 0$. Then Kenig, Ponce and Vega [19] proved the local well-posedness of equation (1.2) in $H^s(\mathbb{R})$ for $s > -3/4$. The L^2-conservation of the solutions of (1.2) leads to the global well-posedness in $H^s(\mathbb{R})$ for $s \geqslant 0$. Note that some global existence results have been obtained by Colliander, Staffilani and Takaoka [13] for special initial data in Sobolev spaces of negative order.

In [22, 23, 24], Molinet and Ribaud introduced some Bourgain-type spaces in proving the well-posedness for the KdV-Burgers (KdVB) equation

$$u_t + u_{xxx} + uu_x = u_{xx} \tag{1.4}$$

and for the Kadomtsev-Petviashvili-Burgers (KPB) equation

$$(u_t + u_{xxx} + uu_x - u_{xx})_x + \varepsilon u_{yy} = 0, \qquad \varepsilon = \pm 1. \tag{1.5}$$

The their main ideas are to use a fixed point argument in suitable Bourgain-type spaces adapted to both linear parts, dispersion and dissipation, of the equation.

In this work, we will apply the ideas of [22, 23, 24] and prove the local existence for the initial value problem associated to (1.1) with initial value $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ when $s_1 > -1/2$ and $s_2 \geqslant 0$. More precisely, similar to [22, 23, 24], following, we introduce a Bourgain-type space associated to the linear part of (1.1). This space is in fact the intersection of the space introduced in [8] and of a Sobolev space. The advantage of this space is that it contains both the dissipative and dispersive parts of the linear symbol of (1.1). See [1, 2, 3, 4, 10, 14, 18, 21, 26, 27, 31] and references therein for discussions and examples of the dispersive and dissipative equations. We also show that the associated Cauchy problem is globally well-posed in $H^{s_1,s_2}(\mathbb{R}^2)$ for $\gamma = 0$ $s_1 > -3/2$ and $s_2 \geqslant 0$. We should also note that the method used here may be applied for more general types of equations (which contain both the dissipation and dispersion, see e.g. [11, 17, 25]).

This paper is organized as follows. In Section 2, we introduce some notations and our main results. In Section 3, we derive linear estimates and some smoothing properties for the operator arising from (1.1) in the Bourgain spaces (Lemma 2). Section 4 is devoted to establish bilinear estimates. In Section 5, using bilinear estimates, a standard fixed point argument and some smoothing properties, we prove the existence of a unique solution of (1.1) in the anisotropic Sobolev space H^{s_1,s_2} with $s_1 > -1/2$ and $s_2 \geqslant 0$; and also the local solution of (1.1), when $\gamma = 0$, extends globally in time in H^{s_1,s_2} with $s_1 > -3/2$ and $s_2 \geqslant 0$.

2. Notations and main results

For the simplicity, throughout the paper we assume that $\gamma = 1$ (if $\gamma \neq 0$) and $\mu = 1$. Before stating our main result, we introduce our notations that are used in this paper.

We denote $\langle\cdot\rangle = 1 + |\cdot|$. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leqslant CB$. Similarly, we will write $A \sim B$ to mean $A \lesssim B$ and $A \gtrsim B$.

For $n \in \mathbb{N}$, we denote by $\widehat{\varphi}$ the Fourier transform of φ, defined as

$$\widehat{\varphi}(\omega) = \int_{\mathbb{R}^n} \varphi(x) \mathrm{e}^{-\mathrm{i}x\cdot\omega} \mathrm{d}x.$$

For $b, s_1, s_2 \in \mathbb{R}$, we denote $H^b = H^b(\mathbb{R})$, $\dot{H}^b = \dot{H}^b(\mathbb{R})$ and $H^{s_1,s_2} = H^{s_1,s_2}(\mathbb{R}^2)$ as the nonhomogeneous Sobolev, the homogeneous Sobolev and the anisotropic Sobolev spaces, respectively, defined by:

$$H^b = \left\{\varphi \in \mathscr{S}'(\mathbb{R});\ \|\varphi\|_{H^b} = \|\langle\tau\rangle^b \widehat{\varphi}(\tau)\|_{L^2_\tau} < \infty\right\},$$
$$\dot{H}^b = \left\{\varphi \in \mathscr{S}'(\mathbb{R});\ \|\varphi\|_{H^b} = \||\tau|^b \widehat{\varphi}(\tau)\|_{L^2_\tau} < \infty\right\},$$
$$H^{s_1,s_2} = \left\{\varphi \in \mathscr{S}'(\mathbb{R}^2);\ \|\varphi\|_{H^{s_1,s_2}} = \|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \widehat{\varphi}(\xi,\eta)\|_{L^2_{\xi,\eta}} < \infty\right\}.$$

Let $U(t)$ be the unitary group in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1, s_2 \in \mathbb{R}$, defined by the free evolution of the KdV equation, i.e. $U(t) = \mathrm{e}^{-t\partial_x^3}$. We also denote by

$$V(t) = \mathrm{e}^{-t(\partial_x^4 + \partial_x^2 - \partial_y^2)}, \qquad t \geqslant 0,$$

the semigroup associated to the linear part of equation (1.1) that we extend to a group on $\mathbb{R}$ by

$$V(t)f = \left(\mathrm{e}^{\mathrm{i}\xi^3 t - |t|(\xi^4 - \xi^2 + \eta^2)} \widehat{f}(\xi,\eta)\right)^{\vee}, \qquad \forall t \in \mathbb{R}.$$

We note that solving equation (1.1) with the initial data $u(x,y,0) = \varphi(x,y)$ is equivalent to solve the following integral equation

$$u(t) = V(t)\varphi - \int_0^t V(t-t') f(u)(t') \mathrm{d}t', \tag{2.1}$$

where $f(u) = (u^2)_x + (u^2)_{xx}$.

Actually, to prove the local existence result, we shall apply a fixed point argument to the following truncated version of (2.1)

$$u(t) = \theta(t)\left(V(t)\varphi - \int_0^t V(t-t')(\theta_T f(u))(t') \mathrm{d}t'\right), \tag{2.2}$$

where θ is a cutoff function satisfying

$$\theta \in C_0^\infty(\mathbb{R}), \quad 0 \leqslant \theta \leqslant 1, \quad \mathrm{supp}(\theta) \subset [-2,2], \quad \text{and} \quad \theta \equiv 1 \quad \text{on } [-1,1] \tag{2.3}$$

and

$$\theta_T(t) = \theta\left(\frac{t}{T}\right),$$

for any $T > 0$. We note that if u is a solution of (2.2), then $\widetilde{u} = u|_{[0,T]}$ will be a solution of (2.1) in $[0,T]$.

Following [8], we introduce a Bourgain-type space which is in relation with both the dissipative and dispersive parts of (1.1) at the same time. We define this space by

$$\mathscr{X}^{b,s_1,s_2} = \left\{u \in \mathscr{S}'(\mathbb{R}^3) \,:\, \|u\|_{\mathscr{X}^{b,s_1,s_2}} < \infty\right\},$$

equipped with the norm

$$\|u\|_{\mathscr{X}^{b,s_1,s_2}} = \left\| \left\langle \mathrm{i}(\tau - \xi^3) + (\xi^4 - \xi^2) + \eta^2 \right\rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \widehat{u}(\xi, \eta, \tau) \right\|_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

For all $T > 0$, we define the localized space associated $\mathscr{X}_T^{b,s_1,s_2}$ as the set of all functions $u : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ such that $\|u\|_{\mathscr{X}_T^{b,s_1,s_2}} < \infty$, where

$$\|u\|_{\mathscr{X}_T^{b,s_1,s_2}} = \inf_{v \in \mathscr{X}^{b,s_1,s_2}} \left\{ \|v\|_{\mathscr{X}^{b,s_1,s_2}} \,:\, v(t) = u(t) \text{ on } [0,T] \right\}.$$

Now we state our main existence results.

THEOREM 1. *Let $s_1 > -1/2$, $s_2 \geqslant 0$ and $s_1' \in (-1/2, \min\{0, s_1\}]$, then for all $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$, there exists $T = T(\|\varphi\|_{H^{s_1',0}(\mathbb{R}^2)})$ such that $T(\rho) \to +\infty$ as $\rho \to 0$, and a unique solution $u \in \mathscr{X}_T^{1/2,s_1,s_2}$ of the initial value problem associated to equation (1.1) with initial data $u(x,y,0) = \varphi(x,y)$. Moreover, we have*

$$u \in C([0,T]; H^{s_1,s_2}(\mathbb{R}^2)) \cap C((0,T); H^\infty(\mathbb{R}^2)) \tag{2.5}$$

and the map

$$S : H^{s_1,s_2}(\mathbb{R}^2) \longrightarrow \mathscr{X}_T^{1/2,s_1,s_2} \cap C([0,T]; H^{s_1,s_2}(\mathbb{R}^2)), \quad \varphi \mapsto u, \tag{2.6}$$

is smooth. Furthermore, if $\varphi \in H^{s_1',s_2'}(\mathbb{R}^2)$ with $s_1' > s_1$ and $s_2' > s_2$, the result holds with s_1' and s_2' instead of s_1 and s_2, respectively, in the same time interval $[0,T]$ with $T = T(\|\varphi\|_{H^{s_1,s_2}(\mathbb{R}^2)})$.

The following theorem give us a global existence under suitable conditions.

THEOREM 2. *Let $\gamma = 0$ and $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$ for $s_1 > -3/2$ and $s_2 \geqslant 0$. Then the local solution u of the initial vale problem associated to equation (1.1) with initial data $u(0) = \varphi$ extends globally in time.*

3. Linear estimates

In this section we study the linear operator θV.

LEMMA 1. *Let* $s\in\mathbb{R}$*, then*

$$\|\theta(t)V(t)\varphi\|_{\mathscr{X}^{1/2,s_1,s_2}} \lesssim \|\varphi\|_{H^{s_1,s_2}}, \tag{3.1}$$

for every $\varphi\in H^{s_1,s_2}$.

Proof. Let $\varphi\in H^{s_1,s_2}$ and $\zeta=(\xi,\eta)$. Then we have

$$\begin{aligned}
&\|\theta(t)V(t)\varphi\|_{\mathscr{X}^{1/2,s_1,s_2}}\\
&=\left\|\left\langle \mathrm{i}(\tau-\xi^3)+(\xi^4-\xi^2)+\eta^2\right\rangle^{1/2}\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\left(\theta(t)\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}\widehat{\varphi}(\zeta)\right)^{\wedge_t}\right\|_{L^2(\mathbb{R}^3)}\\
&=\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\widehat{\varphi}(\zeta)\left\|\left\langle \mathrm{i}(\tau-\xi^3)+(\xi^4-\xi^2)+\eta^2\right\rangle^{1/2}\left(\theta(t)\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}\right)^{\wedge_t}\right\|_{L^2_\tau}\right\|_{L^2_\zeta}\\
&\lesssim I^1+I^2,
\end{aligned} \tag{3.2}$$

where

$$I^1=\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\left\langle(\xi^4-\xi^2)+\eta^2\right\rangle^{1/2}\widehat{\varphi}(\zeta)\|g_\zeta(t)\|_{L^2_t}\right\|_{L^2_\zeta},$$

$$I^2=\left\|\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\widehat{\varphi}(\zeta)\|g_\zeta(t)\|_{H^{1/2}_t}\right\|_{L^2_\zeta},$$

and

$$g_\zeta(t)=\theta(t)\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}. \tag{3.3}$$

First we estimate I^1. We consider two cases:

$$I^1_1=\{|\xi|\geqslant\sqrt{2}\}\quad\text{and}\quad I^1_2=\{|\xi|\leqslant\sqrt{2}\}.$$

Contribution of I^1_1 *to* I^1. In this case, we have $\xi^4-\xi^2+\eta^2\geqslant 2$, then we can obtain

$$\|g_\zeta\|_{L^2}\leqslant\left\|\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}\right\|_{L^2_t}\simeq\frac{1}{\sqrt{\xi^4-\xi^2+\eta^2}}\lesssim\frac{1}{\langle\xi^4-\xi^2+\eta^2\rangle^{1/2}}.$$

Contribution of I^1_2 *to* I^1. In this case we divide I^1_2 into two regions

$$I^1_{21}=\{|\xi|\leqslant\sqrt{2},\ |\eta|\geqslant\sqrt{2}\}$$

and

$$I^1_{22}=\{|\xi|,|\eta|\leqslant\sqrt{2}\}.$$

Contribution of I^1_{21} *to* I^1. Since $\eta^2\geqslant 2$, similar to the previous case, we get

$$\|g_\zeta\|_{L^2}\lesssim\frac{1}{\langle\xi^4-\xi^2+\eta^2\rangle^{1/2}}.$$

Contribution of I^1_{22} to I^1. When $\xi,\eta \in I^1_{22}$, we have $|\xi^4-\xi^2+\eta^2| \leqslant K$, for some $K>0$ (independent of ξ and η). Thus we obtain

$$\|g_\zeta\|_{L^2} \leqslant \|\mathrm{e}^{K|t|}\|_{L^2([-2,2])} \lesssim 1 \lesssim \frac{1}{\langle \xi^4-\xi^2+\eta^2\rangle^{1/2}}.$$

Then, we deduce that

$$I^1 \lesssim \|\varphi\|_{H^{s_1,s_2}}. \tag{3.4}$$

Now we are going to estimate I^2. We consider two cases:

$$I^2_1 = \{|\xi| \geqslant \sqrt{2}\} \quad \text{and} \quad I^2_2 = \{|\xi| \leqslant \sqrt{2}\}.$$

Contribution of I^2_1 to I^2. By using the Young inequality, we see that

$$\begin{aligned}\|g_\zeta\|_{H^{1/2}} &= \|\langle\tau\rangle^{1/2}\widehat{\theta} * (\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})^{\wedge_t}(\tau)\|_{L^2_\tau}\\ &\lesssim \|\langle\tau\rangle^{1/2}\widehat{\theta}\|_{L^1_\tau}\|\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}\|_{L^2_t} + \|\widehat{\theta}\|_{L^1_\tau}\|\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}\|_{\dot{H}^{1/2}_t}\\ &\lesssim \frac{1}{\langle\xi^4-\xi^2+\eta^2\rangle^{1/2}} \lesssim 1.\end{aligned}$$

Contribution of I^2_2 to I^2. In this case we divide I^2_2 into two regions I^1_{21} and I^1_{22} as defined above. The contribution of I^1_{21} to I^2 is similar to the contribution of I^1_{21} to I^1.
Contribution of I^1_{22} to I^2. Since $|\xi|,|\eta| \leqslant \sqrt{2}$, then $|\xi^4-\xi^2+\eta^2| \leqslant 4$. Thus we have

$$\|g_\zeta\|_{H^{1/2}} \leqslant \sum_{j\geqslant 0}\frac{4^j}{j!}\||t|^j\theta(t)\|_{H^{1/2}_t} \lesssim 1,$$

since

$$\||t|^j\theta(t)\|_{H^{1/2}_t} \leqslant \||t|^j\theta(t)\|_{H^1_t} \lesssim j,$$

for $j \geqslant 1$. Therefore we obtain that

$$I^2 \lesssim \|\varphi\|_{H^{s_1,s_2}}. \tag{3.5}$$

Thus, the proof is completed. □

The following lemmas will be useful in proving some smoothing properties in the Bourgain spaces for the operator $t \mapsto \int_0^t V(t-t')f(t)\mathrm{d}t'$.

LEMMA 2. *Let $w \in \mathscr{S}(\mathbb{R}^3)$. Define k_ζ on $\mathbb{R}$ by*

$$k_\zeta(t) = \theta(t)\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}t\tau} - \mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)}}{\mathrm{i}\tau+\xi^4-\xi^2+\eta^2}\widehat{w}(\tau)\mathrm{d}\tau. \tag{3.6}$$

Then it holds for all $\zeta \in \mathbb{R}^2$ that

$$\begin{aligned}\Big\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{k}^t_\zeta(\tau)\Big\|_{L^2_\tau} &\lesssim \Big\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{-1/2}\widehat{w}(\zeta,\tau)\Big\|_{L^2_\tau}\\ &\quad + \int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau.\end{aligned} \tag{3.7}$$

Proof. We decompose k_ζ into

$$\begin{aligned} k_\zeta(t) = \theta(t)\Bigg(\int_{|\tau|\leqslant 1} \frac{e^{it\tau}-1}{i\tau+\xi^4-\xi^2+\eta^2}\widehat{w}(\zeta,\tau)d\tau \\ + \int_{|\tau|\leqslant 1} \frac{1-e^{-|t|(\xi^4-\xi^2+\eta^2)}}{i\tau+\xi^4-\xi^2+\eta^2}\widehat{w}(\zeta,\tau)d\tau \\ + \int_{|\tau|\geqslant 1} \frac{e^{it\tau}}{i\tau+\xi^4-\xi^2+\eta^2}\widehat{w}(\zeta,\tau)d\tau \\ - \int_{|\tau|\geqslant 1} \frac{e^{-|t|(\xi^4-\xi^2+\eta^2)}}{i\tau+\xi^4-\xi^2+\eta^2}\widehat{w}(\zeta,\tau)d\tau\Bigg) \\ = I_1+I_2+I_3+I_4, \end{aligned} \tag{3.8}$$

then we examine the different contributions of (3.8) on the left hand side of (3.7).

Contribution of I_4. Since $|\tau|\geqslant 1$, we have

$$\begin{aligned} \left\|\langle i\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_4^t}(\tau)\right\|_{L^2_\tau}^2 \\ \leqslant \left(\int_{\mathbb{R}}\langle i\tau+\xi^4-\xi^2+\eta^2\rangle|\widehat{g^t_\zeta}(\tau)|^2 d\tau\right)\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|}{\langle i\tau+\xi^4-\xi^2+\eta^2\rangle}\right)^2, \end{aligned}$$

where g_ζ is defined in (3.3). Exactly the same computations as in Lemma 1 lead us to

$$\int_{\mathbb{R}}\langle i\tau+\xi^4-\xi^2+\eta^2\rangle\left|\widehat{g^t_\zeta}(\tau)\right|^2 d\tau \lesssim 1.$$

We conclude then that

$$\left\|\langle i\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_4^t}(\tau)\right\|_{L^2_\tau} \lesssim \int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|}{\langle i\tau+\xi^4-\xi^2+\eta^2\rangle}. \tag{3.9}$$

Contribution of I_3. Since

$$I_3 = \theta(t)\left(\frac{\widehat{w}(\zeta,\tau)}{i\tau+\xi^4-\xi^2+\eta^2}\chi_{\{|\tau|\geqslant 1\}}\right)^{\vee_\tau}(t),$$

we use the Young inequality to obtain that

$$
\begin{aligned}
&\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_3^t}(\tau)\right\|_{L^2_\tau}\\
&\quad=\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{\theta}(\tau')*_{\tau'}\left(\frac{\widehat{w}(\zeta,\tau)\chi_{|\tau'|\geqslant 1}}{\mathrm{i}\tau'+\xi^4-\xi^2+\eta^2}\right)(\tau)\right\|_{L^2_\tau}\\
&\quad\lesssim\left\|\langle \tau'\rangle^{1/2}|\widehat{\theta}|(\tau')*_{\tau'}\left(\frac{\widehat{w}(\zeta,\tau)\chi_{|\tau'|\geqslant 1}}{\mathrm{i}\tau'+\xi^4-\xi^2+\eta^2}\right)(\tau)\right\|_{L^2_\tau}\\
&\qquad\quad+\left\||\widehat{\theta}|(\tau')*_{\tau'}\left(\frac{\widehat{w}(\zeta,\tau)\chi_{|\tau'|\geqslant 1}}{\mathrm{i}\tau'+\xi^4-\xi^2+\eta^2}\right)(\tau)\right\|_{L^2_\tau}\\
&\quad\lesssim\|\theta\|_{H^{1/2}}\left\|\frac{\widehat{w}(\zeta,\tau)\chi_{|\tau'|\geqslant 1}}{\mathrm{i}\tau'+\xi^4-\xi^2+\eta^2}\right\|_{L^1_{\tau'}}+\|\widehat{\theta}\|_{L^1_\tau}\left\|\frac{\widehat{w}(\zeta,\tau)\chi_{|\tau'|\geqslant 1}}{\mathrm{i}\tau'+\xi^4-\xi^2+\eta^2}\right\|_{L^2_{\tau'}}\\
&\quad\lesssim\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau'+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau+\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau'+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^{1/2}.
\end{aligned}
\tag{3.10}
$$

Contribution of I_2. First, note that

$$
\begin{aligned}
\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_2^t}(\tau)\right\|_{L^2_\tau}&\leqslant\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|}{\langle \mathrm{i}\tau'+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^{1/2}\\
&\quad\times\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\left(\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right)^{\wedge_t}(\tau)\right\|_{L^2_\tau}.
\end{aligned}
\tag{3.11}
$$

Since

$$
\begin{aligned}
&\int_{|\tau|\leqslant 1}\frac{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}\mathrm{d}\tau\\
&\quad=\int_{|\tau|\leqslant 1}\frac{\mathrm{d}\tau}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}+\int_{|\tau|\leqslant 1}\frac{\mathrm{d}\tau}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\\
&\quad\lesssim\int_0^1\frac{\mathrm{d}\tau}{\tau^2+(\xi^4-\xi^2+\eta^2)^2}+\frac{1}{|\xi^4-\xi^2+\eta^2|}\\
&\quad\lesssim\frac{1}{|\xi^4-\xi^2+\eta^2|}\int_0^1\frac{1}{1+(\frac{\tau}{|\xi^4-\xi^2+\eta^2|})^2}\mathrm{d}\left(\frac{\tau}{|\xi^4-\xi^2+\eta^2|}\right)+\frac{1}{|\xi^4-\xi^2+\eta^2|}\\
&\quad\lesssim\frac{1}{|\xi^4-\xi^2+\eta^2|}.
\end{aligned}
$$

We deduce from (3.11) and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_2^t}(\tau)\right\|_{L^2_\tau}&\lesssim\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau'+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^{1/2}\\
&\quad\times\frac{1}{\sqrt{|\xi^4-\xi^2+\eta^2|}}\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\left(\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right)^{\wedge_t}\right\|_{L^2_\tau}.
\end{aligned}
\tag{3.12}
$$

Next, we consider two different cases. If $(\xi,\eta)\in\{|\xi|\geqslant 2\}\cup\{|\eta|\geqslant 2\}$, then we have $\xi^4-\xi^2+\eta^2\gtrsim 1$, so

$$\begin{aligned}\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\left(\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right)^{\wedge_t}(\tau)\right\|_{L^2_\tau}&\\ \lesssim \|\theta\|_{H^{1/2}}+\langle\xi^4-\xi^2+\eta^2\rangle^{1/2}\|\theta\|_{L^2}+\|g_\zeta\|_{H^{1/2}}&\\ +\langle\xi^4-\xi^2+\eta^2\rangle^{1/2}\|g_\zeta\|_{L^2}&\\ \lesssim\sqrt{|\xi^4-\xi^2+\eta^2|},&\end{aligned}$$

which implies together with (3.12) that

$$\left\|\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2}\widehat{I_2^t}(\tau)\right\|_{L^2_\tau}\lesssim\left(\int_{\mathbb{R}}\frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau'+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^{1/2}. \tag{3.13}$$

If $|\xi|\leqslant 2$ and $|\eta|\leqslant 2$, then $|\xi^4-\xi^2+\eta^2|\leqslant 4$ and we have

$$\begin{aligned}\left\|\langle \mathrm{i}\tau+\rangle^{1/2}\left(\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right)^{\wedge_t}(\tau)\right\|_{L^2_\tau}&\\ \lesssim\left\|\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right\|_{H^{1/2}_t}.&\end{aligned} \tag{3.14}$$

Then by an argument similar to Lemma 1, we get

$$\begin{aligned}\left\|\theta(t)(1-\mathrm{e}^{-|t|(\xi^4-\xi^2+\eta^2)})\right\|_{H^{1/2}_t}&\lesssim\sum_{j\geqslant 1}\frac{|\xi^4-\xi^2+\eta^2|^j}{j!}\|t^j\theta(t)\|_{H^{1/2}}\\ &\lesssim|\xi^4-\xi^2+\eta^2|\sum_{j\geqslant 0}\frac{|\xi^4-\xi^2+\eta^2|^j}{j!}\\ &\lesssim|\xi^4-\xi^2+\eta^2|,\end{aligned}$$

which together with (3.12) and (3.14) also implies (3.13) in this case.

Contribution of I_1. Since I_1 can be rewritten as

$$I_1=\theta(t)\int_{|\tau|\leqslant}\sum_{j\geqslant 1}\frac{(\mathrm{i}t\tau)^j}{(\mathrm{i}\tau+\xi^4-\xi^2+\eta^2)j!}\widehat{w}(\zeta,\tau),$$

we deduce from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\left\|\left\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\right\rangle^{1/2}\widehat{I_1^t}(\tau)\right\|_{L^2_\tau}\\ &\quad\lesssim\sum_{j\geqslant 1}\frac{1}{j!}\left(\|t^j\theta(t)\|_{H^{1/2}}+\sqrt{\langle\xi^4-\xi^2+\eta^2\rangle}\|t^j\theta(t)\|_{L^2}\right)\\ &\qquad\times\int_{|\tau|\leqslant 1}\frac{|\tau|^j|\widehat{w}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\mathrm{d}\tau\end{aligned}$$

$$\lesssim \sqrt{\langle \xi^4-\xi^2+\eta^2\rangle}\left(\int_{\mathbb{R}} \frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^{1/2}$$
$$\times\left(\int_{|\tau|\leqslant 1} \frac{\tau^2\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}\mathrm{d}\tau\right)^{1/2}$$
$$\lesssim \left\| \frac{\widehat{w}(\zeta,\tau)}{\sqrt{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}}\right\|_{L^2_\tau}, \tag{3.15}$$

where we used the inequality

$$\int_{|\tau|\leqslant 1} \frac{\tau^2\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}\mathrm{d}\tau \lesssim \frac{1}{\langle \xi^4-\xi^2+\eta^2\rangle}.$$

Then, combining (3.8), (3.9), (3.10), (3.13) and (3.15), we get (3.7) which concludes the proof of Lemma 2. □

LEMMA 3. *Let* $s_1,s_2\in\mathbb{R}$. *Then*

$$\left\|\theta(t)\int_0^t V(t-t')v(t')\mathrm{d}t'\right\|_{\mathscr{X}^{1/2,s_1,s_2}}$$
$$\lesssim \|v\|_{\mathscr{X}^{1/2,s_1,s_2}} + \left(\int_{\mathbb{R}^2}\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}\left(\int_{\mathbb{R}}\frac{|(U(-t)v)^\wedge(\zeta,\tau)|}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle}\mathrm{d}\tau\right)^2\mathrm{d}\zeta\right)^{1/2}, \tag{3.16}$$

for all $v\in\mathscr{S}(\mathbb{R}^3)$. *Moreover for any* $0<\delta</1/2$,

$$\left\|\theta(t)\int_0^t V(t-t')v(t')\mathrm{d}t'\right\|_{\mathscr{X}^{1/2,s_1,s_2}} \lesssim \|v\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}}, \tag{3.17}$$

for all $v\in\mathscr{S}(\mathbb{R}^3)$.

Proof. It suffices to prove (3.16). Define

$$w(\cdot,t)=U(-t)v(\cdot,t)\in\mathscr{S}(\mathbb{R}^2). \tag{3.18}$$

We obtain from Fubini's theorem and the Fourier inverse formula, that

$$\theta(t)\int_0^t V(t-t')v(t')\mathrm{d}t' = U(t)(k_\zeta(t))^{\vee_\zeta}, \tag{3.19}$$

where k_ζ is defined in (3.6). Then by using (3.18) and (3.19) and Lemma 2, we deduce that

$$\left\|\theta(t)\int_0^t V(t-t')v(t')\mathrm{d}t'\right\|_{\mathscr{X}^{1/2,s_1,s_2}}$$

$$
\begin{aligned}
&= \Big\| \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \big\| \langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2} \widehat{k}^t_\zeta(\tau) \big\|_{L^2_\tau} \Big\|_{L^2_\zeta} \\
&\lesssim \Big\| \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2} \Big(\int_{\mathbb{R}} \frac{|\widehat{w}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle} \mathrm{d}\tau \\
&\qquad + \Big(\int_{\mathbb{R}} \frac{|\widehat{w}(\zeta,\tau)|}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle} \mathrm{d}\tau \Big)^2 \Big)^{1/2} \Big\|_{L^2_\zeta} \\
&\lesssim \|v\|_{\mathscr{X}^{-1/2,s_1,s_2}} + \Big(\int_{\mathbb{R}^2} \langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2} \Big(\frac{|(U(-t)v)^\wedge(\zeta,\tau)|}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle} \mathrm{d}\tau \Big)^2 \mathrm{d}\zeta \Big)^{1/2}.
\end{aligned}
$$

This completes the proof of (3.16). □

PROPOSITION 1. *Let $s_1, s_2 \in \mathbb{R}$ and $0<\delta<1/2$. Then, for all $f \in \mathscr{X}^{-1/2+\delta,s_1,s_2}$, we have*

$$
t \longmapsto \int_0^t V(t-t')f(t')\mathrm{d}t' \in C(\mathbb{R}^+; H^{s_1+4\delta_1, s_2+2\delta_2}), \tag{3.20}
$$

where $\delta=\delta_1+\delta_2$. Moreover,

$$
\Big\| \int_0^t V(t-t')f(t')\mathrm{d}t' \Big\|_{C([0,T];H^{s_1,s_2})} \lesssim \|f\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}}. \tag{3.21}
$$

Proof. Define $g(x,y,t) = U(-t)f(\cdot,t)(x,y)$. Since U is a strongly continuous unitary group in $H^{s_1,s_2}(\mathbb{R}^2)$, it is enough to show that

$$
F(\zeta,\cdot): t \in \mathbb{R}^+ \longmapsto \langle\xi\rangle^{s_1+4\delta_1}\langle\eta\rangle^{s_2+2\delta} \int_0^t \mathrm{e}^{-(t-t')(\xi^4-\xi^2+\eta^2)} \widehat{g(\cdot,t)}(\zeta)\mathrm{d}t'
$$

is continuous in $L^2_\zeta(\mathbb{R}^2)$, when

$$
\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{-1/2+\delta}\widehat{g}(\zeta,\tau) \in L^2(\mathbb{R}^3).
$$

Similar to (3.19), we can compute, using the Fourier inverse transform in time and Fubini's theorem, that

$$
F(\zeta,t) = \langle\xi\rangle^{s_1+4\delta_1}\langle\eta\rangle^{s_2+2\delta} \int_{\mathbb{R}} \widehat{g}(\zeta,\tau) \frac{\mathrm{e}^{\mathrm{i}t\tau} - \mathrm{e}^{-t(\xi^4-\xi^2+\eta^2)}}{\mathrm{i}\tau+\xi^4-\xi^2+\eta^2} \mathrm{d}\tau.
$$

Fix $t_0 \in \mathbb{R}^+$ and define for all $t \in \mathbb{R}$,

$$
\begin{aligned}
H(\zeta,t) &:= F(\zeta,t) - F(\zeta,t_0) \\
&= \langle\xi\rangle^{s_1+4\delta_1}\langle\eta\rangle^{s_2+2\delta} \int_{\mathbb{R}} \frac{\widehat{g}(\zeta,\tau)}{\mathrm{i}\tau+\xi^4-\xi^2+\eta^2} \\
&\qquad \times \Big[\mathrm{e}^{\mathrm{i}t\tau} - \mathrm{e}^{\mathrm{i}t_0\tau} - \mathrm{e}^{-t(\xi^4-\xi^2+\eta^2)} + \mathrm{e}^{-t_0(\xi^4-\xi^2+\eta^2)} \Big] \mathrm{d}\tau.
\end{aligned}
$$

We will use the Lebesgue dominated convergence theorem to show that

$$\lim_{t\to t_0}\|H(\cdot,t)\|_{L^2(\mathbb{R}^2)}=0. \tag{3.22}$$

First we note that

$$\lim_{t\to t_0}h(\zeta,\tau,t)=0,\qquad \text{a.e.}\quad (\zeta,\tau)\in\mathbb{R}^3, \tag{3.23}$$

where

$$h(\zeta,\tau,t)=\frac{\widehat{g}(\zeta,\tau)}{\mathrm{i}\tau+\xi^4-\xi^2+\eta^2}\left[\mathrm{e}^{\mathrm{i}t\tau}-\mathrm{e}^{\mathrm{i}t_0\tau}-\mathrm{e}^{-t(\xi^4-\xi^2+\eta^2)}+\mathrm{e}^{-t_0(\xi^4-\xi^2+\eta^2)}\right]. \tag{3.24}$$

Moreover, since $t\to t_0$, we can suppose that $0\leqslant t\leqslant T$, and then

$$|h(\zeta,\tau,t)|\leqslant(2+\mathrm{e}^{t/4}+\mathrm{e}^{t_0/4})\frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\lesssim\frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}. \tag{3.25}$$

We deduce from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\int_{\mathbb{R}}\frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\mathrm{d}\tau\\ &\qquad\lesssim\left\|\frac{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\right\|_{L^2_\tau}\left\|\frac{\widehat{g}(\zeta,\tau)}{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}\right\|_{L^2_\tau}.\end{aligned}$$

By our hypotheses on g, we know that

$$\int_{\mathbb{R}^2}\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}\int_{\mathbb{R}}\frac{|\widehat{g}(\zeta,\tau)|^2}{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}\mathrm{d}\tau\mathrm{d}\zeta<+\infty.$$

Hence,

$$\int_{\mathbb{R}}\frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\mathrm{d}\tau\lesssim\left\|\frac{\widehat{g}(\zeta,\tau)}{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}\right\|_{L^2_\tau}, \tag{3.26}$$

for almost every $\zeta\in\mathbb{R}^2$. We use (3.23)-(3.26) and the Lebesgue dominated convergence theorem to conclude that

$$\lim_{t\to t_0}H(\zeta,t)=0,\qquad \text{a.e.}\quad \zeta\in\mathbb{R}^2. \tag{3.27}$$

Next we show that there exists $G\in L^2(\mathbb{R}^2)$ such that

$$|H(\zeta,t)|\leqslant|G(\zeta)|, \tag{3.28}$$

for all $\zeta\in\mathbb{R}^2$ and $t\in\mathbb{R}^+$.

When $|\xi|\geqslant 2$ and $|\eta|\geqslant 2$, we get from the Cauchy-Schwarz inequality and (3.25) that

$$\begin{aligned}&|H(\zeta,t)|\\ &\lesssim\langle\xi\rangle^{s_1+4\delta_1}\langle\eta\rangle^{s_2+2\delta_2}\left\|\frac{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|}\right\|_{L^2_\tau}\left\|\frac{\widehat{g}(\zeta,\tau)}{\langle\mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}\right\|_{L^2_\tau}.\end{aligned}$$

Since $\xi^4-\xi^2+\eta^2 \gtrsim 2$,

$$\left\| \frac{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|} \right\|_{L^2_\tau} \lesssim \left(\int_{\mathbb{R}} \frac{1}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^{1+2\delta}} \mathrm{d}\tau \right)^{1/2} \lesssim \langle \xi \rangle^{-4\delta_1} \langle \eta \rangle^{-2\delta_2}.$$

Now by using the hypotheses our g, we have for all $t \in \mathbb{R}^+$ that

$$|H(\zeta,t)| \lesssim \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \left\| \frac{\widehat{g}(\zeta,\tau)}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}} \right\|_{L^2_\tau} \in L^2(\mathbb{R}^2),$$

which proves (3.28) in this case. When $|\xi| \leqslant 2$ and $|\eta| \leqslant 2$, then we have

$$|\xi^4-\xi^2+\eta^2| \lesssim 2,$$

so

$$\begin{aligned} |H(\zeta,t)| &\lesssim \int_{\mathbb{R}} \frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|} \left| \mathrm{e}^{-t(\xi^4-\xi^2+\eta^2)} - \mathrm{e}^{-t_0(\xi^4-\xi^2+\eta^2)} \right| \mathrm{d}\tau \\ &\quad + \int_{\mathbb{R}} \frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|} \left| \mathrm{e}^{\mathrm{i}t\tau} - \mathrm{e}^{\mathrm{i}t_0\tau} \right| \mathrm{d}\tau \\ &= I + II. \end{aligned}$$

We first estimate II. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} II &\leqslant |t-t_0| \int_{|\tau|\leqslant 1} \frac{|\tau||\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|} \mathrm{d}\tau + 2\int_{|\tau|\geqslant 1} \frac{|\widehat{g}(\zeta,\tau)|}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|} \mathrm{d}\tau \\ &\lesssim \left(\frac{|\widehat{g}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}} \mathrm{d}\tau \right)^{1/2} \\ &\quad \times \left[\left(\int_{|\tau|\leqslant 1} |\tau|^{1-2\delta} \mathrm{d}\tau \right)^{1/2} + \left(\int_{|\tau|\geqslant 1} \langle \tau \rangle^{-1-2\delta} \mathrm{d}\tau \right)^{1/2} \right] \\ &\lesssim \left(\frac{|\widehat{g}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}} \mathrm{d}\tau \right)^{1/2} \in L^2(\mathbb{R}^2). \end{aligned}$$

To estimate I, we use again use the Cauchy-Schwarz inequality to see that

$$\begin{aligned} I &\leqslant |t-t_0| \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}} \mathrm{d}\tau \right)^{1/2} |\xi^4-\xi^2+\eta^2| \\ &\quad \times \left(\int_{\mathbb{R}} \frac{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2} \mathrm{d}\tau \right)^{1/2}. \end{aligned}$$

Hence

$$\left(\int_{\mathbb{R}} \frac{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}\mathrm{d}\tau\right)^{1/2}$$
$$\approx \left(\int_{\mathbb{R}} \frac{1}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^2}\mathrm{d}\tau\right)^{1/2} + \left(\int_{\mathbb{R}} \frac{1}{|\mathrm{i}\tau+\xi^4-\xi^2+\eta^2|^{1+2\delta}}\mathrm{d}\tau\right)^{1/2}$$
$$\approx \frac{1}{\sqrt{|\xi^4-\xi^2+\eta^2|}} + \frac{1}{|\xi^4-\xi^2+\eta^2|^{\delta}}.$$

Then, since $|\xi^4-\xi^2+\eta^2| \lesssim 2$, we conclude that

$$I \lesssim \left(\int_{\mathbb{R}} \frac{|\widehat{g}(\zeta,\tau)|^2}{\langle \mathrm{i}\tau+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}\mathrm{d}\tau\right)^{1/2} \in L^2(\mathbb{R}^2).$$

Thus (3.28) holds in this case. The proof of (3.28) for the cases $\{|\xi| \leqslant 2, |\eta| \geqslant 2\}$ and $\{|\xi| \geqslant 2, |\eta| \leqslant 2\}$ is similar as above. We use (3.27), (3.28) and the dominated convergence theorem to prove (3.22) which concludes the proof of Proposition 1. Estimate (3.21) follows exactly by the same computations. □

Then, we will derive a linear estimate to obtain a contraction factor T^{μ} in the proof of Theorem 1.

LEMMA 4. *For all* $s_1, s_2 \in \mathbb{R}$*, for all* $T > 0$ *and for all* $0 < \delta < 1/2$*, we have that*

$$\|\theta_T w\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}} \lesssim T^{\delta}\|w\|_{\mathscr{X}^{-1/2+2\delta,s_1,s_2}}, \qquad \forall w \in \mathscr{X}^{-1/2+2\delta,s_1,s_2}. \tag{3.29}$$

Proof. By duality, it suffices to prove

$$\|\theta_T v\|_{\mathscr{X}^{1/2-2\delta,-s_1,-s_2}} \lesssim T^{\delta}\|v\|_{\mathscr{X}^{1/2-\delta,-s_1,-s_2}}, \qquad \forall v \in \mathscr{X}^{1/2-\delta,-s_1,-s_2}. \tag{3.30}$$

Let $J_1 = 1-\partial_x^2$ and $J_2 = 1-\partial_y^2$. First, by using the definitions of $\mathscr{X}^{b,s_1,s_2}$ and the unitary group U, the fact that U is a unitary group, the Hölder inequality and the Sobolev embedding theorem, we observe that

$$\begin{aligned}\|\theta_T v\|_{\mathscr{X}^{0,-s_1,-s_2}} &= \|J_1^{-s_1}J_2^{-s_2}\theta_T v\|_{L^2(\mathbb{R}^3)} = \|\theta_T J_1^{-s_1}J_2^{-s_2}U(-t)v\|_{L^2(\mathbb{R}^3)}\\ &\lesssim T^{1/\delta}\|J_1^{-s_1}J_2^{-s_2}U(-t)v\|_{L^2_\zeta(\mathbb{R}^2)L_t^{1/\delta}(\mathbb{R})}\\ &\lesssim T^{1/\delta}\|J_1^{-s_1}J_2^{-s_2}U(-t)v\|_{L^2_\zeta(\mathbb{R}^2)H_t^{1/2-\delta}(\mathbb{R})},\end{aligned}$$

which implies

$$\|\theta_T v\|_{\mathscr{X}^{0,-s}} \lesssim T^{1/2-\delta}\|v\|_{\mathscr{X}^{1/2-\delta,-s}}, \qquad \forall v \in \mathscr{X}^{1/2-\delta,-s}. \tag{3.31}$$

On the other hand, we have

$$\begin{aligned}\|\theta_T v\|_{\mathscr{X}^{1/2-2\delta,-s}} &\lesssim \||\tau-\xi^3|^{1/2-\delta}\langle\zeta\rangle^{-s}\widehat{\theta_T v}(\zeta,\tau)\|_{L^2(\mathbb{R}^3)} \\ &\qquad + \|\langle\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}\langle\zeta\rangle^{-s}\widehat{\theta_T v}(\zeta,\tau)\|_{L^2(\mathbb{R}^3)} \\ &= I + II.\end{aligned}$$

By applying the Plancherel identity, we see that

$$\begin{aligned}II &= \|\langle\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}\langle\zeta\rangle^{-s}\|\theta_T(t)\widehat{v}(\zeta,t)\|_{L^2_t}\|_{L^2_\zeta(\mathbb{R}^2)} \\ &\leqslant \|\langle\xi^4-\xi^2+\eta^2\rangle^{1/2-\delta}\langle\zeta\rangle^{-s}\widehat{v}(\zeta,\tau)\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

To estimate I, it is enough to show that

$$\int_{\mathbb{R}}|\widehat{\theta_T} *_\tau \widehat{v}(\tau)|^2|\tau-a|^{1-2\delta}\mathrm{d}\tau \lesssim \int_{\mathbb{R}}|\widehat{v}(\tau)|^2|\tau-a|^{1-2\delta}\mathrm{d}\tau, \qquad \forall a\in\mathbb{R}.$$

By using again the Plancherel identity and the Leibniz rule for fractional derivatives, we obtain

$$\begin{aligned}\|\widehat{\theta_T} *_\tau \widehat{v}(\tau)|\tau-a|^{1/2-\delta}\|_{L^2_\tau} &= \|D_t^{1/2-\delta}(\mathrm{e}^{-\mathrm{i}at}\theta_T v)\|_{L^2_t} \\ &\lesssim \|\mathrm{e}^{-\mathrm{i}at} v D_t^{1/2-\delta}\theta_T\|^2_{L^2_t} + \|\theta_T\|_{L^\infty_t}\|D_t^{1/2-\delta}(\mathrm{e}^{-\mathrm{i}at}v)\|^2_{L^2_t} \\ &\lesssim \|\mathrm{e}^{-\mathrm{i}at} v D_t^{1/2-\delta}\theta_T\|^2_{L^2_t} + \|\widehat{v}(\tau)|\tau-a|^{1/2-\delta}\|^2_{L^2_\tau}.\end{aligned}$$

To estimate $\|\mathrm{e}^{\mathrm{i}at} v D_t^{1/2-\delta}\theta_T\|_{L^2_t}$, we use the Hölder inequality and the Hardy-Littlewood-Sobolev theorem and obtain

$$\begin{aligned}\|\mathrm{e}^{\mathrm{i}at} v D_t^{1/2-\delta}\theta_T\|_{L^2_t} &\leqslant \|\mathrm{e}^{\mathrm{i}at} v\|_{L_t^{1/\delta}}\|D_t^{1/2-\delta}\theta_T\|_{L_t^{2/(1-2\delta)}} \\ &\lesssim \|D_t^{1/2-\delta}(\mathrm{e}^{-\mathrm{i}at}v)\|_{L^2_t}\|D_t^{1/2-\delta}\theta_T\|_{L_t^{2/(1-2\delta)}}.\end{aligned}$$

Now by using the Hausdorff-Young theorem, we derive

$$\begin{aligned}\|D_t^{1/2-\delta}\theta_T\|_{L_t^{2/(1-2\delta)}} &\lesssim \left(\int_{\mathbb{R}}\left[|\tau|^{1/2-\delta}|T\widehat{\theta}(T\tau)|\right]^{2/(1+2\delta)}\mathrm{d}\tau\right)^{1/2+\delta} \\ &\lesssim \left(\int_{\mathbb{R}}\left[|\tau|^{1/2-\delta}|\widehat{\theta}(\tau)|\right]^{2/(1+2\delta)}\mathrm{d}\tau\right)^{1/2+\delta} \lesssim 1.\end{aligned}$$

Finally an interpolation gives (3.30). □

4. Bilinear estimates

In this section, we derive the crucial bilinear estimate to prove the local existence result.

THEOREM 3. *Let $s_1 > -3/2$ and $s_2 \geqslant 0$, then there exists $\delta > 0$ such that*

$$\|(uv)_x\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}} \lesssim \|u\|_{\mathscr{X}^{1/2,s_1,s_2}} \|v\|_{\mathscr{X}^{1/2,s_1,s_2}}. \tag{4.1}$$

Proof. By duality, it suffices to prove that for all $s_1 > -3/2$ and $s_2 \geqslant 0$, there exists $\delta > 0$ such that

$$I \lesssim \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}, \tag{4.2}$$

where

$$I = \int_{\mathbb{R}^6} \frac{|\xi| \langle\xi\rangle^{s_1} \langle\eta\rangle^{s_2} \langle \mathrm{i}\sigma + \xi^4 - \xi^2 + \eta^2\rangle^{\delta-1/2} f(\xi_1,\eta_1,\tau_1) g(\xi_2,\eta_2,\tau_2) h(\zeta,\tau)}{\langle\xi_1\rangle^{s_1} \langle\eta_1\rangle^{s_2} \langle \mathrm{i}\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2\rangle^{1/2} \langle\xi_2\rangle^{s_1} \langle\eta_2\rangle^{s_2} \langle \mathrm{i}\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2\rangle^{1/2}} \mathrm{d}\nu, \tag{4.3}$$

and

$$\mathrm{d}\nu = \mathrm{d}\zeta \mathrm{d}\tau \mathrm{d}\zeta_1 \mathrm{d}\tau_1, \quad \zeta = (\xi,\eta), \quad \tau_2 = \tau - \tau_1, \quad \zeta_2 = \zeta - \zeta_1,$$
$$\sigma = \tau - \xi^3, \quad \sigma_1 = \tau_1 - \xi_1^3, \quad \sigma = \tau_2 - \xi_2^3.$$

Moreover, we can assume that $s_2 = 0$ since in the case $s_2 \geqslant 0$ we have

$$\langle\eta\rangle^{s_2} \leqslant \langle\eta_1\rangle^{s_2} \langle\eta - \eta_1\rangle^{s_2}, \qquad \forall \eta, \eta_1 \in \mathbb{R}.$$

Case $-3/2 < s_1 < 0$. Let $s = s_1 = -3/2 + \varepsilon$, where $0 < \varepsilon < 3/2$; and choose $0 < \delta \ll \varepsilon$. A symmetry argument shows that it is enough to estimate the contribution to I of the following subset of $\mathbb{R}^6$:

$$\Omega = \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\sigma_1| \geqslant |\sigma_2|\}.$$

Now we divide Ω into $\Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \Omega \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi| > 4\},$$
$$\Omega_2 = \Omega \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi| \leqslant 4\}.$$

Case 1. Contribution of Ω_1 to I. We divide Ω_1 into two regions:

$$\Omega_1^1 = \Omega_1 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi| \leqslant 2|\xi_1|\},$$
$$\Omega_1^2 = \Omega_1 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi| > 2|\xi_1|\}.$$

Case 1.1. Contribution of Ω_1^1 to I. Denote by I_1^1 the contribution of this region to I. We show that

$$\begin{aligned} I_1^1 &\leqslant \sup_{(\zeta_1,\tau_1)\in\mathbb{R}^3} \left(J_1^1(\zeta_1,\tau_1)\right)^{1/2} \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}, \\ &\leqslant C \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{4.4}$$

where

$$J_1^1(\zeta_1,\tau_1) = \frac{\langle\xi_1\rangle^{3/2-\varepsilon}}{\langle \mathrm{i}\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2\rangle} \int_{\Omega_1^1} \frac{\langle\xi^2\rangle \langle\xi_2^2\rangle^{3/2-\varepsilon} \langle \mathrm{i}\sigma_2 + \xi^4 - \xi^2 + \eta^2\rangle^{2\delta-1}}{\langle\xi^2\rangle^{3/2-\varepsilon} \langle \mathrm{i}\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2\rangle} \mathrm{d}\zeta \mathrm{d}\tau.$$

We divide Ω_1^1 into two regions:

$$\Omega_1^{11} = \Omega_1^1 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi_2| > 2\},$$

$$\Omega_1^{12} = \Omega_1^1 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi_2| \leqslant 2\}.$$

Case 1.11. Contribution of Ω_1^{11} to I. Since $|\xi_1| \geqslant 2$, then we have that

$$\begin{aligned}\langle \mathrm{i}\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2\rangle &\gtrsim \langle \xi_1\rangle^{3-2\varepsilon}\langle\sigma_1\rangle^{(1-\varepsilon)/4}\langle\xi_1\rangle^{3\varepsilon/4}\\ &\gtrsim \langle \xi_1\rangle^{3-2\varepsilon}\langle\sigma_2\rangle^{(1-\varepsilon)/4}\langle\xi\rangle^{3\varepsilon/4},\end{aligned} \tag{4.5}$$

$$\langle \mathrm{i}\sigma + \xi^4 - \xi^2 + \eta^2\rangle^{1-2\delta} \gtrsim \langle\sigma\rangle^{1/2-\varepsilon/8-2\delta}\langle\eta\rangle^{1+\varepsilon/4} \tag{4.6}$$

and

$$\langle \mathrm{i}\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2\rangle \gtrsim \langle\sigma_2\rangle^{1/4+\varepsilon/2}\langle\xi_2\rangle^{3-2\varepsilon}; \tag{4.7}$$

so that

$$\begin{aligned}J_1^{11} &\lesssim \int_{\Omega_1^{11}} \frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\sigma_1\rangle^{(1-\varepsilon)/4}\langle\xi\rangle^{1+\varepsilon}\langle\sigma_2\rangle^{1/4+\varepsilon/2}\langle\sigma\rangle^{1/2-\varepsilon/8-2\delta}\langle\eta\rangle^{1+\varepsilon/4}}\\ &\lesssim \int_{\mathbb{R}^3} \frac{\mathrm{d}\tau\mathrm{d}\zeta}{\langle\widetilde{\sigma}\rangle^{1+\varepsilon/8-2\delta}\langle\xi\rangle^{1+\varepsilon}\langle\eta\rangle^{1+\varepsilon/4}} \lesssim 1,\end{aligned}$$

where $|\widetilde{\sigma}| = \min\{|\sigma|,|\sigma_2|\}$.

Case 1.12. Contribution of Ω_1^{12} to I. Similar to (4.5)-(4.7), we have

$$\begin{aligned}J_1^{11} &\lesssim \frac{1}{\langle \mathrm{i}\sigma_1 + \xi_1^4 + \eta_1^2\rangle^{1/4+\varepsilon/2}} \int_{\Omega_1^{12}} \frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\xi\rangle^{1-2\varepsilon}\langle\sigma_2\rangle\langle \mathrm{i}\sigma + \xi^4 - \xi^2 - \eta^2\rangle^{1-2\delta}}\\ &\lesssim \frac{1}{\langle\xi_1\rangle^{1/4+\varepsilon/2-\varepsilon'}\langle\sigma_1\rangle^{\varepsilon'}} \int_{\Omega_1^{12}} \frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\xi\rangle^{1-2\varepsilon}\langle\sigma_2\rangle\langle\xi^4+\eta^2\rangle^{1-2\delta}}\\ &\lesssim \int_{\mathbb{R}^3} \frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\widetilde{\sigma}\rangle^{1+\varepsilon'}\langle\xi\rangle^{1-4\varepsilon'}\langle\xi^4+\eta^2\rangle^{1-2\delta}} \lesssim 1,\end{aligned}$$

where $\varepsilon' = \varepsilon/(2(n+1))$, for $n \gg 1$.

Case 1.2. Contribution of Ω_1^2 to I. We divide Ω_1^2 into the following two regions:

$$\Omega_1^{21} = \Omega_1^2 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\sigma_1| \geqslant |\sigma|\},$$

$$\Omega_1^{22} = \Omega_1^2 \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\sigma_1| \leqslant |\sigma|\}.$$

Case 1.21. Contribution of Ω_1^{21} to I. In Ω_1^{21}, we have $|\xi| \sim |\xi_2|$ and $|\xi_2| > 2$. Thus we divide Ω_1^{21} into two subdomains:

$$\Omega_1^{211} = \Omega_1^{21} \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi_1| \geqslant 2\},$$

$$\Omega_1^{212} = \Omega_1^{21} \cap \{(\tau,\tau_1,\zeta,\zeta_1) \in \mathbb{R}^6 \ : \ |\xi_1| \leqslant 2\}.$$

Case 1.211. Contribution of Ω_1^{211} to I. In this case, we have

$$\xi_1^4-\xi_1^2+\eta_1^2\gtrsim\xi_1^4+\eta_1^2\gtrsim 1.$$

Therefore, we obtain

$$\begin{aligned}
J_1^{211}&\lesssim\frac{1}{\langle\mathrm{i}\sigma_1+\xi_1^4+\eta_1^2\rangle^{1/4+\varepsilon/2}}\int_{\Omega_1^{211}}\frac{\langle\mathrm{i}\sigma+\xi^4-\xi^2+\eta^2\rangle^{2\delta-1}}{\langle\xi\rangle^{1-2\varepsilon}\langle\mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle^{1/4+\varepsilon/2}}\mathrm{d}\zeta\mathrm{d}\tau\\
&\lesssim\int_{\Omega_1^{211}}\frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\sigma_1\rangle^{1/4+\varepsilon/2}\langle\xi\rangle^{1-2\varepsilon}\langle\sigma_2\rangle^{(1-\varepsilon)/4}\langle\xi^4\rangle^{3\varepsilon/4}\langle\sigma\rangle^{1/2-2\delta-\varepsilon/8}\langle\eta\rangle^{1+\varepsilon/4}}\\
&\lesssim\int_{\mathbb{R}^3}\frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\widetilde{\sigma}\rangle^{1-2\delta+\varepsilon/8}\langle\xi\rangle^{1+\varepsilon}\langle\eta\rangle^{1+\varepsilon/4}}\lesssim 1.
\end{aligned}$$

Case 1.212. Contribution of Ω_1^{212} to I. Since $|\xi_1|\lesssim 1$, then we obtain

$$\begin{aligned}
J_1^{212}&\lesssim\frac{1}{\langle\sigma_1\rangle}\int_{\Omega_1^{212}}\frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\xi\rangle^{1-2\varepsilon}\langle\sigma_2\rangle^{(1-\varepsilon)/4}\langle\xi^4\rangle^{3\varepsilon/4}\langle\sigma\rangle^{1/2-2\delta-\varepsilon/8}\langle\eta\rangle^{1+\varepsilon/4}}\\
&\lesssim\int_{\mathbb{R}^3}\frac{\mathrm{d}\zeta\mathrm{d}\tau}{\langle\widetilde{\sigma}\rangle^{7/4-2\delta-3\varepsilon/8}\langle\xi\rangle^{1+\varepsilon}\langle\eta\rangle^{1+\varepsilon/4}}\lesssim 1.
\end{aligned}$$

Case 1.22. Contribution of Ω_1^{22} to I. We show that for I_1^{22}, the contribution of the Ω_1^{22} to I,

$$\begin{aligned}
I_1^{22}&\leqslant\sup_{(\zeta,\tau)\in\mathbb{R}^3}\left(\mathscr{I}_1^{22}(\zeta,\tau)\right)^{1/2}\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)}\\
&\leqslant C\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)},
\end{aligned}\tag{4.8}$$

where

$$\begin{aligned}
\mathscr{I}_1^{22}=&\frac{\xi^2}{\langle\xi\rangle^{3-2\varepsilon}\langle\mathrm{i}\sigma+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}\\
&\times\int_{\Omega_1^{22}}\frac{\langle\xi_1\rangle^{3-2\varepsilon}\langle\xi_2\rangle^{3-2\varepsilon}}{\langle\mathrm{i}\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle\langle\mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle}\mathrm{d}\zeta_1\mathrm{d}\tau_1.
\end{aligned}$$

We divide Ω_1^{22} into the following two regions:

$$\Omega_1^{221}=\Omega_1^{22}\cap\{(\zeta,\zeta_1,\tau,\tau_1)\in\mathbb{R}^6\ :\ |\eta_1|\leqslant 2\},$$

$$\Omega_1^{222}=\Omega_1^{22}\cap\{(\zeta,\zeta_1,\tau,\tau_1)\in\mathbb{R}^6\ :\ |\eta_1|\geqslant 2\}.$$

Case 1.221. Contribution of Ω_1^{221} to I. If $|\xi_1|\leqslant 2$, It is readily seen that

$$\begin{aligned}
\mathscr{I}_1^{221}\lesssim&\frac{\langle\mathrm{i}\sigma+\xi^4-\xi^2+\eta^2\rangle^{2\delta-1}}{\langle\xi\rangle^{1-2\varepsilon}}\\
&\times\int_{|\eta_1|\leqslant 2}\int_{|\xi_1|\leqslant 2}\int_{\mathbb{R}}\frac{\langle\mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle^{-1/4-\varepsilon/2}\mathrm{d}\zeta_1\mathrm{d}\tau_1}{\langle\mathrm{i}\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle}\\
\lesssim&\ 1.
\end{aligned}$$

If $|\xi_1| \geqslant 2$, then

$$\begin{aligned}\mathscr{J}_1^{221} &\lesssim \frac{1}{\langle\xi\rangle^{1-2\varepsilon}\langle \mathrm{i}\sigma+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}} \\ &\quad\times \int_{|\eta_1|\leqslant 2}\int_{|\xi_1|\leqslant 2}\int_{\mathbb{R}} \frac{\langle \mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle^{-1/4-\varepsilon/2}\mathrm{d}\zeta_1\mathrm{d}\tau_1}{\langle \mathrm{i}\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle^{1/4+\varepsilon/2}} \\ &\lesssim \int_{\mathbb{R}^2} \frac{\mathrm{d}\tau_1\mathrm{d}\xi_1}{\langle\sigma_2\rangle^{5/4+\varepsilon/2-2\delta}\langle\xi_1\rangle^2} \lesssim 1.\end{aligned}$$

Case 1.222. Contribution of Ω_1^{222} to I. In this case we have

$$\xi_1^4-\xi^2+\eta_1^2 \gtrsim \xi_1^4+\eta_1^2 \geqslant 1.$$

Therefore if $|\xi_1| \leqslant 2$, then

$$\begin{aligned}\mathscr{J}_1^{222} &\lesssim \frac{1}{\langle\xi\rangle^{1-2\varepsilon}\langle\sigma\rangle^{1/2-2\delta}}\int_{\Omega_1^{222}} \frac{\mathrm{d}\zeta_1\mathrm{d}\tau_1}{\langle \mathrm{i}\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle\langle \mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle^{1/4+\varepsilon/2}} \\ &\lesssim \int_{\mathbb{R}^2}\frac{\mathrm{d}\eta_1\mathrm{d}\tau_1}{\langle\eta_1^2\rangle\langle\sigma_2\rangle^{5/4-2\delta+\varepsilon/2}} \lesssim 1,\end{aligned}$$

while for $|\xi_1| \geqslant 2$ we have

$$\begin{aligned}\mathscr{J}_1^{222} &\lesssim \frac{1}{\langle\xi\rangle^{1-2\varepsilon}\langle \mathrm{i}\sigma+\xi^4-\xi^2+\eta^2\rangle^{1/2-2\delta}} \\ &\quad\times \int_{\Omega_1^{222}} \frac{\langle \mathrm{i}\sigma++\xi_2^4-\xi_2^2+\eta_2^2\rangle^{-1/4-\varepsilon/2}\mathrm{d}\zeta_1\mathrm{d}\tau_1}{\langle \mathrm{i}\sigma_1++\xi_1^4-\xi_1^2+\eta_1^2\rangle^{1/4+\varepsilon/2}} \\ &\lesssim \int_{\mathbb{R}^3}\frac{\mathrm{d}\zeta_1\mathrm{d}\tau_1}{\langle\xi_1\rangle^{1-2\varepsilon}\langle\sigma\rangle^{3/4-2\delta}\langle\xi_1^4+\eta_1^2\rangle^{1/4+\varepsilon/2}\langle\sigma_2\rangle^{1/4+\varepsilon/2}\langle\xi_1\rangle^{\varepsilon}} \lesssim 1.\end{aligned}$$

Case 2. Contribution of Ω_2 to I. We show that (4.4) holds by proving that

$$J_2 = \frac{\langle\xi_1\rangle^{3-2\varepsilon}}{\langle \mathrm{i}\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle}\int_{\Omega_2}\frac{\langle\xi^2\rangle\langle\xi_2\rangle^{3-2\varepsilon}\langle \mathrm{i}\sigma_2+\xi^4-\xi^2+\eta^2\rangle^{2\delta-1}}{\langle\xi\rangle^{3-2\varepsilon}\langle \mathrm{i}\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle}\mathrm{d}\zeta\mathrm{d}\tau, \quad (4.9)$$

is bounded. First we note that if $|\xi_1| \leqslant 6$ and $|\eta| \leqslant 2$, then $J_2 \lesssim 1$. Now if $|\xi_1| \leqslant 6$ and $|\eta| \geqslant 2$, then $|\xi_2| \lesssim 1$ and we see that

$$J_2 \lesssim \int_{\mathbb{R}^2}\frac{\mathrm{d}\tau\mathrm{d}\eta}{\langle\sigma_1\rangle\langle\sigma_2\rangle\langle\sigma\rangle^{1/2-2\delta-\varepsilon}\langle\eta\rangle^{1+2\varepsilon}} \lesssim 1.$$

If $|\xi_1| \geqslant 6$ and $|\eta| \leqslant 2$, then $|\xi_2| \geqslant 2$ and

$$J_2 \lesssim \int_{\mathbb{R}}\frac{\mathrm{d}\tau}{\langle\sigma_1\rangle^{1/4+\varepsilon/2}\langle\sigma_2\rangle^{1/4+\varepsilon/2}\langle\sigma\rangle^{1/2-2\delta}} \lesssim 1.$$

Finally, $|\xi_1| \geqslant 6$ and $|\eta| \geqslant 2$, then $|\xi_2| \geqslant 2$ and consequently we have

$$J_2 \lesssim \int_{\mathbb{R}^2} \frac{\mathrm{d}\tau \mathrm{d}\eta}{\langle \sigma_1 \rangle^{1/4+\varepsilon/2} \langle \sigma_2 \rangle^{1/4+\varepsilon/2} \langle \sigma \rangle^{1/2-2\delta-\varepsilon/8} \langle \eta \rangle^{1+2\varepsilon/4}} \lesssim 1.$$

Case $s \geqslant 0$. By using again the fact $\langle \xi \rangle^s \leqslant \langle \xi_1 \rangle^s \langle \xi_2 \rangle^s$, for all $s \geqslant 0$, it suffices to prove (4.2) for $s = 0$. By symmetry one can assume that $|\sigma_1| \geqslant |\sigma_2|$. We prove that $J \lesssim 1$, where J is defined in (4.9). First we consider the case $|\xi| \geqslant 4$. We also assume $2|\xi_1| \geqslant |\xi|$. Hence $|\xi_1| \geqslant 2$ and

$$\begin{aligned} J &\lesssim \frac{1}{\langle \mathrm{i}\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle} \int_{\mathbb{R}^3} \frac{\langle \xi^2 \rangle}{\langle \mathrm{i}\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle \langle \mathrm{i}\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \mathrm{d}\zeta \mathrm{d}\tau \\ &\lesssim \frac{1}{\langle \mathrm{i}\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle^{1/2}} \int_{\mathbb{R}^3} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle \langle \mathrm{i}\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \\ &\lesssim \int_{\mathbb{R}^3} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle^{3/2} \langle \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \lesssim 1. \end{aligned}$$

If $2|\xi_1| \leqslant |\xi|$ holds, then $|\xi| \sim |\xi_2| \geqslant 2$ and

$$\begin{aligned} J &\lesssim \frac{1}{\langle \sigma_1 \rangle} \int_{\mathbb{R}^3} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \mathrm{i}\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{1/2} \langle \mathrm{i}\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \\ &\lesssim \int_{\mathbb{R}^3} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle^{3/2} \langle \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \lesssim 1. \end{aligned}$$

Next we consider the case $|\xi| \leqslant 4$. If $|\eta| > 4$, then

$$J \lesssim \frac{1}{\langle \sigma_1 \rangle} \int_{|\xi| \leqslant 4} \int_{\mathbb{R}^2} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle \langle \eta^2 \rangle^{1-2\delta}} \lesssim \int_{\mathbb{R}^2} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle^2 \langle \eta^2 \rangle^{1-2\delta}} \lesssim 1, \tag{4.10}$$

while for $|\eta| \leqslant 4$ we get

$$J \lesssim \frac{1}{\langle \sigma_1 \rangle} \int_{|\xi| \leqslant 4} \int_{|\eta| \leqslant 4} \int_{\mathbb{R}} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle \langle \sigma \rangle^{1-2\delta}} \lesssim \int_{\mathbb{R}} \frac{\mathrm{d}\zeta \mathrm{d}\tau}{\langle \sigma_2 \rangle^2 \langle \sigma \rangle^{1-2\delta}} \lesssim 1. \tag{4.11}$$

This completes the proof of Theorem 3. □

THEOREM 4. *Let $s_1 > -1/2$ and $s_2 \geqslant 0$, then there exists $\delta > 0$ such that*

$$\|(uv)_{xx}\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}} \lesssim \|u\|_{\mathscr{X}^{1/2,s_1,s_2}} \|v\|_{\mathscr{X}^{1/2,s_1,s_2}}. \tag{4.12}$$

Proof. We follow the same strategy as in the proof of Theorem 3. By duality, estimate (4.12) is equivalent to prove that for all $s_1 > -1/2$ and $s_2 \geqslant 0$, there exists $\delta > 0$ such that

$$I \lesssim \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}, \tag{4.13}$$

where

$$I=\int_{\mathbb{R}^6}\frac{|\xi|^2\langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle i\sigma+\xi^4-\xi^2+\eta^2\rangle^{\delta-1/2}f(\xi_1,\eta_1,\tau_1)g(\xi_2,\eta_2,\tau_2)h(\zeta,\tau)}{\langle\xi_1\rangle^{s_1}\langle\eta_1\rangle^{s_2}\langle i\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle^{1/2}\langle\xi_2\rangle^{s_1}\langle\eta_2\rangle^{s_2}\langle i\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle^{1/2}}\mathrm{d}\nu, \tag{4.14}$$

and

$$\mathrm{d}\nu=\mathrm{d}\zeta\mathrm{d}\tau\mathrm{d}\zeta_1\mathrm{d}\tau_1,\quad \zeta=(\xi,\eta),\quad \tau_2=\tau-\tau_1,\quad \zeta_2=\zeta-\zeta_1,$$
$$\sigma=\tau-\xi^3,\quad \sigma_1=\tau_1-\xi_1^3,\quad \sigma=\tau_2-\xi_2^3.$$

Moreover, we can assume that $s_2=0$ since in the case $s_2\geqslant 0$ we have

$$\langle\eta\rangle^{s_2}\leqslant\langle\eta_1\rangle^{s_2}\langle\eta-\eta_1\rangle^{s_2},\qquad \forall\eta,\eta_1\in\mathbb{R}.$$

Case $0>s_1>-1/2$. We can write $s=s_1=-1/2+\varepsilon$, where $0<\varepsilon<1/2$. Choose $0<\delta\ll\varepsilon$. By a symmetry argument, it is enough to estimate the contribution to I of the following subset of $\mathbb{R}^6$:

$$\Omega=\{(\zeta,\tau,\zeta_1,\tau_1)\in\mathbb{R}^6\,:\,|\sigma_1|\geqslant|\sigma_2|\}.$$

Now we divide Ω into the following regions:

$$\Omega_1=\{(\zeta,\tau,\zeta_1,\tau_1)\in\mathbb{R}^6\,:\,|\xi|>4,\ |\xi|\leqslant 2|\xi_1|\},$$

$$\Omega_2=\{(\zeta,\tau,\zeta_1,\tau_1)\in\mathbb{R}^6\,:\,|\xi|>4,\ |\xi|>2|\xi_1|,\ |\sigma_1|\geqslant|\sigma|\},$$
$$\Omega_3=\{(\zeta,\tau,\zeta_1,\tau_1)\in\mathbb{R}^6\,:\,|\xi|>4,\ |\xi|>2|\xi_1|,\ |\sigma|\geqslant|\sigma_1|\},$$
$$\Omega_4=\{(\zeta,\tau,\zeta_1,\tau_1)\in\mathbb{R}^6\,:\,|\xi|\leqslant 4\}.$$

Case 1. Contribution of Ω_1 to I. Denote by I_1 the contribution of this region to I. We show that

$$\begin{aligned}I_1&\leqslant\sup_{(\zeta_1,\tau_1)\in\mathbb{R}^3}(J_1(\zeta_1,\tau_1))^{1/2}\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)},\\&\leqslant C\|f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{4.15}$$

where

$$J_1=\frac{\langle\xi_1\rangle^{1-2\varepsilon}}{\langle i\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle}\times\int_{\Omega_1}\frac{\langle\xi^4\rangle\langle\xi_2\rangle^{1-2\varepsilon}}{\langle\xi\rangle^{1-2\varepsilon}\langle i\sigma_2+\xi_2^4-\xi_2^2+\eta_2^2\rangle\langle i\sigma_2+\xi^4-\xi^2+\eta^2\rangle^{1-2\delta}}\mathrm{d}\zeta\mathrm{d}\tau.$$

We note that in Ω_1 we have $|\xi_1|\geqslant|\xi|/2\geqslant 2$ and so that $\xi^4-\xi^2\sim\xi^4$ and $\xi_1^4-\xi_1^2\sim\xi_1^4$. First we consider the case $|\xi_2|\leqslant 2$. Then we have

$$J_1\lesssim\frac{1}{\langle i\sigma_1+\xi_1^4-\xi_1^2+\eta_1^2\rangle^{3/4+\varepsilon/2}}$$

$$
\begin{aligned}
&\times \int_{\Omega_1} \frac{\langle \xi^4 \rangle^{3/4+\varepsilon/2}}{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} d\zeta d\tau \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \sigma_2 \rangle \langle \sigma \rangle^{\varepsilon'} \langle \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta-\varepsilon'}} \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \widetilde{\sigma} \rangle^{1+\varepsilon'} \langle \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta-\varepsilon'}} \lesssim 1,
\end{aligned}
$$

where $0 < \varepsilon' \ll 1$ is small enough and $|\widetilde{\sigma}| = \min\{|\sigma|, |\sigma_2|\}$.

If the case $|\xi_2| \geqslant 2$ holds, then $\xi_2^4 - \xi_2^2 + \eta_2^2 \sim \xi_2^4 + \eta_2^2 \gtrsim 1$. Thus we get

$$
\begin{aligned}
J_1 &\lesssim \frac{1}{\langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle^{3/4+\varepsilon/2}} \int_{\Omega_1} \frac{\langle \xi \rangle^{3+2\varepsilon} \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{2\delta-1}}{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{3/4+\varepsilon/2}} d\zeta d\tau \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{3/4+\varepsilon/2} \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \xi^4 + \eta^2 \rangle^{3/4+\varepsilon/4-2\delta} \langle \widetilde{\sigma} \rangle^{1+\varepsilon/4}} \lesssim 1,
\end{aligned}
$$

where $|\widetilde{\sigma}| = \min\{|\sigma|, |\sigma_2|\}$.

Case 2. Contribution of Ω_2 to I. Consider Ω_2 in the integral of J_1 (instead of Ω_1) and denote it by J_2. In this case, we have $|\xi| \sim |\xi_2| \gtrsim |\xi_1|$ and $|\xi_2| \geqslant 2$. If $|\xi_1| \geqslant 2$, then it is straightforward to see that

$$
\begin{aligned}
J_2 &\lesssim \frac{1}{\langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle^{3/4+\varepsilon/2}} \int_{\Omega_2} \frac{\langle \xi^4 \rangle^{3/4+\varepsilon/2} \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{2\delta-1}}{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{3/4+\varepsilon/2}} d\zeta d\tau \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \sigma \rangle^{3/4+\varepsilon/2} \langle i\sigma + \xi^4 + \eta^2 \rangle^{1-2\delta}} \lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \sigma \rangle^{1+\varepsilon/4} \langle \xi^4 + \eta^2 \rangle^{3/4-2\delta+\varepsilon/4}} \lesssim 1.
\end{aligned}
$$

Next suppose that $|\xi_1| \leqslant 2$. Then we get

$$
\begin{aligned}
J_2 &\lesssim \frac{1}{\langle \sigma_1 \rangle} \int_{\Omega_2} \frac{\langle \xi^4 \rangle^{3/4+\varepsilon/2}}{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{3/4+\varepsilon/2} \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} d\zeta d\tau \\
&\lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \sigma \rangle \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \lesssim \int_{\mathbb{R}^3} \frac{d\zeta d\tau}{\langle \sigma \rangle^{1+\varepsilon'} \langle \xi^4 + \eta^2 \rangle^{1-2\delta-\varepsilon'}} \lesssim 1,
\end{aligned}
$$

where $0 < \varepsilon' \ll 1$ is small enough.

Case 3. Contribution of Ω_3 to I. For this case, we prove that for I_3, the contribution of the Ω_3 to I,

$$
\begin{aligned}
I_3 &\leqslant \sup_{(\zeta,\tau)\in\mathbb{R}^3} \left(\mathscr{J}_3(\zeta,\tau)\right)^{1/2} \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} \\
&\leqslant C \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{4.16}
$$

where

$$\mathscr{I}_3 = \frac{\langle \xi^4 \rangle}{\langle \xi \rangle^{1-2\varepsilon} \langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \times \int_{\Omega_3} \frac{\langle \xi_1 \rangle^{1-2\varepsilon} \langle \xi_2 \rangle^{1-2\varepsilon}}{\langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle \langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle} d\zeta_1 d\tau_1.$$

If $|\xi_1| \leqslant 2$, we have

$$\begin{aligned}\mathscr{I}_3 &\lesssim \frac{\langle \xi \rangle^{3+2\varepsilon}}{\langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \times \int_{\Omega_3} \frac{d\zeta_1 d\tau_1}{\langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle \langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{3/4+\varepsilon/2}} \\ &\lesssim \int_{|\xi_1| \leqslant 2} \int_{|\eta_1| \leqslant 2} \int_{\mathbb{R}} \frac{d\zeta_1 d\tau_1}{\langle \sigma_1 \rangle^{1-2\delta} \langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle} \lesssim 1,\end{aligned}$$

while for $|\xi_1| \geqslant 2$ we get $\xi_1^4 - \xi_1^2 + \eta_1^2 \gtrsim 1$ and consequently

$$\begin{aligned}\mathscr{I}_3 &\lesssim \frac{\langle \xi^4 \rangle^{3/4+\varepsilon/2}}{\langle i\sigma + \xi^4 - \xi^2 + \eta^2 \rangle^{1-2\delta}} \int_{\Omega_3} \frac{\langle i\sigma_2 + \xi_2^4 - \xi_2^2 + \eta_2^2 \rangle^{-3/4-\varepsilon/2}}{\langle i\sigma_1 + \xi_1^4 - \xi_1^2 + \eta_1^2 \rangle^{3/4+\varepsilon/2}} d\zeta_1 d\tau_1 \\ &\lesssim \int_{\mathbb{R}^3} \frac{d\zeta_1 d\tau_1}{\langle \sigma_1 \rangle^{1-2\delta} \langle \sigma_1 \rangle^{3/4} \langle \xi_1^4 + \eta_1^2 \rangle^{3/4+\varepsilon/4}} \lesssim 1.\end{aligned}$$

Contribution of Ω_4 to I is followed by the same lines of the proof of Theorem 3.

The proof of case $s \geqslant 0$ is similar to Theorem 3 and we omit it. □

5. Proof of the local well-posedness

Now we are ready to prove our local well-posedness result.

Proof of Theorem 1. For $\varphi \in H^{s_1,s_2}(\mathbb{R}^2)$, with $s_1 > -1/2$, $s_1' \in (0, \min\{0, s_1\}]$, $s_2 \geqslant 0$ and $T \leqslant 1$, we define

$$\Gamma_T(u)(t) = \theta(t) \left(V(t)\varphi - \int_0^t V(t-t')(\theta_T f(u))(t') dt' \right), \tag{5.1}$$

where $f(u) = (u^2)_x + (u^2)_{xx}$. Our goal is to use the Picard fixed point theorem to find a solution $\Gamma_T(u) = u$.

We introduce the Bourgain spaces defined by

$$Z_1 = \left\{ u \in \mathscr{X}^{1/2,s_1,s_2};\ \|u\|_{Z_1} = \|u\|_{\mathscr{X}^{1/2,s_1,0}} + \ell_1 \|u\|_{\mathscr{X}^{1/2,s_1,s_2}} \right\}$$

and

$$Z_2 = \left\{ u \in \mathscr{X}^{1/2,s_1,0};\ \|u\|_{Z_2} = \|u\|_{\mathscr{X}^{1/2,s_1',0}} + \ell_2 \|u\|_{\mathscr{X}^{1/2,s_1,0}} \right\},$$

where

$$\ell_1 = \frac{\|\varphi\|_{H^{s_1,0}}}{\|\varphi\|_{H^{s_1,s_2}}}, \qquad \ell_2 = \frac{\|\varphi\|_{H^{s_1',0}}}{\|\varphi\|_{H^{s_1,0}}}.$$

The goal to introduce two Bourgain spaces is to show in a first time that there exists $T_1 = T(\|\varphi\|_{H^{s_1,0}})$ and a solution u of (5.1) in a ball of Z_1, and then to solve (5.1) in Z_2 in order to check that the time of existence $T = T(\|\varphi\|_{H^{s_1',0}})$ with $s_1' \in (0, \min\{0, s_1\}]$.

We note that by Theorems 3 and 4, it is readily seen that

$$\begin{aligned} \|(uv)_x + (uv)_{xx}\|_{\mathscr{X}^{-1/2+\delta,s_1,s_2}} \lesssim \|u\|_{\mathscr{X}^{1/2,s_1',0}}\|v\|_{\mathscr{X}^{1/2,s_1,s_2}} + \|u\|_{\mathscr{X}^{1/2,s_1,s_2}}\|v\|_{\mathscr{X}^{1/2,s_1',0}} \\ + \|u\|_{\mathscr{X}^{1/2,s_1,0}}\|v\|_{\mathscr{X}^{1/2,s_1',s_2}} + \|u\|_{\mathscr{X}^{1/2,s_1',s_2}}\|v\|_{\mathscr{X}^{1/2,s_1,0}}. \end{aligned} \tag{5.2}$$

A classical argument, similar to [22, 23, 24] and using Lemmas 1, 3 and 4 and inequality (5.2), we deduce the existence and the uniqueness of the solution for (5.1).

The continuity of map $\varphi \mapsto u$ from H^{s_1,s_2} to $\mathscr{X}^{1/2,s_1,s_2}$ follows from classical argument, while the continuity and regularity from H^{s_1,s_2} to $C([0,T], H^{s_1,s_2})$ follows again from Proposition 1 and the bilinear estimates. The analyticity of the flow-map is a direct consequence of the implicit function theorem. □

We also prove the global well-posedness for the initial value problem associated to equation (1.1), under suitable conditions.

Proof of Theorem 2. First one can observe that when $\gamma = 0$, Theorem 1 holds for the initial data $\varphi \in H^{s_1,s_2}$ with $s_1 > -3/2$ and $s_2 \geqslant 0$, by Theorem 3. Now we define $T^* = T^*(\|\varphi\|_{H^s(\mathbb{R}^2)})$ by

$$T^* = \sup\Big\{T > 0 \;:\; \exists! \text{ solution of (1.1)} \\ \text{belongs to } C([0,T]; H^{s_1,s_2}(\mathbb{R}^2)) \cap \mathscr{X}_T^{1/2,s_1,s_2} \text{ with initial data } \varphi\Big\}.$$

Since u is smooth, we deduce that u solves the Cauchy problem (1.1) with $u(0) = \varphi$, in a classical sense. This allows us to take the L^2 scalar product of (1.1) with u and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^2)} = -\|u_{xx}\|^2_{L^2(\mathbb{R}^2)} - \langle u_{xx}, u\rangle_{L^2(\mathbb{R}^2)} - \|u_y\|^2_{L^2(\mathbb{R}^2)} \leqslant \frac{1}{4}\|u\|^2_{L^2(\mathbb{R}^2)}, \tag{5.3}$$

where we used the Cauchy-Schwarz inequality. Thus by the Gronwall inequality, we deduce the following *a priori* estimate

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leqslant \|\varphi\|_{L^2(\mathbb{R}^2)} \mathrm{e}^{T^*/2} =: M, \qquad \forall t \in (0, T^*). \tag{5.4}$$

When $s = 0$, since the time existence T is a decreasing function of the norm of the initial data $\|\psi\|_{L^2(\mathbb{R}^2)}$ such that $T \to +\infty$ when $\|\psi\|_{L^2(\mathbb{R}^2)} \to 0$, we know that there exists a time $T_1 > 0$ such that for all $\psi \in L^2(\mathbb{R}^2)$, with $\|\psi\|_{L^2(\mathbb{R}^2)} \leqslant M$, there exists a unique solution v of (1.1) satisfying $v(0) = \psi$ and $v \in C([0,T_1];L^2(\mathbb{R}^2)) \cap \mathscr{X}^{1/2,0,0}$. Now, choose $0 < T < T_1$, apply this result with $\psi = u(T^* - \varepsilon)$ and define

$$\widetilde{u}(t) = \begin{cases} u(t), & t \in [0,T^* - \varepsilon], \\ v(t - T^* + \varepsilon), & t \in [T^* - \varepsilon, T^* - \varepsilon + T_1]. \end{cases} \tag{5.5}$$

Then, by the uniqueness result, $\widetilde{u}$ is a solution of (1.1) in the time interval $[0,T^* - \varepsilon + T_1]$; so that T^* cannot be finite.

When $-3/2 < s_1 < 0$, we can argue the same: the smoothness property implies that the solution u belongs to $C((0,T^*);L^2(\mathbb{R}^2))$ and $v \in \mathscr{X}^{1/2,0,0} \subset \mathscr{X}^{1/2,s_1,s_2}$, consequently we can apply the uniqueness result also in this case. When $s_1 > 0$, the result is a direct consequence of the case $s_1 = 0$ and the fact that the time existence only depends of $\|\varphi\|_{L^2(\mathbb{R}^2)}$. $\square$

Acknowledgements. The author would like to thank Luc Molinet, Jerry L. Bona and Felipe Linares for their valuable comments. The author also wishes to thank the unknown referee for his/her valuable suggestions which helped to improve the paper.

REFERENCES

[1] N. AKHMEDIEV, A. ANKIEWICZ, *Dissipative Soliton*, in: Lecture Notes in Physics **661**, Springer Verlag, Berlin, Heidelberg, 2005.

[2] A. B. AL'SHIN, M. O. KORPUSOV, A. G. SVESHNIKOV, *Blow-up in nonlinear Sobolev type equations*, De Gruyter Ser. Nonlinear Anal. Appl. **15**, Walter de Gruyter & Co., Berlin, 2011.

[3] J. ANGULO, *Nonlinear dispersive equations. Existence and stability of solitary and periodic travelling wave solutions*, Math. Surveys Monogr. **156**, Amer. Math. Soc., Providence, RI, 2009.

[4] H. BAHOURI, J.-Y. CHEMIN, R. DANCHIN, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften **243**, Springer, 2011.

[5] D. E. BAR, A. A. NEPOMNYASHCHY, *Stability of periodic waves generated by long-wavelength instabilities in isotropic and anisotropic systems*, Physica D, **132** (1999), 411–427.

[6] J. L. BONA, W. G. PRITCHARD, L. R. SCOTT, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London Series A, **302** (1981), 457–510.

[7] J. BOURGAIN, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. Schrödinger equations*, GAFA, **3** (1993), 107–156.

[8] J. BOURGAIN, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations II. The KdV equation*, GAFA, **3** (1993), 209–262.

[9] J. BOURGAIN, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA, **3** (1993), 315–341.

[10] J. BOURGAIN, W.-M. WANG, *in Mathematical Aspects of Nonlinear Dispersive Equations*, edited by J. Bourgain, C. E. Kenig, and S. Klainerman (Princeton University Press, Princeton, NJ, 2007.

[11] W. CHEN, J. LI, *On the low regularity of the modified Korteweg-de Vries equation with a dissipative term*, J. Differential Equations, **240** (2007), 125–144.

[12] B. I. COHEN, J. A. KROMMES, W .M. TANG, M. N. ROSENBLUTH, *Nonlinear saturation of the dissipative trapped-ion mode by mode coupling*, Nuclear Fusion, **16** (1976), 971–992.

[13] J. COLLIANDER, G. STAFFILANI, H. TAKAOKA, *Global well-posedness of the previous KdV below L^2*, Math. Res. Lett., **6** (1999), 755–778.

[14] D. B. DIX, *Large-Time Behavior of Solutions of Linear Dispersive Equations*, in: Lecture Notes in Mathematics **1668**, Springer-Verlag, Berlin, 1997.

[15] A. ESFAHANI, *The Cauchy problem for the dissipation-modified Kadomtsev-Petviashvili equation*, J. Math. Anal. Appl., **390** (2012), 439–455.

[16] J. HAMMACK, H. SEGUR, *The Kortweg-de Vries equation and water waves. II. Comparison with experiments*, J. Fluid Mech., **65** (1974), 289–313.

[17] J. HAN, L. PENG, *The well posedness of the dissipative Korteweg-de Vries equations with low regularity data*, Nonlinear Analysis, **69** (2008), 171–188.

[18] N. HAYASHI, E. I. KAIKINA, P. I. NAUMKIN, I. A. SHISHMAREV, *Asymptotics for Dissipative Nonlinear Equations*, Lecture Notes in Mathematics **1884**, Springer-Verlag, 2006.

[19] C. E. KENIG, G. PONCE, L. VEGA, *A bilinear estimate with applications to the KdV*, J. Amer. Math. Soc., **9** (1996), 573–603.

[20] C. E. KENIG, G. PONCE, L. VEGA, *Quadratic forms for the 1-D semilinear Schrödinger equation*, Trans. Amer. Math. Soc., **348** (1996), 3323–3353.

[21] F. LINARES, G. PONCE, *Introduction to nonlinear dispersive equations*, Springer, 2009.

[22] L. MOLINET, F. RIBAUD, *The Cauchy problem for dissipative Korteweg-de Vries equations in Sobolev spaces of negative order*, Indiana Univ. Math. J., **50** (2001), 1745–1776.

[23] L. MOLINET, F. RIBAUD, *The global Cauchy problem in Bourgain's type spaces for a dispersive dissipative semilinear equation*, SIAM J. Math. Anal., **33** (2002), 1269–1296.

[24] L. MOLINET, F. RIBAUD, *On the low regularity of the Korteweg-de Vries-Burgers equation*, Int. Math. Res. Not., **37** (2002), 1979–2005.

[25] M. OTANI, *Well-posedness of the generalized Benjamin-Ono-Burgers equations in Sobolev spaces of negative order*, Osaka J. Math., **43** (2006), 935–965.

[26] P. L. SACHDEV, C. S. RAO, *Large time asymptotics for solutions of nonlinear partial differential equations*, Springer Monographs in Mathematics, Springer, New York, 2010.

[27] T. TAO, *Nonlinear Dispersive Equations: Local and Global Analysis*, in: CBMS Regional Conference Series in Mathematics, 2006.

[28] J. TOPPER, T. KAWAHARA, *Approximate equations for long nonlinear waves on a viscous fluid*, J. Phys. Soc. Japan, **44** (1978), 663–666.

[29] S. K. TURITSYN, J. J. RASMUSSEN, M. A. RAADU, *Stability of weak double layers*, Royal Institute of Technology, Stockholm, TRITA-EPP-91 -01, 1991.

[30] N. J. ZABUSKY, C. GALVIN, *Shallow-water waves, the Korteweg-de Vries equation and solitons*, J. Fluid Mech., **47** (1971), 811–824.

[31] E. ZAUDERER, *Partial Differential Equations of Applied Mathematics*, 33rd ed., John Wiley & Sons, Inc., New York, 2006.

(Received September 4, 2011)
(Revised April 15, 2012)

Amin Esfahani
School of Mathematics and Computer Science
Damghan University
Damghan, P.O. Box 36715-364
Iran
e-mail: amin@impa.br, esfahani@du.ac.ir