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# THE ANTIMAXIMUM PRINCIPLE AND THE EXISTENCE OF A SOLUTION FOR THE GENERALIZED $p$–LAPLACE EQUATIONS WITH INDEFINITE WEIGHT

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*Abstract.* This paper treats the antimaximum principle and the existence of a solution for quasilinear elliptic equation $-\text{div}\,(a(x,|\nabla u|)\nabla u) = \lambda m(x)|u|^{p-2}u + h(x)$ in $\Omega$ under the Neumann boundary condition. Here, a map $a(x,|y|)y$ on $\overline{\Omega}\times\mathbb{R}^N$ is strictly monotone in the second variable and satisfies certain regularity conditions. This equation contains the $p$-Laplacian problem as a special case.

## 1. Introduction

In this paper, we study the antimaximum principle (AMP) and consider the existence of a solution for the following quasilinear elliptic equation:

$$(P;\lambda,m,h) \qquad \begin{cases} -\text{div}\,(a(x,|\nabla u|)\nabla u) = \lambda m(x)|u|^{p-2}u + h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial\Omega$, $\nu$ denotes the outward unit normal vector on $\partial\Omega$, $a$ is a positive function on $\overline{\Omega}\times(0,+\infty)$, $\lambda\in\mathbb{R}$, $1<p<\infty$, $m\in L^\infty(\Omega)$ and $h\in L^\infty(\Omega)_+$. Here, we set a map $A(x,y) := a(x,|y|)y$ for $(x,y)\in\overline{\Omega}\times\mathbb{R}^N$ and, then $A$ is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption $(A)$). The equation $(P;\lambda,m,h)$ contains the corresponding $p$-Laplacian problem as a special case. However, in general, we do not suppose that the operator $A$ is $(p-1)$-homogeneous in the second variable.

*Throughout this paper, we assume that*

$$|\{m>0\}| := |\{x\in\Omega;\, m(x)>0\}| > 0 \tag{1.1}$$

where $|X|$ denotes the Lebesgue measure of a measurable set $X$. In this paper, we deal with the following four cases concerning the weight function $m\in L^\infty(\Omega)$ under (1.1):

(i) $m\not\equiv 0$ and $m(x)\geqslant 0$ for a.e. $x\in\Omega$; (ii) $\int_\Omega m\,dx>0$ and $|m<0|>0$;
(iii) $\int_\Omega m\,dx = 0$; (iv) $\int_\Omega m\,dx<0$.

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Here, we say that $u \in W^{1,p}(\Omega)$ is a (weak) solution of $(P;\lambda,m,h)$ if

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx = \lambda\int_\Omega m|u|^{p-2}u\varphi\,dx + \int_\Omega h\varphi\,dx$$

for all $\varphi \in W^{1,p}(\Omega)$.

In [8], the study of AMP started by Clement and Peletier. They proved that there exists $\delta > 0$ for every $\lambda \in (\lambda_1, \lambda_1+\delta)$ such that any solution is negative in $\Omega$ for $-\Delta u = \lambda u + h$ in $\Omega$ under the Dirichlet or Neumann boundary condition, where $\lambda_1$ denotes the first eigenvalue of $-\Delta$. This situation is called as that "AMP holds at right of $\lambda_1$". Although the above $\delta$ depends on $h$ in general, they presented also the existence of such $\delta$ independent of $h$ in the case of $N=1$ under the Neumann boundary condition. When we can take $\delta$ independent of $h$, we say that "AMP holds uniformly at right of $\lambda_1$". The AMP was extended in [16] to the case having the (indefinite) weight. Moreover, many authors have studied the AMP for the Laplace equation and other equations (cf. [2], [3], [5], [6], [10], [11], [22]). In the case of the $p$-Laplacian, Godoy et al ([13] and [14]) presented the several results concerning AMP for $-\Delta_p u = \lambda m|u|^{p-2}u + h$ in $\Omega$ under the Dirichlet and Neumann boundary conditions. First purpose of this paper is to prove similar results to one of [14] for the generalized $p$-Laplace equation $(P;\lambda,m,h)$.

On the other hand, it is obvious that the AMP has no effect if a solution does not exist. However, there are few existence results of a solution to our equation (and also the $p$-Laplace equation). For example, if $\lambda < 0$ and $m \equiv 1$ holds, then the standard argument guarantees the existence of a solution. In [14], it is shown that the equation $-\Delta_p u = m|u|^{p-2}u + h$ in $\Omega$ has a unique positive solution provided $0 < \lambda < \lambda^*(m)$, $\int_\Omega m\,dx < 0$ and $0 \not\equiv h \in L^\infty(\Omega)_+$, where $\lambda^*(m)$ is the principal eigenvalue defined in Section 2.1. To the Laplace problems under the Dirichlet boundary condition, the existence results are well known (cf. [1]).

Therefore, second purpose is to show that $(P;\lambda,m,h)$ has at least one solution under some condition to $\lambda$ by variational methods. In particular, in the case where $A$ is asymptotically $(p-1)$ homogeneous (see the condition $(AH)$ in Section 4.3), $(P;\lambda,m,h)$ has at least one solution if $\lambda$ exists between the principal eigenvalue and the second eigenvalue (Theorem 6 and see Remark 7).

*Throughout this paper, we assume that the map $A$ satisfies the following assumption $(A)$:*

$(A)$ $A(x,y) = a(x,|y|)y$, where $a(x,t) > 0$ for all $(x,t) \in \overline{\Omega}\times(0,+\infty)$ and

(i) $A \in C^0(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)\cap C^1(\overline{\Omega}\times(\mathbb{R}^N\setminus\{0\}),\mathbb{R}^N)$;

(ii) there exists $C_1 > 0$ such that

$$|D_yA(x,y)| \leqslant C_1|y|^{p-2} \quad \text{for every } x\in\overline{\Omega}, \text{ and } y\in\mathbb{R}^N\setminus\{0\};$$

(iii) there exists $C_0 > 0$ such that

$$D_yA(x,y)\xi\cdot\xi \geqslant C_0|y|^{p-2}|\xi|^2 \quad \text{for every } x\in\overline{\Omega},\ y\in\mathbb{R}^N\setminus\{0\} \text{ and } \xi\in\mathbb{R}^N;$$

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x,y)| \leqslant C_2(1+|y|^{p-1}) \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\};$$

(v) there exist $C_3 > 0$ and $1 \geqslant t_0 > 0$ such that

$$|D_x A(x,y)| \leqslant C_3 |y|^{p-1} \, (-\log|y|\,)$$

for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

*Throughout this paper, we assume* $C_0 \leqslant p-1 \leqslant C_1$ because we can take such desired $C_0$ and $C_1$ anew if necessary.

A similar hypothesis to $(A)$ is considered in the study of quasilinear elliptic problems (cf. [21, Example 2.2.] and [9], [20], [19]). It is easily seen that many examples as in the above references satisfy the condition $(AH)$. In particular, for $A(x,y) = |y|^{p-2}y$, that is, $\mathrm{div}A(x,\nabla u)$ stands for the usual $p$-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p-1$ in $(A)$. Conversely, in the case where $C_0 = C_1 = p-1$ holds in $(A)$, by the inequalities in Remark 1 (ii) and (iii) in Section 2, we see $a(x,t) = |t|^{p-2}$ whence $A(x,y) = |y|^{p-2}y$.

In section 2.1, we recall several results concerning the weighted eigenvalue problems for the $p$-Laplacian. Then, in Section 3, we show that the AMP holds at some $\lambda$ for our equation. Finally, we present the existence results to our equation (in Section 4).

## 2. Preliminaries

In what follows, the norm on $W^{1,p}(\Omega)$ is $\|u\|^p := \|\nabla u\|_p^p + \|u\|_p^p$, where $\|u\|_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \leqslant q \leqslant \infty)$. Setting

$$G(x,y) := \int_0^{|y|} a(x,t)t\,dt,$$

then we can easily see that

$$\nabla_y G(x,y) = A(x,y) \quad \text{and} \quad G(x,0) = 0 \tag{2.1}$$

for every $x \in \overline{\Omega}$.

REMARK 1. It is easily seen that the following assertions hold under condition $(A)$:

(i) for all $x \in \overline{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in $y$;

(ii) $|A(x,y)| \leqslant \frac{C_1}{p-1}|y|^{p-1}$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;

(iii) $A(x,y)y \geqslant \frac{C_0}{p-1}|y|^p$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;

(iv) $G(x,y)$ is convex in $y$ for all $x$ and satisfies the following inequalities:

$$A(x,y)y \geqslant G(x,y) \geqslant \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x,y) \leqslant \frac{C_1}{p(p-1)}|y|^p \tag{2.2}$$

for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$,

where $C_0$ and $C_1$ are the positive constants in $(A)$.

REMARK 2. Let $m \in L^\infty(\Omega)$ and $h \in L^\infty(\Omega)_+$. Then, we remark the following:

(i) If $u \in W^{1,p}(\Omega)$ is a solution of $(P;\lambda,m,h)$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0<\alpha<1$ and $\partial u/\partial\nu = 0$ on $\partial\Omega$;

(ii) If $u \in W^{1,p}(\Omega)$ is a non-trivial solution of $(P;\lambda,m,h)$ such that $u \geqslant 0$, then $\min_{\overline{\Omega}} u > 0$ holds;

*Proof.* For readers' convenience, we give a sketch of the proof. (i): Let $u \in W^{1,p}(\Omega)$ be a solution of $(P;\lambda,m,h)$. Then, because $u \in L^\infty(\Omega)$ by the Moser iteration process (cf. Appendix in [19]), we see that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0<\alpha<1)$ by the regularity result in [17]. Furthermore, by [7, Theorem 3], $u$ satisfies the boundary condition

$$0 = \frac{\partial u}{\partial \nu_A} = A(\cdot,\nabla u)\,\nu = a(\cdot,|\nabla u|)\frac{\partial u}{\partial \nu} \quad \text{in } W^{-1/q,q}(\partial\Omega)$$

for every $1<q<\infty$ (see [7] for the definition of $W^{-1/q,q}(\partial\Omega)$). Since $u \in C^{1,\alpha}(\overline{\Omega})$ and $a(x,t)>0$ for every $t \neq 0$, $u$ satisfies the Neumann boundary condition, that is, $\frac{\partial u}{\partial \nu}(x) = 0$ for every $x \in \partial\Omega$.

(ii): Let $u \in W^{1,p}(\Omega)$ be a solution of $(P;\lambda,m,h)$ satisfying $u \geqslant 0$ and $u \not\equiv 0$. Then, we have

$$-\mathrm{div}A(x,\nabla u) + |\lambda|\|m\|_\infty u^{p-1} \geqslant h \geqslant 0 \quad \text{in } \Omega.$$

By noting that $u \in C^{1,\alpha}(\overline{\Omega})$ $(0<\alpha<1)$ by (i), we have $u(x)>0$ for every $x \in \Omega$ by Theorem B in [19, Appendix]. Due to the strong maximum principle (see Theorem A in [19, Appendix]), we easily see that $u(x)>0$ for every $x \in \partial\Omega$ (note $\partial u/\partial\nu = 0$ on $\partial\Omega$ by (i)). This yields $\min_{\overline{\Omega}} u > 0$ because of $u \in C^{1,\alpha}(\overline{\Omega})$ by (i).

### 2.1. The weighted eigenvalue problems for the $p$-Laplacian

The following lemmas can be easily shown by way of contradiction. Here, we omit the proofs.

LEMMA 1. ([14, Lemma 2.3.]) *Assume $\int_\Omega m\,dx < 0$. Then, there exists a constant $c>0$ such that $\int_\Omega |\nabla u|^p\,dx \geqslant c\|u\|_p^p$ for every $u \in W^{1,p}(\Omega)$ with $\int_\Omega m|u|^p\,dx > 0$.*

LEMMA 2. ([14, Lemma 2.8.]) *Assume that* $\int_\Omega m\,dx \neq 0$ *and* $\xi > 0$. *Then, there exists a constant* $b(m,\xi) > 0$ *such that*

$$\int_\Omega |\nabla u|^p\,dx - \xi \int_\Omega m|u|^p\,dx \geqslant b(m,\xi)\int_\Omega |u|^p\,dx$$

*for every* $u \in B(m) := \{u \in W^{1,p}(\Omega)\,;\, \int_\Omega m|u|^p\,dx \leqslant 0\}$.

LEMMA 3. *Assume that* $m \geqslant 0$ *in* $\Omega$. *Then, for every* $\xi > 0$ *there exists* $d(\xi) > 0$ *such that*

$$\int_\Omega |\nabla u|^p\,dx + \xi \int_\Omega m|u|^p\,dx \geqslant d(\xi)\int_\Omega |u|^p\,dx$$

*for every* $u \in W^{1,p}(\Omega)$.

LEMMA 4. *Let* $N < p$, $\lambda > 0$ *and* $\Lambda \subset \mathbb{R}$ *be a compact set. Define*

$$\tilde{B}(m) := \left\{ u \in W^{1,p}(\Omega)\,;\, \int_\Omega m|u|^p\,dx \geqslant 0 \text{ and } u \text{ vanishes somewhere in } \overline{\Omega} \right\}.$$

*Then, there exists a* $C > 0$ *such that for every* $\varepsilon \in \Lambda$, *the following inequality holds:*

$$\int_\Omega |\nabla u|^p\,dx + \lambda \int_\Omega (m+\varepsilon)|u|^p\,dx \geqslant C\|u\|_p^p \quad \text{for every } u \in \tilde{B}(m+\varepsilon).$$

*Proof.* By way of contradiction, we suppose that there exist $\{\varepsilon_n\} \subset \Lambda$ and $u_n \in \tilde{B}(m+\varepsilon_n)$ such that

$$\int_\Omega |\nabla u_n|^p\,dx + \lambda \int_\Omega (m+\varepsilon_n)|u_n|^p\,dx < \frac{1}{n}\|u_n\|_p^p.$$

Set $v_n := u_n/\|u_n\|_p$. Then, because we have

$$\int_\Omega |\nabla v_n|^p\,dx \leqslant \int_\Omega |\nabla v_n|^p\,dx + \lambda \int_\Omega (m+\varepsilon_n)|v_n|^p\,dx < \frac{1}{n} \tag{2.3}$$

by $v_n \in \tilde{B}(m+\varepsilon_n)$, we may assume that $v_n$ weakly converges to some $v_0$ in $W^{1,p}(\Omega)$ and $v_n(x)$ converges to $v_0(x)$ uniformly in $x \in \overline{\Omega}$ (note $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact). Moreover, it is easily seen that $v_0$ vanishes somewhere in $\overline{\Omega}$ because $v_n$ vanishes somewhere in $\overline{\Omega}$ by $u_n \in \tilde{B}(m+\varepsilon_n)$. Since $\Lambda$ is compact, we may assume that $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$ for some $\varepsilon_0 \in \Lambda$ by choosing a subsequence. Consequently, $v_0 \in \tilde{B}(m+\varepsilon_0)$ holds. On the other hand, by taking the limit inferior in (2.3), we have $\int_\Omega |\nabla v_0|^p\,dx \leqslant 0$. This implies that $v_0$ is a constant function such that $\|v_0\|_p = 1$. This contradicts to the fact that $v_0$ vanishes somewhere in $\overline{\Omega}$.

Now, we state several known results relative to the following weighted eigenvalue problems for the $p$-Laplacian:

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of (2.4) if the equation (2.4) has a non-trivial solution. First, we recall the following principal eigenvalue $\lambda^*(m)$ which plays an important role for studying AMP.

$$\lambda^*(m) := \inf\left\{ \int_\Omega |\nabla u|^p\, dx\,;\, u \in W^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p\, dx = 1 \right\}. \tag{2.5}$$

PROPOSITION 1. ([14, Proposition 2.2.]) *The following assertions hold;*

(i) *If* $\int_\Omega m\, dx \geqslant 0$ *holds, then* $\lambda^*(m) = 0$*;*

(ii) *If* $\int_\Omega m\, dx < 0$ *holds, then* $\lambda^*(m) > 0$ *is a simple eigenvalue and it admits a positive eigenfunction. In addition, the interval* $(0, \lambda^*(m))$ *contains no eigenvalues of* (2.4).

Moreover, we recall a second value $\overline{\lambda}(m)$ defined by

$$\overline{\lambda}(m) := \inf\left\{ \begin{array}{l} \int_\Omega |\nabla u|^p\, dx;\ u \in W^{1,p}(\Omega),\ \int_\Omega m|u|^p\, dx = 1 \\ \qquad\qquad\qquad \text{and } u \text{ vanishes on some ball in } \Omega \end{array} \right\}. \tag{2.6}$$

In the case of $N < p$ (note that $W^{1,p}(\Omega)$ is compactly imbedded into $C(\overline{\Omega})$), we introduce $\tilde{\lambda}(m)$ as follows:

$$\tilde{\lambda}(m) := \inf\left\{ \begin{array}{l} \int_\Omega |\nabla u|^p\, dx;\ u \in W^{1,p}(\Omega),\ \int_\Omega m|u|^p\, dx = 1 \\ \qquad\qquad\qquad \text{and } u \text{ vanishes somewhere in } \overline{\Omega} \end{array} \right\}. \tag{2.7}$$

It is easily shown that $\overline{\lambda}(m) = \tilde{\lambda}(m)$ (see section 3 in [14]).

Concerning $\lambda^*(m)$ and $\overline{\lambda}(m)$, the following result is shown in [14].

LEMMA 5. ([14, Lemma 3.1.]) *If* $p \leqslant N$, *then* $\lambda^*(m) = \overline{\lambda}(m)$. *If* $p > N$, *then* $\lambda^*(m) < \overline{\lambda}(m)$. *Moreover, if* $p > N$, *then* $(\lambda^*(m), \overline{\lambda}(m)]$ *has no eigenvalues of* (2.4).

To prove lemma above, we need the following lemma proved by the same argument as in [13, Claim 4.1] or [5, Lemma 3.1.] (Note that Lemma 4 guarantees the boundedness of a minimizing sequence of $\tilde{\lambda}(m)$). Here, we omit the proof.

LEMMA 6. *Assume that* $p > N$. *Then,* $\tilde{\lambda}(m)$ *is attained. Furthermore, a minimizer for* $\tilde{\lambda}(m)$ *vanishes at exactly one point in* $\overline{\Omega}$.

LEMMA 7. *Let* $N < p$. *Then,* $\tilde{\lambda}(m+\varepsilon') < \tilde{\lambda}(m+\varepsilon) < \tilde{\lambda}(m)$ *for every* $\varepsilon' > \varepsilon > 0$ *holds. Moreover,* $\lim_{\varepsilon \to +0} \tilde{\lambda}(m+\varepsilon) = \tilde{\lambda}(m)$.

*Proof.* We choose a minimizer $u$ for $\tilde{\lambda}(m)$ because Lemma 6 guarantees the existence of it. Then, for every $\varepsilon > 0$, we have

$$\tilde{\lambda}(m+\varepsilon) \leqslant \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega (m+\varepsilon)|u|^p\, dx} < \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega m|u|^p\, dx} = \int_\Omega |\nabla u|^p\, dx = \tilde{\lambda}(m)$$

by the definition of $\tilde{\lambda}(m+\varepsilon)$. By applying the same argument to a minimizer for $\tilde{\lambda}(m+\varepsilon)$, we obtain $\tilde{\lambda}(m+\varepsilon') < \tilde{\lambda}(m+\varepsilon)$ for $\varepsilon' > \varepsilon > 0$.

Now, we shall prove

$$\lim_{\varepsilon\to+0} \tilde{\lambda}(m+\varepsilon) = \tilde{\lambda}(m).$$

Let $\{\varepsilon_n\}$ be any sequence such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n\to\infty$. Because we know $\limsup_{n\to\infty}\tilde{\lambda}(m+\varepsilon_n) \leqslant \tilde{\lambda}(m)$ by the first assertion, it suffices only to prove $\liminf_{n\to\infty}\tilde{\lambda}(m+\varepsilon_n) \geqslant \tilde{\lambda}(m)$. We take a minimizer $u_n$ for $\tilde{\lambda}(m+\varepsilon_n)$. Then, it follows from Lemma 4 with $\lambda = 1$ and $\Lambda = [0,\sup_n \varepsilon_n]$ that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$ (note that $\|\nabla u_n\|_p^p = \tilde{\lambda}(m+\varepsilon_n) < \tilde{\lambda}(m)$). Thus, we may assume, by choosing a subsequence, that there exists $u_0 \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ and $u_n \to u_0$ in $C(\overline{\Omega})$. Then, we see that $\int_\Omega m|u_0|^p\,dx = 1$ and $u_0$ vanishes somewhere in $\overline{\Omega}$ since $u_n(x)$ converges to $u_0(x)$ uniformly in $x\in\overline{\Omega}$. Hence, by the definition of $\tilde{\lambda}(m)$, we obtain

$$\liminf_{n\to\infty}\tilde{\lambda}(m+\varepsilon_n) = \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^p\,dx \geqslant \int_\Omega |\nabla u_0|^p\,dx \geqslant \tilde{\lambda}(m),$$

whence our claim is shown. Because $\{\varepsilon_n\}$ is an arbitrary sequence, our conclusion is proved.

Finally, we recall the second eigenvalue of (2.4). The following result is shown in [4] (Although they handle the asymmetry case, it is sufficient to consider the case of $m\equiv n$ in this paper).

$$J(u) := \int_\Omega |\nabla u|^p\,dx \quad \text{for } u\in W^{1,p}(\Omega), \quad \widetilde{J} := J|_{S(m)} \tag{2.8}$$

$$S(m) := \left\{u\in W^{1,p}(\Omega);\ \int_\Omega m|u|^p\,dx = 1\right\}, \tag{2.9}$$

$$\Sigma(m) := \{\gamma\in C([0,1],S(m));\ \gamma(0)\in P\cap S(m),\ \gamma(1)\in(-P)\cap S(m)\}, \tag{2.10}$$

$$c(m) := \inf_{\gamma\in\Sigma(m)}\max_{t\in[0,1]}\widetilde{J}(\gamma(t)), \tag{2.11}$$

where $P := \{u\in W^{1,p}(\Omega);\ u(x)\geqslant 0 \text{ for a.e. } x\in\Omega\}$.

LEMMA 8. ([4, Theorem 3.2.]) *We have: $c(m)$ is an eigenvalue of* (2.4) *which satisfies $\lambda^*(m) < c(m)$. Moreover, there is no eigenvalues of* (2.4) *between $\lambda^*(m)$ and $c(m)$.*

REMARK 3. we remark that $\tilde{\lambda}(m) < c(m)$ (if $N<p$). Indeed, if $\tilde{\lambda}(m)\geqslant c(m)$ under $N<p$, then it contradicts to the fact that $(\lambda^*(m),\tilde{\lambda}(m)]$ (note $\tilde{\lambda}(m) = \overline{\lambda}(m)$) contains no eigenvalues of (2.4) by Lemma 5 since $c(m)$ is an eigenvalue of (2.4) and $\lambda^*(m) < c(m)$.

LEMMA 9. *For every $\varepsilon' > \varepsilon > 0$, we have $c(m+\varepsilon') < c(m+\varepsilon) < c(m)$. In addition, $\lim_{\varepsilon\to+0} c(m+\varepsilon) = c(m)$ holds.*

*Proof.* First, we shall prove that $c(m+\varepsilon) < c(m)$ for $\varepsilon > 0$. Because $c(m)$ is an eigenvalue, we can choose a solution $u \in W^{1,p}(\Omega)$ with $\|u\| = 1$ for $-\Delta_p u = c(m)m|u|^{p-2}u$ in $\Omega$, $\partial u/\partial \nu = 0$ on $\partial\Omega$. Then, we note that $u$ is a sign-changing function because any eigenfunction corresponding to $\lambda$ other than the principal eigenvalue changes sign (refer to [15, Proposition 4.3.] or see Proposition 2 with $C_0 = C_1 = p-1$ and $h \equiv 0$). Thus, we have $0 < \|\nabla u_\pm\|_p^p = c(m)\int_\Omega m u_\pm^p\, dx$ by taking $\pm u_\pm$ as test function. Set a continuous path $\gamma_0$ by

$$\gamma_0(t) := \frac{(1-t)u_+ - tu_-}{\left((1-t)^p \int_\Omega m u_+^p\, dx + t^p \int_\Omega m u_-^p\, dx\right)^{1/p}} \quad \text{for } t \in [0,1].$$

Then, it is easily seen that $\gamma_0 \in \Sigma(m)$ and

$$c(m) = \|\nabla \gamma_0(t)\|_p^p > \frac{\|\nabla \gamma_0(t)\|_p^p}{1+\varepsilon\|\gamma_0(t)\|_p^p} = \frac{\|\nabla \gamma_0(t)\|_p^p}{\int_\Omega (m+\varepsilon)|\gamma_0(t)|^p\, dx}$$

for every $t \in [0,1]$ (note $\|\gamma_0(t)\|_p > 0$). By setting

$$\gamma_\varepsilon := \frac{\gamma_0}{\left(\int_\Omega (m+\varepsilon)|\gamma_0|^p\, dx\right)^{1/p}} \in \Sigma(m+\varepsilon),$$

we obtain $c(m) > c(m+\varepsilon)$ by the definition of $c(m+\varepsilon)$. By considering $m+\varepsilon$ and $(m+\varepsilon)+(\varepsilon'-\varepsilon)$, we obtain $c(m+\varepsilon) > c(m+\varepsilon')$ for $\varepsilon' > \varepsilon$ due to the above assertion.

Let $\{\varepsilon_n\}$ be any sequence such that $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. By the first assertion, we can get $\limsup_{n\to\infty} c(m+\varepsilon_n) \leqslant c(m)$. Now, we shall prove

$$\lambda_0 := \liminf_{n\to\infty} c(m+\varepsilon_n) \geqslant c(m).$$

We may put $\lambda_0 = \lim_{n\to\infty} c(m+\varepsilon_n) \geqslant 0$ by choosing a subsequence (note $c(m+\varepsilon_n) > \lambda^*(m+\varepsilon_n) \geqslant 0$). By the same reason as in the first part, there exists a sign-changing solution $u_n \in W^{1,p}(\Omega)$ with $\|u_n\| = 1$ for $-\Delta_p u_n = c(m+\varepsilon_n)(m+\varepsilon_n)|u_n|^{p-2}u_n$ in $\Omega$, $\partial u_n/\partial \nu = 0$ on $\partial\Omega$. In addition, by the standard argument (refer to Proposition 3) and the boundedness of $\|u_n\|$, we may assume that $u_n$ converges to some $u_0$ in $C^1(\overline{\Omega})$ by choosing a subsequence. Hence, $u_0$ is a solution with $\|u_0\| = 1$ of $-\Delta_p u = \lambda_0 m|u|^{p-2}u$ in $\Omega$, $\partial u/\partial \nu = 0$ on $\partial\Omega$. This means that $u_0$ is an eigenfunction corresponding to $\lambda_0$ with weight $m$. Because $u_n$ changes sign and $u_n \to u_0$ in $C^1(\overline{\Omega})$, $u_0$ vanishes somewhere in $\overline{\Omega}$. Hence, we can see that $\lambda_0 \neq 0$ and $\lambda_0 \neq \lambda^*(m)$ because any eigenfunction corresponding to the principal eigenvalue (that is, $0$ or $\lambda^*(m)$) is positive or negative in $\overline{\Omega}$ (see [15, Proposition 4.2.]). This implies that $\lambda_0 \geqslant c(m)$ by Lemma 8, and so $\liminf_{n\to\infty} c(m+\varepsilon_n) \geqslant c(m)$. As a result, our conclusion is shown since $\{\varepsilon_n\}$ is an arbitrary sequence.

LEMMA 10. *Assume* $\int_\Omega m\, dx > 0$. *Then,*

$$X(m) := \left\{u \in W^{1,p}(\Omega)\, ;\, \int_\Omega m|u|^{p-2}u\, dx = 0 \text{ and } \int_\Omega m|u|^p\, dx = 1\right\} \neq \emptyset \qquad (2.12)$$

*and*

$$c(m) \geqslant \lambda_{X(m)} := \inf_{u \in X(m)} \int_\Omega |\nabla u|^p \, dx > 0 \tag{2.13}$$

*hold, where* $c(m)$ *is the second eigenvalue defined by* (2.11).

*Proof.* Since $c(m) > 0$ is an eigenvalue of (2.4), there exists a non-trivial solution $u$ for $-\Delta_p u = c(m) m |u|^{p-2} u$ in $\Omega$, $\partial u / \partial \nu = 0$ on $\partial\Omega$. Then, by taking $\varphi \equiv 1$ or $u$ as test function, we have $\int_\Omega m|u|^{p-2} u \, dx = 0$ (note $c(m) > 0$) and $\int_\Omega m|u|^p \, dx > 0$ because $u$ changes sign (so $\|\nabla u\|_p > 0$, cf. [14, Proposition 2.4.] or Proposition 2 with $C_0 = C_1 = p-1$), and hence $u / (\int_\Omega m|u|^p \, dx)^{1/p}$ belongs to $X(m)$. As a result, it is easily seen that $c(m) \geqslant \lambda_{X(m)}$ holds.

Now, we shall prove $\lambda_{X(m)} > 0$ by way of contradiction. So, we assume that there exists a $\{u_n\} \subset X(m)$ such that $\|\nabla u_n\|_p \to 0$ as $n \to \infty$. Then, since Lemma 2 (with $\int_\Omega (-m)|u_n|^p \, dx = -1$) guarantees the boundedness of $\|u_n\|$, by choosing a subsequence if necessary, we may suppose that $u_n$ strongly converges to some constant function $u_0$ in $W^{1,p}(\Omega)$ (note $\|\nabla u_0\|_p = 0$). Hence, $|u_0| = 1/(\int_\Omega m \, dx)^{1/p}$ holds because of $\int_\Omega m|u_0|^p \, dx = 1$. On the other hand, by taking the limit in the equality

$$0 = \int_\Omega m|u_n|^{p-2} u_n \, dx = u_0 \int_\Omega m|u_n|^{p-2} u_n \, dx = \int_\Omega m|u_n|^{p-2} u_n u_0 \, dx,$$

we obtain $\int_\Omega m|u_0|^p \, dx = 0$. This is a contradiction.

REMARK 4. In the case of $p \geqslant 2$ and $m \equiv 1$, it is proved that $c(m) = \lambda_{X(m)}$ holds (see [12, Theorem 6.2.29]).

## 2.2. Elementary results

Here, we define a positive constant $A_p$ by

$$A_p := \frac{C_1}{p-1} \left( \frac{C_1}{C_0} \right)^{p-1} \geqslant 1, \tag{2.14}$$

which is equal to 1 in the case of $A(x,y) = |y|^{p-2} y$ (i.e. the special case of the $p$-Laplacian ) because we can choose $C_0 = C_1 = p-1$.

LEMMA 11. *Let* $\varepsilon > 0$. *For every* $u$, $\varphi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)$ *with* $u \geqslant 0$ *and* $\varphi \geqslant 0$ *in* $\Omega$, *we have*

$$\int_\Omega A(x, \nabla u) \nabla \left( \frac{\varphi^p}{(u+\varepsilon)^{p-1}} \right) dx \leqslant A_p \|\nabla \varphi\|_p^p.$$

*Proof.* Let $\varepsilon > 0$ and let $u$, $\varphi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)$ satisfy $u \geqslant 0$ and $\varphi \geqslant 0$ in $\Omega$. Then, we have

$$A(x, \nabla u) \nabla \left( \frac{\varphi^p}{(u+\varepsilon)^{p-1}} \right)$$

$$
\begin{aligned}
&= p\left(\frac{\varphi}{u+\varepsilon}\right)^{p-1} A(x,\nabla u)\nabla\varphi - (p-1)\left(\frac{\varphi}{u+\varepsilon}\right)^{p} A(x,\nabla u)\nabla u \\
&\leqslant \frac{pC_1}{p-1}\left(\frac{\varphi}{u+\varepsilon}\right)^{p-1} |\nabla u|^{p-1}|\nabla\varphi| - C_0\left(\frac{\varphi}{u+\varepsilon}\right)^{p} |\nabla u|^{p} \\
&= \left\{\left(\frac{pC_0}{p-1}\right)^{1/p} \frac{\varphi}{u+\varepsilon}|\nabla u|\right\}^{p-1} \left(\frac{p}{p-1}\right)^{1/p} C_1 C_0^{(1-p)/p}|\nabla\varphi| \\
&\qquad - C_0\left(\frac{\varphi}{u+\varepsilon}\right)^{p}|\nabla u|^{p} \\
&\leqslant A_p|\nabla\varphi|^{p} \quad \text{in } \Omega
\end{aligned}
$$

by (ii) and (iii) in Remark 1 and Young's inequality.

PROPOSITION 2. *Let $h \in L^\infty(\Omega)_+$. If one of the following holds, then the equation $(P;\lambda,m,h)$ has no solutions $u \not\equiv 0$ such that $u(x) \geqslant 0$ for a.e. $x \in \Omega$:*

(i) *$m \geqslant 0$ in $\Omega$ and $\lambda > 0$ if $h \not\equiv 0$ or $\lambda \neq 0$ if $h \equiv 0$;*

(ii) *$\int_\Omega m\,dx > 0$, $|m<0| > 0$ and $\lambda \notin [-A_p\lambda^*(-m), 0]$;*

(iii) *$\int_\Omega m\,dx = 0$ and $\lambda \neq 0$ if $h \equiv 0$ or $\lambda \in \mathbb{R}$ if $h \not\equiv 0$;*

(iv) *$\int_\Omega m\,dx < 0$ and $\lambda \notin [0, A_p\lambda^*(m)]$.*

*Proof.* Let $u$ be a non negative solution of $(P;\lambda,m,h)$ with $u \not\equiv 0$. Then, $u \in C^{1,\alpha}(\overline{\Omega})$ (some $0<\alpha<1$) and $\min_{\overline{\Omega}} u > 0$ by Remark 2.

(i): By taking $\varphi \equiv 1$ as test function, we have

$$0 = \lambda \int_\Omega m u^{p-1}\,dx + \int_\Omega h\,dx.$$

In the case of $h \not\equiv 0$, this yields $\lambda < 0$ because of

$$\lambda \int_\Omega m u^{p-1}\,dx = -\int_\Omega h\,dx < 0 \quad \text{and} \quad \int_\Omega m u^{p-1}\,dx > 0.$$

In the case of $h \equiv 0$, then we see that $\lambda = 0$ occurs.

(ii) $\sim$ (iv): By Lemma 11, we obtain

$$
\begin{aligned}
A_p\|\nabla\varphi\|_p^p &\geqslant \int_\Omega A(x,\nabla u)\nabla\left(\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\right)dx \\
&= \lambda\int_\Omega m\left(\frac{u}{u+\varepsilon}\right)^{p-1}\varphi^p\,dx + \int_\Omega h\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\,dx \\
&\geqslant \lambda\int_\Omega m\left(\frac{u}{u+\varepsilon}\right)^{p-1}\varphi^p\,dx
\end{aligned}
$$

for every $\varepsilon > 0$ and $\varphi \in W^{1,p}(\Omega)\cap C^1(\Omega)\cap L^\infty(\Omega)$ satisfying $\varphi \geqslant 0$ in $\Omega$. Thus, by $\varepsilon \downarrow 0$ (note $u > 0$ in $\overline{\Omega}$), we have $A_p\|\nabla\varphi\|_p^p \geqslant \lambda \int_\Omega m\varphi^p\,dx$ for every $\varphi \in W^{1,p}(\Omega)\cap C^1(\Omega)\cap L^\infty(\Omega)$ satisfying $\varphi \geqslant 0$ in $\Omega$. By combining the above inequality and an argument as in [14, Proposition 2.4.], we can easily prove our assertion (note $\lambda m = (-\lambda)(-m)$).

PROPOSITION 3. *Let $f_n\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying*

$$|f_n(x,t)| \leqslant D(1+|t|^{r-1}) \quad \text{for every } x\in\Omega,\ t\in\mathbb{R}$$

*with some positive constant $D$ independent of $n$ and $r\in[p,p^*)$, where $p^*=\infty$ if $N\leqslant p$, $p^* = pN/(N-p)$ if $N>p$. Assume that $A_n\colon \Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a map satisfying* $(A)$ (i), (ii), (iii) and (iv) *with positive constants $C_1'$, $C_0'$ and $C_2'$ independent of $n$. If $u_n$ is a solution for*

$$-\mathrm{div}A_n(x,\nabla u) = f_n(x,u) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

*and $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, then there exist a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ and $u_0\in C^1(\overline{\Omega})$ such that $u_{n_l}\to u_0$ in $C^1(\overline{\Omega})$ as $l\to\infty$.*

*Proof.* Since $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, we may assume that $u_n$ weakly converges to some $u_0$ in $W^{1,p}(\Omega)$ by choosing a subsequence.

We can show that there exists a $C>0$ depend only $|\Omega|$, $p$, $N$, $D$, $C_0'$, $C_1'$ and the embedding constant of $W^{1,p}(\Omega)$ to $L^{\bar{p}^*}(\Omega)$ such that

$$\|u_n\|_\infty \leqslant C\max\{1, \|u_n\|^{(\bar{p}^*-p)/(\bar{p}^*-r)}\}$$

by the Moser iteration process (refer to Theorem C in [19]), where $\bar{p}^* = p^*$ if $N>p$ and $\bar{p}^* > r$ is an any constant if $N\leqslant p$. Since $D$, $C_1'$ and $C_0'$ are independent of $n$, $\|u_n\|_\infty$ is bounded. Therefore, the regularity result in [17] guarantees that there exist $\gamma\in(0,1)$ and $M>0$ independent of $n$ such that $u_n\in C^{1,\gamma}(\overline{\Omega})$ and $\|u_n\|_{C^{1,\gamma}(\overline{\Omega})}\leqslant M$ (where we use the fact that $C_2'$ is independent of $n$ also). Since the inclusion of $C^{1,\gamma}(\overline{\Omega})$ to $C^1(\overline{\Omega})$ is compact, $u_n$ converges $u_0$ in $C^1(\overline{\Omega})$ (note that $u_n\rightharpoonup u_0$ in $W^{1,p}(\Omega)$).

## 3. Antimaximum principle

In this section, we assume that $\int_\Omega m\,dx\geqslant 0$ without loss of generality by noting $\lambda m = (-\lambda)(-m)$.

THEOREM 1. *Assume $\int_\Omega m\,dx>0$ (resp. $\int_\Omega m\,dx = 0$). Then, for any $0\not\equiv h\in L^\infty(\Omega)_+$ there exists $\delta=\delta(h)>0$ such that any solution $u$ of $(P;\lambda,m,h)$ satisfies $u<0$ in $\overline{\Omega}$ provided $0<\lambda<\delta$ (resp. $0<|\lambda|<\delta$).*

*Proof.* Because of $\lambda m = (-\lambda)(-m)$, it is sufficient to prove that for any $0 \not\equiv h \in L^\infty(\Omega)_+$ there exists $\delta = \delta(h) > 0$ such that any solution $u$ of $(P;\lambda,m,h)$ satisfies $u < 0$ in $\overline{\Omega}$ provided $0 < \lambda < \delta$. By way of contradiction, we may assume that there exist $0 \not\equiv h \in L^\infty(\Omega)_+$, $\{\lambda_n\}$ and a solution $u_n \in W^{1,p}(\Omega)$ of $(P:\lambda_n,m,h)$ such that $\lambda_n \downarrow 0$ and $u_n \geqslant 0$ somewhere in $\overline{\Omega}$. Note that $u_n \in C^1(\overline{\Omega})$ by Remark 2.

Moreover, we note that $\|u_n\|$ is bounded if $\|u_n\|_p$ is bounded by the following inequality

$$\begin{aligned} \frac{C_0}{p-1}\|\nabla u_n\|_p^p \leqslant \int_\Omega A(x,\nabla u_n)\nabla u_n\,dx = \lambda_n\int_\Omega m|u_n|^p + \int_\Omega hu_n\,dx \\ \leqslant \lambda_n\|m\|_\infty\|u_n\|_p^p + \|h\|_\infty\|u_n\|_1, \end{aligned} \tag{3.1}$$

where we use (iii) in Remark 1. Hence, by applying Proposition 3 to $A_n(x,y) = A(x,y)$ or $A_n(x,y) := A(x,y\|u_n\|_p)/\|u_n\|_p^{p-1}$, we see that $u_n$ or $u_n/\|u_n\|_p$ has a convergent subsequence in $C^1(\overline{\Omega})$ in the case where $\|u_n\|_p$ is bounded or not, respectively. Therefore, by the same argument as in [14, Theorem 3.2.], we can obtain a contradiction.

It follows from the following proposition that we can not take such $\delta$ independent of $h$ as in Theorem 1.

PROPOSITION 4. *Assume that $N \geqslant p$ and $\int_\Omega m\,dx \geqslant 0$. Then, for any $\varepsilon > 0$ there exists $0 \not\equiv h \in L^\infty(\Omega)_+$ such that for any $\lambda \geqslant \varepsilon$ the equation $(P;\lambda,m,h)$ has no solution $u$ satisfying $u \leqslant 0$ in $\overline{\Omega}$ and $|\{x \in \Omega_m; u(x) = 0\}| = 0$, where $\Omega_m := \{x \in \Omega; m(x) \neq 0\}$.*

*Proof.* By using Lemma 11 instead of [14, Lemma 2.5.] as in the argument of [14, Theorem 3.5.], we shall give the proof. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for any $0 \neq h \in L^\infty(\Omega)_+$, there exist $\lambda_h \geqslant \varepsilon_0$ and $u_h$ being a solution of $(P;\lambda_h,m,h)$ with $u_h \leqslant 0$ in $\overline{\Omega}$ and $|\{x \in \Omega_m; u_h(x) = 0\}| = 0$. Fix $0 < \delta < \varepsilon_0/A_p$, where $A_p$ is the positive constant defined by (2.14). Because we know $\overline{\lambda}(m) = \lambda^*(m) = 0$ in the case of $N \geqslant p$ by Lemma 5, there exists $\varphi \in W^{1,p}(\Omega)$ such that $\varphi = 0$ on some (open) ball $B \subset \Omega$,

$$\int_\Omega m|\varphi|^p\,dx = 1 \quad \text{and} \quad \int_\Omega |\nabla\varphi|^p\,dx < \delta.$$

By considering $|\varphi|$ instead of $\varphi$, we may assume that $\varphi \geqslant 0$ in $\Omega$. Here, we choose $h \in C_0^\infty(\Omega)$ such that $h \geqslant 0$, $h \not\equiv 0$ and supp $h \subset B$. By the above contradictory hypothesis, we can obtain $\lambda_h \geqslant \varepsilon_0$ and $u_h \in W^{1,p}(\Omega)$ being a solution of $(P;\lambda_h,m,h)$ with $u_h \leqslant 0$ in $\overline{\Omega}$ and $|\{x \in \Omega_m; u_h(x) = 0\}| = 0$. Set $v = -u_h$, then $v$ is non negative solution of

$$-\operatorname{div} A(x,\nabla v) = \lambda_h m v^{p-1} - h \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

since $A$ is odd in the second variable. Let $\varphi_M := \max\{\varphi, M\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $M > 0$. Then, for this $\varphi_M$ and $v$, the inequality as in Lemma 11 holds because we see that $p(\varphi_M/(v+\varepsilon))^{p-1}\nabla\varphi_M-(p-1)(\varphi_M/(v+\varepsilon))^p\nabla v\in L^p(\Omega)^N$ (note $v=-u_h\in C^1(\overline{\Omega})$ (see Remark 2)). Thus, we obtain

$$A_p\|\nabla\varphi_M\|_p^p \geqslant \int_\Omega A(x,\nabla v)\nabla\left(\frac{\varphi_M^p}{(v+\varepsilon)^{p-1}}\right)dx=\lambda_h\int_{\Omega_m} m\left(\frac{v}{v+\varepsilon}\right)^{p-1}\varphi_M^p\,dx$$

for every $\varepsilon>0$ and $M>0$ by supp $h\cap$supp $\varphi_M=$ supp $h\cap$supp $\varphi=\emptyset$. Because $v>0$ a.e. on $\Omega_m$, by taking $\varepsilon\downarrow 0$ and $M\to\infty$ in the above inequality, we obtain

$$\varepsilon_0\leqslant\lambda_h=\lambda_h\int_\Omega m\varphi^p\,dx=\lambda_h\int_{\Omega_m} m\varphi^p\,dx\leqslant A_p\|\nabla\varphi\|_p^p<A_p\delta<\varepsilon_0.$$

This is a contradiction.

REMARK 5. For the usual $p$-Laplace equation under the Dirichlet boundary condition, it is known that AMP holds at right of the principal eigenvalue $\lambda_1(m)$ and at left of $-\lambda_1(-m)$ (see [14]). However, in generall, we do not know wherther AMP holds near $\pm\lambda_1(\pm m)$ or not for the equation

$$-\operatorname{div}A(x,\nabla u)=\lambda m|u|^{p-2}u+h\quad\text{in }\Omega,\quad u=0\quad\text{on }\partial\Omega, \tag{3.2}$$

A major cause is that $C_0\lambda_1(m)/(p-1)<A_p\lambda_1(m)$ occurs in the case of $C_0<C_1$ because $\lambda_1(m)$ is positive. On the other hand, by the same argument as in the proof of Proposition 2, we can prove that equation (3.2) has no positive solutions provided $\lambda\notin[-A_p\lambda_1(-m),A_p\lambda_1(m)]$ and $0\neq h\in L^\infty(\Omega)_+$.

### 3.1. The case of $N<p$

THEOREM 2. *Assume that $N<p$ and $0\not\equiv h\in L^\infty(\Omega)_+$. Then, the following assertions hold:*

(i) *Suppose $\int_\Omega m\,dx>0$ and $\lambda$ satisfies $0<\lambda\leqslant C_0\tilde{\lambda}(m)/(p-1)$. Then, any solution $u$ of $(P;\lambda,m,h)$ satisfies $u<0$ in $\overline{\Omega}$.*

*In addition, if $|\{m<0\}|>0$ and $(C_1/C_0)^p\lambda^*(-m)<\tilde{\lambda}(-m)$, then the same conclusion holds for every $\lambda$ satisfying*

$$-\frac{C_0}{p-1}\tilde{\lambda}(-m)\leqslant\lambda<-A_p\lambda^*(-m);$$

(ii) *Suppose $\int_\Omega m\,dx=0$ and $\lambda$ satisfies*

$$0<\lambda\leqslant\frac{C_0\tilde{\lambda}(m)}{p-1}\quad\text{or}\quad-\frac{C_0\tilde{\lambda}(-m)}{p-1}\leqslant\lambda<0.$$

*Then, any solution $u$ of $(P;\lambda,m,h)$ satisfies $u<0$ in $\overline{\Omega}$.*

*Proof.* By $\lambda m = (-\lambda)(-m)$ and $\lambda^*(m) = 0$ if $\int_\Omega m\,dx \geqslant 0$, it is sufficient to prove that any solution of $(P;\lambda,m,h)$ is negative in $\overline{\Omega}$ under the hypothesis that $0 \not\equiv h \in L^\infty(\Omega)_+$, $A_p\lambda^*(m) < \lambda \leqslant C_0\tilde{\lambda}(m)/(p-1)$ with any $m \in L^\infty(\Omega)$ with (1.1). By way of contradiction, we shall prove our assertion above. So, we assume that there exist $m \in L^\infty(\Omega)$ with (1.1), $0 \not\equiv h \in L^\infty(\Omega)_+$, $\lambda \in (A_p\lambda^*(m), C_0\tilde{\lambda}(m)/(p-1)]$ and $u \in W^{1,p}(\Omega)$ being a solution of $(P:\lambda,m,h)$ with $u \geqslant 0$ somewhere in $\overline{\Omega}$. By taking $\varphi = -u_-$ as test function, we have

$$\begin{aligned}\frac{C_0}{p-1}\|\nabla u_-\|_p^p &\leqslant \int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx \\ &= \lambda\int_\Omega m u_-^p\,dx - \int_\Omega h u_-\,dx \\ &\leqslant \lambda \int_\Omega m u_-^p\,dx. \end{aligned} \tag{3.3}$$

Then, we can see that $\int_\Omega m u_-^p\,dx > 0$. Indeed, if $\int_\Omega m u_-^p\,dx = 0$ (note $\lambda > 0$), then $\nabla u_- \equiv 0$ and $\int_\Omega h u_-\,dx = 0$ holds, whence $u_- \equiv 0$. Thus, $u \geqslant 0$ and $u \not\equiv 0$ by $h \not\equiv 0$. This contradicts to Proposition 2 because of $\lambda > A_p\lambda^*(m)$.

As a result, we can get a contradiction easily by the following inequality (obtained by (3.3))

$$\frac{\|\nabla u_-\|_p^p}{\int_\Omega m u_-^p\,dx} \leqslant \frac{p-1}{C_0}\lambda \leqslant \tilde{\lambda}(m),$$

the definition of $\tilde{\lambda}(m)$, Lemma 6 and a similar argument to [5, Theorem 2.1.].

THEOREM 3. *Assume that $N < p$ and $0 \not\equiv h \in L^\infty(\Omega)_+$. Then, the following assertions hold:*

(i) *Let $\int_\Omega m\,dx > 0$. Then, there exists $\delta = \delta(h) > 0$ for every $\lambda$ satisfying*

$$C_0\tilde{\lambda}(m)/(p-1) < \lambda < C_0\tilde{\lambda}(m)/(p-1) + \delta$$

*such that any solution $u$ of $(P;\lambda,m,h)$ satisfies $u < 0$ in $\overline{\Omega}$.*

*In addition, if $|\{m < 0\}| > 0$ and $(C_1/C_0)^p\lambda^*(-m) < \tilde{\lambda}(-m)$, then there exists $\delta' = \delta'(h) > 0$ such that the same conclusion holds for every $\lambda$ satisfying*

$$-\frac{C_0}{p-1}\tilde{\lambda}(-m) - \delta' < \lambda < -\frac{C_0}{p-1}\tilde{\lambda}(-m);$$

(ii) *Let $\int_\Omega m\,dx = 0$. Then, there exists $\delta = \delta(h) > 0$ for every $\lambda$ satisfying*

$$\frac{C_0\tilde{\lambda}(m)}{p-1} < \lambda < \frac{C_0\tilde{\lambda}(m)}{p-1} + \delta \quad \text{or} \quad -\frac{C_0\tilde{\lambda}(-m)}{p-1} - \delta' < \lambda < -\frac{C_0\tilde{\lambda}(-m)}{p-1}$$

*such that any solution $u$ of $(P;\lambda,m,h)$ satisfies $u < 0$ in $\overline{\Omega}$.*

*Proof.* By the same reason as in the proof of Theorem 2, it is sufficient to prove that for any $m \in L^\infty(\Omega)$ with (1.1) and $0 \not\equiv h \in L^\infty(\Omega)_+$, there exists a $\delta > 0$ such that any solution of $(P;\lambda,m,h)$ is negative in $\overline{\Omega}$ if

$$C_0\frac{\tilde{\lambda}(m)}{p-1} < \lambda < C_0\frac{\tilde{\lambda}(m)}{p-1} + \delta$$

under the hypothesis $A_p\lambda^*(m) < C_0\tilde{\lambda}(m)/(p-1)$ (note $A_p\lambda^*(m) < C_0\tilde{\lambda}(m)/(p-1)$ if and only if $(C_1/C_0)^p\lambda^*(m) < \tilde{\lambda}(m)$), where $A_p$ is the positive constant defined by (2.14). Thus, by way of contradiction, we assume that there exist $m \in L^\infty(\Omega)$ with (1.1), $0 \not\equiv h \in L^\infty(\Omega)_+$, $\{\lambda_n\}$ and $\{u_n\} \subset W^{1,p}(\Omega)$ such that $\lambda_n \downarrow C_0\tilde{\lambda}(m)/(p-1)$ and $u_n$ is a solution of $(P;\lambda_n,m,h)$ satisfying $u_n \geqslant 0$ somewhere in $\overline{\Omega}$.

If $\|u_n\|_p$ is bounded, then we can obtain a subsequence $\{u_{n_l}\}$ convergent to some $u_0$ in $C^1(\overline{\Omega})$ by Proposition 3 with $A_n = A$. This implies that $u_0$ is a solution of $(P;\lambda,m,h)$ with $u_0 \geqslant 0$ somewhere in $\Omega$ for $\lambda = C_0\tilde{\lambda}(m)/(p-1)$. This contradicts to Theorem 2.

Thus, we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Set $v_n := u_n/\|u_n\|_p$. Then, by a similar inequality to (3.1), we can get the boundedness of $\|v_n\|$. So, we may suppose, by choosing a subsequence, that there exists $v \in W^{1,p}(\Omega)$ such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_n(x) \to v(x)$ uniformly in $x \in \overline{\Omega}$. We note that $v \geqslant 0$ somewhere in $\overline{\Omega}$ because $v_n \geqslant 0$ somewhere in $\overline{\Omega}$. Moreover, we can obtain (note $\lambda_n \to C_0\tilde{\lambda}(m)/(p-1)$):

$$\|\nabla v_+\|_p^p \leqslant \tilde{\lambda}(m)\int_\Omega mv_+^p\,dx \quad \text{and} \quad \|\nabla v_-\|_p^p \leqslant \tilde{\lambda}(m)\int_\Omega mv_-^p\,dx \tag{3.4}$$

by taking the limit inferior in the following inequalities

$$\frac{C_0}{p-1}\|\nabla v_{n+}\|_p^p \leqslant \int_\Omega A(x,\nabla u_n)\frac{\nabla u_{n+}}{\|u_n\|_p^p}\,dx = \lambda_n\int_\Omega mv_{n+}^p\,dx + \int_\Omega h\frac{v_{n+}}{\|u_n\|_p^{p-1}}\,dx,$$
$$\frac{C_0}{p-1}\|\nabla v_{n-}\|_p^p \leqslant \int_\Omega A(x,\nabla u_n)\frac{-\nabla u_{n-}}{\|u_n\|_p^p}\,dx = \lambda_n\int_\Omega mv_{n-}^p\,dx - \int_\Omega h\frac{v_{n-}}{\|u_n\|_p^{p-1}}\,dx,$$

where we use (iii) in Remark 1.

Here, we shall consider by dividing into three cases:

(a) $\int_\Omega mv_+^p\,dx > 0$; (b) $\int_\Omega mv_+^p\,dx = 0$ and $\int_\Omega mv_-^p\,dx = 0$;

(c) $\int_\Omega mv_+^p\,dx = 0$ and $\int_\Omega mv_-^p\,dx > 0$.

Case (a): If $v_+ > 0$ in $\overline{\Omega}$, then $v_n$ is positive in $\overline{\Omega}$ for sufficiently large $n$ because $v = v_+ > 0$ in $\overline{\Omega}$ and $v_n(x) \to v(x)$ uniformly in $x \in \overline{\Omega}$. This means that $u_n$ is a positive solution of $(P;\lambda_n,m,h)$ for sufficiently large $n$. This contradicts to Proposition 2 because of $\lambda_n > C_0\tilde{\lambda}(m)/(p-1) > A_p\lambda^*(m)$. So, we suppose that $v_+$ vanishes somewhere in $\overline{\Omega}$. Then, it follows from (3.4) that $v_+/(\int_\Omega mv_+^p\,dx)^{1/p}$ is a minimizer for $\tilde{\lambda}(m)$. Thus $v_+$ vanishes at exactly one point $x_0 \in \overline{\Omega}$ by Lemma 6, whence $v = v_+$ occurs. Now we shall prove that

$$\frac{C_0\tilde{\lambda}(m)}{p-1}\int_\Omega m\varphi^p\,dx \leqslant A_p\|\nabla\varphi\|_p^p \quad \text{for every } \varphi \in W^{1,p}(\Omega) \text{ with } \varphi \geqslant 0. \tag{3.5}$$

If (3.5) is shown, then we have a contradiction because we can choose some $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geqslant 0$, $\int_\Omega m\varphi^p\,dx = 1$ and $\|\nabla\varphi\|_p^p < \lambda^*(m) + \delta$ for $\delta > 0$ satisfying $A_p\delta < C_0\tilde{\lambda}(m)/(p-1) - A_p\lambda^*(m)$ (note $C_0\tilde{\lambda}(m)/(p-1) > A_p\lambda^*(m)$).

To prove (3.5), we fix $\varepsilon > 0$ and $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$. For sufficiently large $n$, we have $v_n + \varepsilon \geqslant \varepsilon/2$ in $\overline{\Omega}$, and hence $u_n + \varepsilon\|u_n\|_p \geqslant \varepsilon\|u_n\|_p/2 (> 0)$ in $\overline{\Omega}$ since $v_n$ converges to $v = v_+$ uniformly in $\overline{\Omega}$. Thus, Lemma 11 yields the following inequality (note $u_n \in C^1(\overline{\Omega})$):

$$\begin{aligned} A_p\|\nabla\varphi\|_p^p &\geqslant \int_\Omega A(x,\nabla u_n)\nabla\left(\frac{\varphi^p}{(u_n+\varepsilon\|u_n\|_p)^{p-1}}\right)dx \qquad (3.6)\\ &= \lambda_n\int_\Omega m\left(\frac{u_n}{u_n+\varepsilon\|u_n\|_p}\right)^{p-1}\varphi^p\,dx + \int_\Omega h\frac{\varphi^p}{(u_n+\varepsilon\|u_n\|_p)^{p-1}}\,dx\\ &\geqslant \lambda_n\int_\Omega m\left(\frac{v_n}{v_n+\varepsilon}\right)^{p-1}\varphi^p\,dx. \end{aligned}$$

Hence, by taking the limit in the above inequality, we have

$$\frac{C_0\tilde{\lambda}(m)}{p-1}\int_\Omega m\left(\frac{v}{v+\varepsilon}\right)^{p-1}\varphi^p\,dx \leqslant A_p\|\nabla\varphi\|_p^p.$$

Moreover, by taking $\varepsilon \downarrow 0$, we can get (3.5) since $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ and $v(x) > 0$ if $x \neq x_0$.

Case (b): In this case, it follows from (3.4) that $\nabla v \equiv 0$ holds, and so $v$ is a constant function with $\|v\|_p = 1$. Because $v \geqslant 0$ somewhere in $\overline{\Omega}$, we see $v = 1/|\Omega|^{1/p}$. Then, by the same reason as in the first part of the case (a), we have a contradiction.

Case (c): In this case, we can see that $v$ is non positive in $\overline{\Omega}$ (that is, $v = -v_-$) since $\nabla v_+ \equiv 0$ by (3.4) and $\int_\Omega mv_+^p\,dx = 0 < \int_\Omega mv_-^p\,dx$.

If $v = -v_-$ does not vanish in $\overline{\Omega}$, then $u_n < 0$ in $\overline{\Omega}$ for sufficiently large $n$. This yields a contradiction because $u_n \geqslant 0$ somewhere in $\overline{\Omega}$.

Thus, we may assume that $v_-$ vanishes somewhere in $\overline{\Omega}$. Then, (3.4) implies that $v_-/(\int_\Omega mv_-^p\,dx)^{1/p}$ is a minimizer for $\tilde{\lambda}(m)$. By considering

$$\int_\Omega A(x,-\nabla u_n)\nabla\left(\frac{\varphi^p}{(-u_n+\varepsilon\|u_n\|_p)^{p-1}}\right)dx \leqslant A_p\|\nabla\varphi\|_p^p$$

instead of (3.6) as in the proof of case (a), we have the same inequality (3.5) for every $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geqslant 0$ (note that $A$ is odd in the second variable and $-u_n(x) + \varepsilon\|u_n\|_p \to \infty$ uniformly in $x \in \overline{\Omega}$). As a result, we can get a contradiction by the same reason as in the last part of the case (a).

## 4. Existence of a solution

### 4.1. Existence of a positive solution

THEOREM 4. *Let $0 \not\equiv h \in L^\infty(\Omega)_+$. If one of the following cases holds, then $(P;\lambda,m,h)$ has a positive solution:*

(i) $m \geqslant 0$ *in* $\Omega$ *and* $\lambda < 0$*;*

(ii) $\int_\Omega m\,dx > 0,\ |\{m<0\}| > 0$ *and* $0 > \lambda > -C_0\lambda^*(-m)/(p-1)$*;*

(iii) $\int_\Omega m\,dx < 0$ *and* $0 < \lambda < C_0\lambda^*(m)/(p-1)$*,*

*where* $\lambda^*(m)$ *is the principal eigenvalue obtained by* (2.5).

To prove the existence of a positive solution, we define a $C^1$ functional $I_\lambda^+$ on $W^{1,p}(\Omega)$ as follows:

$$I_\lambda^+(u) := \int_\Omega G(x,\nabla u)\,dx - \frac{\lambda}{p}\int_\Omega m u_+^p\,dx - \int_\Omega hu\,dx + \frac{1}{p}\|u_-\|_p^p \tag{4.1}$$

for $\lambda \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, where $G(x,y) := \int_0^{|y|} a(x,t)t\,dt$ (see (2.1) for details).

REMARK 6. We remark that non-trivial critical points of $I_\lambda^+$ correspond to positive solutions for $(P;\lambda,m,h)$. Indeed, if $u$ is a critical point of $I_\lambda^+$, then we have

$$\frac{C_0}{p-1}\|\nabla u_-\|_p^p + \|u_-\|_p^p \leqslant \int_\Omega A(x,\nabla u)(-\nabla u_-)\,dx + \int_\Omega hu_-\,dx + \|u_-\|_p^p = 0$$

by taking $-u_-$ as test function. Thus, $u_- \equiv 0$, and hence $u \geqslant 0$. As a result,

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx = \lambda\int_\Omega m u^{p-1}\varphi\,dx + \int_\Omega h\varphi\,dx$$

holds for every $\varphi \in W^{1,p}(\Omega)$. Because of $u \not\equiv 0$, $u$ is a positive solution of $(P;\lambda,m,h)$ by Remark 2.

LEMMA 12. *Let* $0 \not\equiv h \in L^\infty(\Omega)_+$*. If either* $m \geqslant 0$ *and* $\lambda < 0$ *or* $\int_\Omega m\,dx < 0$ *and* $0 < \lambda < C_0\lambda^*(m)/(p-1)$ *holds, then* $I_\lambda^+$ *is bounded from below, coercive and weakly lower semi-continuous (w.l.s.c.) on* $W^{1,p}(\Omega)$.

*Proof.* Note that $\Phi(u) := \int_\Omega G(x,\nabla u)\,dx$ is w.l.s.c. on $W^{1,p}(\Omega)$ (cf. [18, Theorem 1.2.]) because $\Phi$ is convex and continuous on $W^{1,p}(\Omega)$. Thus, $I_\lambda^+$ is also w.l.s.c. on $W^{1,p}(\Omega)$ since the inclusion of $W^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact.

Now, we prove that $I_\lambda^+$ is bounded from below and coercive on $W^{1,p}(\Omega)$.

Case of $m \geqslant 0$ and $\lambda < 0$: By Lemma 3 and (2.2), we can obtain

$$\begin{aligned}
I_\lambda^+(u) &\geqslant \frac{C_0}{p(p-1)}\|\nabla u\|_p^p + \frac{|\lambda|}{p}\int_\Omega m u_+^p\,dx - \|h\|_\infty\|u\|_1 + \frac{1}{p}\|u_-\|_p^p \\
&\geqslant \frac{C_0}{p(p-1)}\|u_-\|^p + \frac{C_0}{2p(p-1)}\left(\|\nabla u_+\|_p^p + \frac{2(p-1)|\lambda|}{C_0}\int_\Omega m u_+^p\,dx\right) \\
&\quad + \frac{C_0}{2p(p-1)}\|\nabla u_+\|_p^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \\
&\geqslant \frac{C_0}{p(p-1)}\|u_-\|^p + \frac{C_0\min\{d(\xi),1\}}{2p(p-1)}\|u_+\|^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p}
\end{aligned}$$

for every $u \in W^{1,p}(\Omega)$ (note $p-1 \geqslant C_0$), where $d(\xi) > 0$ is a constant obtained by Lemma 3 with $\xi = 2(p-1)|\lambda|/C_0$. This implies that $I_\lambda^+$ is bounded from below, coercive because of $p > 1$.

Case of $\int_\Omega m\,dx < 0$ and $0 < \lambda < C_0\lambda^*(m)/(p-1)$: For every $u \in W^{1,p}(\Omega)$ with $\int_\Omega m u_+^p\,dx > 0$, we have

$$\begin{aligned} I_\lambda^+(u) &\geqslant \frac{1}{p}\left(\frac{C_0}{p-1} - \frac{\lambda}{\lambda^*(m)}\right)\|\nabla u_+\|_p^p + \frac{C_0}{p(p-1)}\|\nabla u_-\|_p^p - \|h\|_\infty\|u\|_1 + \frac{1}{p}\|u_-\|_p^p \\ &\geqslant \frac{1}{2p}\left(\frac{C_0}{p-1} - \frac{\lambda}{\lambda^*(m)}\right)\|\nabla u_+\|_p^p + \frac{c}{2p}\left(\frac{C_0}{p-1} - \frac{\lambda}{\lambda^*(m)}\right)\|u_+\|_p^p \\ &\quad + \frac{C_0}{p(p-1)}\|u_-\|^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \end{aligned} \tag{4.2}$$

by (2.2), the definition of $\lambda^*(m)$, Lemma 1 and $C_0 \leqslant p-1$, where $c$ is a positive constant independent of $u$ obtained in Lemma 1. Next, we deal with $u \in W^{1,p}(\Omega)$ with $\int_\Omega m u_+^p\,dx \leqslant 0$. Take a $\delta$ such that $0 < \delta < \lambda$. Then, we obtain for any $u \in W^{1,p}(\Omega)$ with $\int_\Omega m u_+^p\,dx \leqslant 0$

$$\begin{aligned} I_\lambda^+(u) &\geqslant \frac{C_0}{2p(p-1)}\left(\|\nabla u_+\|_p^p - \frac{2(p-1)\delta}{C_0}\int_\Omega m u_+^p\,dx\right) + \frac{C_0}{2p(p-1)}\|\nabla u_+\|_p^p \\ &\quad + \frac{\delta-\lambda}{p}\int_\Omega m u_+^p\,dx + \frac{C_0}{p(p-1)}\|u_-\|^p - \|h\|_\infty\|u\|_1 \\ &\geqslant \frac{C_0 b(m,\xi)}{2p(p-1)}\|u_+\|_p^p + \frac{C_0}{2p(p-1)}\|\nabla u_+\|_p^p \\ &\quad + \frac{C_0}{p(p-1)}\|u_-\|^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \end{aligned} \tag{4.3}$$

by (2.2) and Lemma 2, where $b(m,\xi)$ is a positive constant obtained in Lemma 2 with $\xi = 2(p-1)\delta/C_0$. Consequently, our conclusion follows from (4.2) and (4.3).

PROOF OF THEOREM 4. By the properties of $I_\lambda^+$ obtained as in Lemma 12, $I_\lambda^+$ has a global minimizer in all cases as in Theorem 4 (cf. [18, Theorem 1.1.]), where, we use $\lambda m = (-\lambda)(-m)$ in the case (ii). Thus, we see that $(P;\lambda,m,h)$ has a positive solution by Remark 6.

## 4.2. Other existence results for the general case

THEOREM 5. *Assume $0 \not\equiv h \in L^\infty(\Omega)_+$. If one of the following conditions holds, then $(P;\lambda,m,h)$ has a solution:*

(i) *$m \geqslant 0$ in $\Omega$ and $0 < \lambda < C_0 c(m)/(p-1)$;*

(ii) *$\int_\Omega m\,dx > 0$ and $0 < \lambda < C_0\lambda_{X(m)}/(p-1)$;*

(iii) *$N < p$ and $A_p\lambda^*(m) < \lambda < C_0\tilde{\lambda}(m)/(p-1)$,* *where* $c(m)$, $\lambda_{X(m)}$, $A_p$ *and* $\tilde{\lambda}(m)$ *are positive constants defined by* (2.11), (2.13), (2.14) *and* (2.7), *respectively.*

To show the existence of a solution, we define a $C^1$ functional $I_\lambda$ on $W^{1,p}(\Omega)$ as follows:

$$I_\lambda(u) := \int_\Omega G(x,\nabla u)\,dx - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx - \int_\Omega hu\,dx \tag{4.4}$$

for $\lambda \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, where $G(x,y) := \int_0^{|y|} a(x,t)t\,dt$ (see (2.1) for details). Note that critical points of $I_\lambda$ correspond to solutions of $(P;\lambda,m,h)$ (see Remark 2).

First, we shall prove that $I_\lambda$ has the mountain pass geometry.

LEMMA 13. *Assume that* $h \in L^\infty(\Omega)_+$, $\int_\Omega m\,dx \neq 0$ *and*

$$\frac{C_1\lambda^*(m)}{p-1} < \lambda < \frac{C_0\,c(m)}{p-1}.$$

*Then,* $I_\lambda$ *is bounded from below on* $E(m)$ *defined by*

$$E(m) := \left\{u \in W^{1,p}(\Omega)\,;\, \|\nabla u\|_p^p \geqslant c(m)\int_\Omega m|u|^p\,dx\right\}. \tag{4.5}$$

*Furthermore, there exist* $u_0$, $u_1 \in W^{1,p}(\Omega)$ *such that*

$$\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{E(m)} I_\lambda \leqslant \max_{t\in[0,1]} I_\lambda(\gamma(t))$$

*for every* $\gamma \in \Gamma$, *where*

$$\Gamma := \left\{\gamma \in C([0,1], W^{1,p}(\Omega))\,;\, \gamma(0) = u_0,\ \gamma(1) = u_1\right\}.$$

*Proof.* First, we shall prove $\inf_{E(m)} I_\lambda > -\infty$. For every $u \in W^{1,p}(\Omega)$ with

$$\int_\Omega m|u|^p\,dx \leqslant 0,$$

we have

$$\begin{aligned} I_\lambda(u) &\geqslant \frac{C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx - \|h\|_\infty\|u\|_1 \\ &\geqslant \frac{C_0 b(m,\xi)}{p(p-1)}\|u\|_p^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} > -\infty \end{aligned} \tag{4.6}$$

by (2.2), Lemma 2 and Hölder's inequality, where $b(m,\xi)$ is a positive constant obtained in Lemma 2 with $\xi = (p-1)\lambda/C_0$. Thus, $I_\lambda$ is bounded from below on $B(m)$, where $B(m)$ is a set defined by

$$B(m) := \{u \in W^{1,p}(\Omega)\,;\, \int_\Omega m|u|^p\,dx \leqslant 0\} \subset E(m). \tag{4.7}$$

Here, we choose a constant $\delta$ such that $1 > \delta > \lambda(p-1)/(c(m)C_0)$.

Let $m \geqslant 0$ in $\Omega$. In this case, for every $u \in E(m)$, we have

$$
\begin{aligned}
I_\lambda(u) &\geqslant \frac{C_0(1-\delta)}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right)\int_\Omega m|u|^p\,dx - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \\
&= \frac{C_0(1-\delta)}{p(p-1)}\left\{\|\nabla u\|_p^p + \frac{p-1}{C_0(1-\delta)}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right)\int_\Omega m|u|^p\,dx\right\} \\
&\quad - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \\
&\geqslant d(\xi')\frac{C_0(1-\delta)}{p(p-1)}\|u\|_p^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} > -\infty
\end{aligned}
\tag{4.8}
$$

by (2.2), the definition of $E(m)$ and Lemma 3, where $d(\xi') > 0$ is a constant obtained in Lemma 3 with

$$\xi' = \frac{p-1}{C_0(1-\delta)}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right).$$

Similarly, in the other cases (that is, $m$ changes sign), for every $u \in E(m)$ with

$$\int_\Omega m|u|^p\,dx > 0,$$

we obtain

$$
\begin{aligned}
I_\lambda(u) &\geqslant \frac{C_0(1-\delta)}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right)\int_\Omega m|u|^p\,dx - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \\
&= \frac{C_0(1-\delta)}{p(p-1)}\left\{\|\nabla u\|_p^p - \frac{p-1}{C_0(1-\delta)}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right)\int_\Omega (-m)|u|^p\,dx\right\} \\
&\quad - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} \\
&\geqslant b(-m,\xi')\frac{C_0(1-\delta)}{p-1}\|u\|_p^p - \|h\|_\infty\|u\|_p|\Omega|^{(p-1)/p} > -\infty
\end{aligned}
\tag{4.9}
$$

by (2.2), the definition of $E(m)$, Lemma 2 (note $\int_\Omega (-m)|u|^p\,dx < 0$) and Hölder's inequality, where $b(-m,\xi')$ is a positive constant obtained in Lemma 2 with

$$\xi' = \frac{p-1}{C_0(1-\delta)}\left(\frac{C_0\delta c(m)}{p-1}-\lambda\right).$$

Consequently, we see that $I_\lambda$ is bounded from below on $E(m)$ by (4.6) and (4.8) or (4.9).

Fix a positive constant $\varepsilon$ such that $C_1(\lambda^*(m)+\varepsilon)/(p-1) < \lambda$. Then, by the definition of $\lambda^*(m)$, we can choose a non negative function $v_0 \in W^{1,p}(\Omega)$ (note that we can use $|v_0|$ instead of $v_0$ if necessary) such that

$$\int_\Omega m v_0^p\,dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p^p < \lambda^*(m)+\varepsilon.$$

Then, for sufficiently large $T > 0$, we have

$$\begin{aligned} I_\lambda(\pm T v_0) &\leqslant \frac{C_1 T^p}{p(p-1)} \|\nabla v_0\|_p^p - \frac{\lambda T^p}{p} + T \int_\Omega h v_0 \, dx \\ &< -\frac{T^p}{p}\left(\lambda - \frac{C_1}{p-1}(\lambda^*(m)+\varepsilon)\right) + T\int_\Omega h v_0\, dx < \inf_{E(m)} I_\lambda \end{aligned} \tag{4.10}$$

by (2.2), $\lambda - C_1(\lambda^*(m)+\varepsilon)/(p-1) > 0$ and $p > 1$. Hence, we set $u_0 := T v_0$ and $u_1 := -T v_0$ for $T > 0$ satisfying (4.10).

Now, we shall prove

$$\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geqslant \inf_{E(m)} I_\lambda \quad \text{for every } \gamma \in \Gamma.$$

Fix any $\gamma \in \Gamma$. If $\gamma([0,1]) \cap B(m) \neq \emptyset$, then

$$\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geqslant \inf_{B(m)} I_\lambda \geqslant \inf_{E(m)} I_\lambda$$

because of $B(m) \subset E(m)$ (see (4.7)). So, we may assume that $\gamma([0,1]) \cap B(m) = \emptyset$, namely $\int_\Omega m|\gamma(t)|^p\, dx > 0$ for every $t \in [0,1]$. Set

$$\tilde{\gamma}(t) := \frac{\gamma(t)}{(\int_\Omega m|\gamma(t)|^p\, dx)^{1/p}},$$

and then $\tilde{\gamma} \in \Sigma(m)$ (see (2.10) for the definition of $\Sigma(m)$). By the definition of $c(m)$, we have $\max_{t\in[0,1]} \|\nabla \tilde{\gamma}(t)\|_p^p \geqslant c(m)$. This implies that there exists $u_\gamma \in \gamma([0,1])$ such that

$$\|\nabla u_\gamma\|_p^p \geqslant c(m) \int_\Omega m|u_\gamma|^p\, dx,$$

whence $u_\gamma \in E(m)$. As a result, we obtain

$$\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geqslant I_\lambda(u_\gamma) \geqslant \inf_{E(m)} I_\lambda.$$

LEMMA 14. *Assume that*

$$h \in L^\infty(\Omega)_+, \ \int_\Omega m\, dx = 0 \quad \textit{and} \quad 0 < \lambda < \frac{C_0\, c(m)}{p-1}.$$

*Then, there exists* $\varepsilon_0 > 0$ *such that* $\lambda < C_0\, c(m+\varepsilon_0)/(p-1)$ *and* $I_\lambda$ *is bounded from below on* $E(m+\varepsilon_0)$ *defined by* (4.5) *with* $m+\varepsilon_0$. *Furthermore, there exist* $u_0,\ u_1 \in W^{1,p}(\Omega)$ *such that*

$$\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{E(m+\varepsilon_0)} I_\lambda \leqslant \max_{t\in[0,1]} I_\lambda(\gamma(t))$$

*for every* $\gamma \in C([0,1], W^{1,p}(\Omega))$ *with* $\gamma(0) = u_0$ *and* $\gamma(1) = u_1$.

*Proof.* By Lemma 9 and $\lambda < C_0 c(m)/(p-1)$, we can choose $\varepsilon_0 > 0$ satisfying $\lambda < C_0 c(m+\varepsilon_0)/(p-1)$. For every $u \in W^{1,p}(\Omega)$, we have

$$I_\lambda(u) \geqslant \frac{C_0}{p(p-1)}\|\nabla u\|_p^p - \frac{\lambda}{p}\int_\Omega (m+\varepsilon_0)|u|^p\,dx + \frac{\varepsilon_0\lambda}{p}\|u\|_p^p - \|h\|_\infty\|u\|_1.$$

Thus, by the same argument as in the proof of Lemma 13 with $m+\varepsilon_0$ instead of $m$, we can show that $I_\lambda$ is bounded from below on $E(m+\varepsilon_0)$ (note $\varepsilon_0\lambda > 0$). Moreover, by choosing a non negative function $v_0 \in W^{1,p}(\Omega)$ such that

$$\int_\Omega m v_0^p\,dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p^p < \lambda^*(m) + \varepsilon = \varepsilon$$

for $0 < \varepsilon < \lambda(p-1)/C_1$, we have

$$I_\lambda(\pm T v_0) \leqslant -\frac{T^p}{p}\left(\lambda - \frac{C_1\varepsilon}{p-1}\right) + T\int_\Omega h v_0\,dx < \inf_{E(m+\varepsilon_0)} I_\lambda$$

for sufficiently large $T > 0$, where we use (2.2) in the first integral. The last assertion can be proved by the same argument as in the proof of Lemma 13 with $m+\varepsilon_0$ instead of $m$.

PROOF OF THEOREM 5. By Proposition 6 in the last subsection 4.4, we will see that $I_\lambda$ satisfies the Palais-Smale condition in all cases. Hence, the mountain pass theorem guarantees the existence of a critical point of $I_\lambda$ since $I_\lambda$ has the mountain pass geometry by Lemma 13 (if $\int_\Omega m\,dx \neq 0$) or Lemma 14 (if $\int_\Omega m\,dx = 0$), where we use $A_p \geqslant C_1/(p-1)$ when $\int_\Omega m\,dx < 0$ and $N < p$. Therefore, $(P;\lambda,m,h)$ has at least one solution.

### 4.3. Asymptotically $(p-1)$ homogeneous case

In this subsection, we deal with the special case where the map $A(x,y)$ is asymptotically $(p-1)$ homogeneous in the following sense:

$(AH)$ there exist a positive function $a_\infty \in C^1(\overline{\Omega},\mathbb{R})$ and a continuous function $\tilde{a}(x,t)$ on $\overline{\Omega}\times\mathbb{R}$ such that

$$A(x,y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x,|y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$
$$\lim_{t\to+\infty}\frac{\tilde{a}(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega},$$

and $A$ satisfies the hypothesis $(A)$.

Under this hypothesis, we obtain the following existence result.

THEOREM 6. *Assume that* $(AH)$, $m \in L^\infty(\Omega)$ *and* $0 \not\equiv h \in L^\infty(\Omega)_+$. *If*

$$\lambda^*(m)\sup_{x\in\Omega} a_\infty(x) < \lambda < c(m)\inf_{x\in\Omega} a_\infty(x),$$

*then* $(P;\lambda,m,h)$ *has at least one solution.*

Under the hypothesis $(AH)$, we define

$$\widetilde{G}(x,y) := \int_0^{|y|} \tilde{a}(x,t)t\,dx.$$

Then, the functional $I_\lambda$ is written by

$$I_\lambda(u) = \frac{1}{p}\int_\Omega a_\infty |\nabla u|^p\,dx + \int_\Omega \widetilde{G}(x,\nabla u)\,dx - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx - \int_\Omega hu\,dx$$

for $u \in W^{1,p}(\Omega)$.

Now, we shall prove that $I_\lambda$ has the mountain pass geometry.

LEMMA 15. *Assume that $(AH)$, $h \in L^\infty(\Omega)_+$, $\int_\Omega m\,dx \neq 0$ and*

$$\lambda^*(m)\sup_{x\in\Omega} a_\infty(x) < \lambda < c(m)\inf_{x\in\Omega} a_\infty(x).$$

*Then, $I_\lambda$ is bounded from below on $E(m)$ defined by (4.5). Furthermore, there exist $u_0,\ u_1 \in W^{1,p}(\Omega)$ such that*

$$\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{E(m)} I_\lambda \leqslant \max_{t\in[0,1]} I_\lambda(\gamma(t))$$

*for every $\gamma \in \Gamma$, where*

$$\Gamma := \left\{\gamma \in C([0,1], W^{1,p}(\Omega));\, \gamma(0) = u_0,\ \gamma(1) = u_1\right\}.$$

*Proof.* By the property of the function $\tilde{a}$ as in $(AH)$ and Young's inequality, for every $\varepsilon > 0$ there exist constants $C_\varepsilon > 0$ and $C'_\varepsilon > 0$ such that

$$\left|\widetilde{G}(x,y)\right| \leqslant \frac{\varepsilon}{2}|y|^p + C_\varepsilon|y| \leqslant \varepsilon|y|^p + C'_\varepsilon \tag{4.11}$$

for every $x \in \Omega$ and $y \in \mathbb{R}^N$. Therefore, we have

$$I_\lambda(u) \geqslant \frac{\underline{a} - p\varepsilon}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx - \|h\|_\infty\|u\|_1 - C'_\varepsilon|\Omega|$$

for every $u \in W^{1,p}(\Omega)$, where $\underline{a} := \inf_{x\in\Omega} a_\infty(x)$. Here, we choose $\varepsilon > 0$ and $0 < \delta < 1$ such that $\lambda < (\underline{a} - p\varepsilon)\delta c(m)$. By a similar argument to Lemma 13, we can show that $I_\lambda$ is bounded from below on $E(m)$.

Next, we shall prove the existence of desired $u_0$ and $u_1$. Take $\varepsilon' > 0$ satisfying

$$\lambda > (\overline{a} + p\varepsilon')(\lambda^*(m) + \varepsilon'),$$

where $\overline{a} := \sup_{x\in\Omega} a_\infty(x)$. Choose a function $v_0 \in W^{1,p}(\Omega)$ such that

$$\int_\Omega m|v_0|^p\,dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p^p < \lambda^*(m) + \varepsilon'.$$

Then, for sufficiently large $T>0$, we have

$$I_\lambda(\pm T v_0) \leqslant -\frac{T^p}{p}\{\lambda-(\overline{a}+p\varepsilon')(\lambda^*(m)+\varepsilon')\}+T\int_\Omega h|v_0|\,dx+C'_{\varepsilon'}|\Omega|$$
$$< \inf_{E(m)} I_\lambda,$$

where we use (4.11) with $\varepsilon=\varepsilon'$. Thus, by setting $u_0:=Tv_0$ and $u_1:=-Tv_0$ for such $T>0$, our claim is shown. Finally, it follows from the same argument as in Lemma 13 that every $\gamma\in\Gamma$ links $E(m)$.

By combining the proof of Lemma 15 with one of Lemma 14, we can show the following lemma in the case of $\int_\Omega m\,dx=0$. Here, we omit the proof.

LEMMA 16. *Assume that* $(AH)$, $h\in L^\infty(\Omega)_+$, $\int_\Omega m\,dx=0$ *and*

$$\lambda^*(m)\sup_{x\in\Omega} a_\infty(x)<\lambda<c(m)\inf_{x\in\Omega} a_\infty(x).$$

*Then, there exists* $\varepsilon_0>0$ *such that* $\lambda<c(m+\varepsilon_0)\inf_{x\in\Omega}a_\infty(x)$ *and* $I_\lambda$ *is bounded from below on* $E(m+\varepsilon_0)$ *defined by* (4.5) *with* $m+\varepsilon_0$. *Furthermore, there exist* $u_0$, $u_1\in W^{1,p}(\Omega)$ *such that*

$$\max\{I_\lambda(u_0),I_\lambda(u_1)\}<\inf_{E(m+\varepsilon_0)} I_\lambda \leqslant \max_{t\in[0,1]} I_\lambda(\gamma(t))$$

*for every* $\gamma\in C([0,1],W^{1,p}(\Omega))$ *with* $\gamma(0)=u_0$ *and* $\gamma(1)=u_1$.

PROOF OF THEOREM 6. It suffices to prove the existence of a critical point of $I_\lambda$ because critical points of $I_\lambda$ correspond to solutions of $(P;\lambda,m,h)$. By Proposition 6 in the last subsection 4.4, we will see that $I_\lambda$ satisfies the Palais-Smale condition if $\lambda$ is not an eigenvalue of

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right)=\lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega. \tag{4.12}$$

Hence, by admitting that $\lambda$ is not an eigenvalue of (4.12), the mountain pass theorem guarantees the existence of a critical point of $I_\lambda$ since $I_\lambda$ has the mountain pass geometry by Lemma 15 or 16 in the case of $\int_\Omega m\,dx\neq 0$ or $\int_\Omega m\,dx=0$, respectively.

Now, we shall prove that the equation (4.12) has no non-trivial solution provided $\lambda^*(m)\sup_{x\in\Omega}a_\infty(x)<\lambda<c(m)\inf_{x\in\Omega}a_\infty(x)$ by way of contradiction. So, we assume that there exists a non-trivial solution $v\in W^{1,p}(\Omega)$ of (4.12). By taking $\pm v_\pm$ as test function, we have

$$\inf_{x\in\Omega} a_\infty(x)\|\nabla v_\pm\|_p^p \leqslant \lambda\int_\Omega m v_\pm^p\,dx \leqslant \sup_{x\in\Omega} a_\infty(x)\|\nabla v_\pm\|_p^p. \tag{4.13}$$

We shall show that $\int_\Omega m v_+^p\,dx>0$ and $\int_\Omega m v_-^p\,dx>0$. If $\int_\Omega m v_+^p\,dx=0$ holds, then $v=-v_-$ or $v=c>0$ with $c=\|v_+\|_p$ occurs because of $\|\nabla v_+\|_p=0$ obtained by

(4.13). Thus, $-v$ or $v$ is a positive solution of (4.12) belonging to $C^1(\overline{\Omega})$ such that $\min_{\overline{\Omega}} v_- = \min_{\overline{\Omega}}(-v) > 0$ or $\min_{\overline{\Omega}} v > 0$, respectively (see Remark 2 with $h \equiv 0$). Then, by applying an argument as in Proposition 2 (with $h \equiv 0$) to the equation (4.12), we obtain the inequality

$$\lambda \int_\Omega m\varphi^p\,dx = \int_\Omega a_\infty |\nabla \psi|^{p-2}\nabla\psi\nabla\left(\frac{\varphi^p}{\psi^{p-1}}\right)dx$$
$$\leqslant \int_\Omega a_\infty |\nabla\varphi|^p\,dx \leqslant \sup_{x\in\Omega} a_\infty(x)\|\nabla\varphi\|_p^p$$

for every $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$, where $\psi = v_-$ if $v < 0$ or $\psi = v_+$ if $v > 0$. By the density of $C^1(\overline{\Omega})$, we have

$$\lambda \int_\Omega m\varphi^p\,dx \leqslant \sup_{x\in\Omega} a_\infty(x)\|\nabla\varphi\|_p^p \quad \text{for every } \varphi \in W^{1,p}(\Omega) \text{ with } \varphi \geqslant 0.$$

This implies that $\lambda \leqslant \lambda^*(m)\sup_{x\in\Omega} a_\infty(x)$ (refer to Proposition 2). This is a contradiction.

Similarly, if $\int_\Omega m v_-^p\,dx = 0$, then we can get a contradiction since $v = v_+$ or $v = -c < 0$ holds. Therefore, our claim is shown. As a result, we can define a continuous path $\gamma_0 \in \Sigma(m)$ (see (2.10) for the definition of $\Sigma(m)$) by

$$\gamma_0(t) := \frac{(1-t)v_+ - tv_-}{\left((1-t)^p\int_\Omega m v_+^p\,dx + t^p\int_\Omega m v_-^p\,dx\right)^{1/p}}.$$

Hence, we have a contradiction to the definition of $c(m)$ because

$$\tilde{J}(\gamma_0(t)) = \|\nabla\gamma_0(t)\|_p^p \leqslant \frac{\lambda}{\inf_{x\in\Omega} a_\infty(x)} < c(m) \quad \text{for every } t \in [0,1]$$

holds by (4.13), where $\tilde{J}$ is the functional defined by (2.8).

REMARK 7. Let $\lambda^*(a_\infty, m)$ and $c(a_\infty, m)$ be the principal eigenvalue or the second eigenvalue of

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{4.14}$$

respectively. Namely,

$$\lambda^*(a_\infty, m) := \inf\left\{\int_\Omega a_\infty|\nabla u|^p\,dx;\ \int_\Omega m|u|^p\,dx = 1\right\},$$
$$c(a_\infty, m) := \inf_{\gamma\in\Sigma(m)}\max_{t\in[0,1]}\int_\Omega a_\infty|\nabla\gamma(t)|^p\,dx.$$

Then, in the assumption of Theorem 6, we can replace

$$\lambda^*(m)\sup_{x\in\Omega} a_\infty(x) < \lambda < c(m)\inf_{x\in\Omega} a_\infty(x)$$

with $\lambda^*(a_\infty, m) < \lambda < c(a_\infty, m)$. In [23], the present author provides the existence result in the more general cases.

### 4.4. Palais-Smale condition

In this section, we prove that $I_\lambda$ satisfies the Palais-Smale condition under the several situation. The following result is proved in [20]. It plays an important role for our poof.

PROPOSITION 5. ([20, Proposition 1]) *Let* $A\colon W^{1,p}(\Omega)\to W^{1,p}(\Omega)^*$ *be the map defined by*

$$\langle A(u),v\rangle=\int_\Omega A(x,\nabla u)\nabla v\,dx$$

*for* $u,\ v\in W$. *Then,* $A$ *is maximal monotone, strictly monotone and has the* $(S)_+$ *property, that is, any sequence* $\{u_n\}$ *weakly convergent to* $u$ *with*

$$\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\leqslant 0\quad \textit{strongly converges to } u.$$

First, we state the result for the Palais-Smale condition in the general case.

PROPOSITION 6. *Let* $h\in L^\infty(\Omega)_+$. *If one of the following cases holds, then* $I_\lambda$ *satisfies the Palais-Smale condition:*

(i) $m\geqslant 0$ *in* $\Omega$ *and* $0<\lambda<C_0c(m)/(p-1)$,

(ii) $\int_\Omega m\,dx>0$ *and* $0<\lambda<C_0\lambda_{X(m)}/(p-1)$,

(iii) $N<p$ *and* $A_p\lambda^*(m)<\lambda<C_0\tilde\lambda(m)/(p-1)$,

*where* $c(m)$, $\lambda_{X(m)}$, $A_p$ *and* $\tilde\lambda(m)$ *are positive constants defined by* (2.11), (2.13), (2.14) *and* (2.7), *respectively.*

*Proof.* Let $\{u_n\}$ be a Palais-Smale sequence of $I_\lambda$, namely,

$$I_\lambda(u_n)\to c\quad\text{and}\quad \|I'_\lambda(u_n)\|_{W^*}\to 0\quad\text{as } n\to\infty$$

for some $c\in\mathbb{R}$. It is sufficient to prove the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$ because the operator $A$ defined in Proposition 5 has the $(S)_+$ property and the inclusion from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact. Then, by noting the following inequality

$$\begin{aligned}\frac{C_0}{p(p-1)}\|\nabla u_n\|_p^p\leqslant\int_\Omega G(x,\nabla u_n)\,dx&=I_\lambda(u_n)+\frac{\lambda}{p}\int_\Omega m|u_n|^p\,dx+\int_\Omega hu_n\,dx\\&\leqslant I_\lambda(u_n)+\lambda\|m\|_\infty\|u_n\|_p^p/p+\|h\|_\infty\|u_n\|_1\end{aligned}\tag{4.15}$$

by (2.2), it is sufficient to prove only the boundedness of $\{u_n\}$ in $L^p(\Omega)$. We shall prove it by contradiction. So, we may assume $\|u_n\|_p\to\infty$ by choosing a subsequence. Put $v_n:=u_n/\|u_n\|_p$. Then, we may suppose that there exists a $v\in W^{1,p}(\Omega)$ such that

$$v_n\rightharpoonup v\quad\text{in } W^{1,p}(\Omega)\quad\text{and hence}\quad v_n\to v\quad\text{in } L^p(\Omega)$$

since (4.15) ensures the boundedness of $\{v_n\}$ in $W^{1,p}(\Omega)$. By taking the limit inferior in the following inequality

$$\frac{C_0}{p-1}\|\nabla v_{n+}\|_p^p \leqslant \int_\Omega A(x,\nabla u_n)\frac{\nabla u_{n+}}{\|u_n\|_p^p}\,dx$$
$$= \lambda\int_\Omega m v_{n+}^p\,dx + \int_\Omega h\frac{v_{n+}}{\|u_n\|_p^{p-1}}\,dx + \left\langle I'_\lambda(u_n), \frac{v_{n+}}{\|u_n\|_p^{p-1}}\right\rangle$$

(where we use Remark 1 (iii) in the first inequality), we have

$$\frac{C_0}{p-1}\|\nabla v_+\|_p^p \leqslant \lambda\int_\Omega m v_+^p\,dx. \tag{4.16}$$

Similarly, we also get

$$\frac{C_0}{p-1}\|\nabla v_-\|_p^p \leqslant \lambda\int_\Omega m v_-^p\,dx. \tag{4.17}$$

Here, we note that it is sufficient to prove the two inequalities $\int_\Omega m v_+^p\,dx > 0$ and $\int_\Omega m v_-^p\,dx > 0$. Indeed, if we can show the above inequalities, then we can define a continuous path $\gamma_0 \in \Sigma(m)$ (see (2.10) for the definition of $\Sigma(m)$) by

$$\gamma_0(t) := \frac{(1-t)v_+ - tv_-}{\left((1-t)^p\int_\Omega m v_+^p\,dx + t^p\int_\Omega m v_-^p\,dx\right)^{1/p}}.$$

For this continuous path, by an easy estimate with (4.16) and (4.17), we have

$$\tilde{J}(\gamma_0(t)) = \|\nabla\gamma_0(t)\|_p^p \leqslant \frac{p-1}{C_0}\lambda < c(m) \quad \text{for every } t\in[0,1],$$

where $\tilde{J}$ is the functional defined by (2.8). This contradicts to the definition of $c(m)$. So, we shall prove

$$\int_\Omega m v_+^p\,dx > 0 \quad \text{and} \quad \int_\Omega m v_-^p\,dx > 0$$

in each case of (i) $\sim$ (iii) by noting (4.16) and (4.17).

Case (i): If $\int_\Omega m v_+^p\,dx = 0$, then $v_+$ is a constant function by (4.16). Moreover, because of $\int_\Omega m\,dx > 0$, we see that $v_+ \equiv 0$, and so $v \leqslant 0$ in $\Omega$. Then, by the equality

$$o(1) = \langle I'_\lambda(u_n), 1/\|u_n\|_p^{p-1}\rangle = \lambda\int_\Omega m|v_n|^{p-2}v_n\,dx + \int_\Omega h/\|u_n\|_p^{p-1}\,dx,$$

we have

$$\int_\Omega m|v|^{p-2}v\,dx = -\int_\Omega m v_-^{p-1} = 0$$

(note $\lambda > 0$). This yields that $m(x)v_-^{p-1}(x) = 0$ for a.e. $x\in\Omega$ (note $m \geqslant 0$ in $\Omega$). Thus, $m(x)v_-^p(x) = 0$ for a.e. $x\in\Omega$. Therefore, (4.17) shows that $v_-$ is a constant function, and so

$$v = -v_- \equiv 0 \quad \text{by} \quad \int_\Omega m v_-^p\,dx = 0.$$

This contradicts to $\|v\|_p = 1$. Hence, we have $\int_\Omega m v_+^p\,dx > 0$. Similarly, we see that $\int_\Omega m v_-^p\,dx > 0$.

Case (ii): First, let $\int_\Omega m v_+^p\,dx = 0$ occur. Then, by the same argument as in case (i), we have $v \leqslant 0$ in $\Omega$ and

$$\int_\Omega m|v|^{p-2}v\,dx = -\int_\Omega m v_-^{p-1} = 0.$$

If $\int_\Omega m v_-^p\,dx > 0$ holds, then $v_-/(\int_\Omega m v_-^p\,dx)^{1/p} \in X(m)$ and we have

$$\frac{\|\nabla v_-\|_p^p}{\int_\Omega m v_-^p\,dx} \leqslant (p-1)\frac{\lambda}{C_0} < \lambda_{X(m)}$$

by (4.17). This contradicts to the definition of $\lambda_{X(m)}$.

On the other hand, if $\int_\Omega m v_-^p\,dx = 0$, then $v_-$ is a constant function by (4.17). Hence we obtain a contradiction in this case also since

$$0 = \int_\Omega m v_-^p\,dx = v_-^p \int_\Omega m\,dx = \frac{1}{|\Omega|}\int_\Omega m\,dx > 0$$

(note $\|v\|_p = 1$ and also that $v_-$ is a constant function). Consequently, we have shown $\int_\Omega m v_+^p\,dx > 0$.

Similarly, we can prove that $\int_\Omega m v_-^p\,dx > 0$ by a similar argument above with $v_+$ instead of $v_-$.

Case (iii): We consider by dividing into the following three cases:

(a) $\int_\Omega m v_+^p\,dx = 0 = \int_\Omega m v_-^p\,dx$;

(b) $\int_\Omega m v_+^p\,dx > 0 = \int_\Omega m v_-^p\,dx$;

(c) $\int_\Omega m v_+^p\,dx = 0 < \int_\Omega m v_-^p\,dx$.

In the case of (a), it follows from (4.16) and (4.17) that $v$ is a constant function. Thus, $v = 1/|\Omega|^{1/p}$ or $v = -1/|\Omega|^{1/p}$ occurs. First, we shall deal with the case of $v = 1/|\Omega|^{1/p} > 0$. Thus, we may assume that $u_n \geqslant \|u_n\|_p/2|\Omega|^{1/p}$ in $\overline{\Omega}$ for sufficiently large $n$ (note $N < p$ and so $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact). So, we obtain

$$\|\nabla\left(1/u_n^{p-1}\right)\|_p \leqslant \frac{2^p(p-1)\|\nabla v_n\|_p}{|\Omega|\|u_n\|_p^{p-1}} \quad \text{and} \quad \|\left(1/u_n^{p-1}\right)\|_p \leqslant \frac{2^{p-1}|\Omega|}{\|u_n\|_p^{p-1}}, \tag{4.18}$$

and so $1/u_n^{p-1} \in W^{1,p}(\Omega)$ for such sufficiently large $n$. Here, we fix any $\varphi \in C^1(\overline{\Omega})$ such that $\varphi \geqslant 0$ in $\Omega$. By taking the limit in the following inequality

$$\begin{aligned} A_p\|\nabla\varphi\|_p^p &\geqslant \int_\Omega A(x,\nabla u_n)\nabla\left(\frac{\varphi^p}{u_n^{p-1}}\right)dx \\ &= \lambda\int_\Omega m\varphi^p\,dx + \int_\Omega \frac{h\varphi^p}{u_n^{p-1}}\,dx + \langle I_\lambda'(u_n), \varphi^p/u_n^{p-1}\rangle \end{aligned}$$

(note $\|\varphi^p/u_n^{p-1}\| = o(1)$ by (4.18)), where the first inequality is shown by Proposition 2, we have

$$A_p\|\nabla\varphi\|_p^p \geqslant \lambda \int_\Omega m\varphi^p\,dx$$

for every $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$ in $\Omega$. Since $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, we obtain

$$A_p\|\nabla\varphi\|_p^p \geqslant \lambda \int_\Omega m\varphi^p\,dx$$

for every $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geqslant 0$ in $\Omega$. Because we can choose $\varphi_k \in W^{1,p}(\Omega)$ such that $\varphi_k \geqslant 0$ in $\Omega$, $\int_\Omega m\varphi_k^p\,dx = 1$ and $\|\nabla\varphi_k\|_p^p < \lambda^*(m) + 1/k$ (we consider $|\varphi_k|$ instead of $\varphi_k$ if necessary), we have a contradiction.

In the case of $v = -1/|\Omega|^{1/p} < 0$ also, we have a contradiction by using $-u_n$ instead of $u_n$ as in the above argument (note that $A$ is odd in the second variable).

In the case of (b), it is easily seen that $v = v_+ \geqslant 0$ holds by (4.17) and

$$\int_\Omega mv_-^p\,dx = 0 < \int_\Omega mv_+^p\,dx.$$

Since we obtain

$$\frac{\|\nabla v_+\|_p^p}{\int_\Omega mv_+^p\,dx} \leqslant (p-1)\frac{\lambda}{C_0} < \tilde{\lambda}(m)$$

by (4.16) and $\int_\Omega mv_+^p\,dx > 0$, it follows that $v_+$ has no zero points in $\overline{\Omega}$ from the definition of $\tilde{\lambda}(m)$. This means that $v > 0$ in $\overline{\Omega}$. Thus, we may assume that $u_n \geqslant \delta\|u_n\|_p/2$ in $\overline{\Omega}$ for sufficiently large $n$, where $\delta = \min_{\overline{\Omega}} v(x)$ because the inclusion of $W^{1,p}(\Omega)$ to $C(\overline{\Omega})$ is compact. So, we can get a contradiction by the same argument as in the case of (b) under $v > 0$.

In the case of (c), we see that $v < 0$ in $\overline{\Omega}$ by a similar argument to the case of (b). This yields a contradiction by a similar argument to the case of (a) under $v < 0$.

To deal with the case of $(AH)$, we prepare the following result.

LEMMA 17. *Assume $(AH)$ and let $\{u_n\} \subset W^{1,p}(\Omega)$ be a Palais-Smale sequence for $I_\lambda$ with $\|u_n\|_p \to \infty$ as $n \to \infty$. Then, $v_n := u_n/\|u_n\|_p$ has a subsequence strongly convergent to a solution $v$ for*

$$-\operatorname{div}\left(a_\infty(x)|\nabla v|^{p-2}\nabla v\right) = \lambda m|v|^{p-2}v \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{4.19}$$

*where $a_\infty$ is the positive function as in $(AH)$.*

*Proof.* By the same argument as in the proof of Proposition 6, we can show the boundedness of $\|v_n\|$ and obtain the inequality

$$\frac{C_0}{p-1}\|\nabla v_n\|_p^p \leqslant \lambda \int_\Omega m|v_n|^p\,dx + o(1) \quad \text{as } n \to \infty. \tag{4.20}$$

So, we may suppose, by choosing a subsequence, that there exists a $v \in W^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{in } W^{1,p}(\Omega) \quad \text{and hence} \quad v_n \to v \quad \text{in } L^p(\Omega).$$

To prove that $v_n$ strongly converges to $v$ in $W^{1,p}(\Omega)$, it suffices to show

$$\lim_{n\to\infty} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v)\, dx = 0 \tag{4.21}$$

because the $p$-Laplace operator has the $(S)_+$ property. To obtain (4.21), we shall get the following

$$\lim_{n\to\infty} \left| \frac{1}{\|u_n\|_p^{p-1}} \int_\Omega \tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v)\, dx \right| = 0, \tag{4.22}$$

where $\tilde{a}$ is the function as in $(AH)$. Here, we fix an any $\varepsilon > 0$. By the property of the function $\tilde{a}$, there exist $R > 0$ and $C > 0$ such that

$$|\tilde{a}(x,t)| \leqslant \varepsilon |t|^{p-2} \text{ if } |t| \geqslant R \text{ and } \quad |\tilde{a}(x,t)| \leqslant C \text{ if } |t| \leqslant R. \tag{4.23}$$

Therefore, we obtain

$$\begin{aligned}
&\left| \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n}{\|u_n\|_p^{p-1}} \nabla (v_n - v)\, dx \right| \qquad (4.24)\\
&\leqslant \int_{|\nabla u_n| \geqslant R} \varepsilon (|\nabla v_n|^p + |\nabla v_n|^{p-1} |\nabla v|)\, dx + \int_{|\nabla u_n| \leqslant R} \frac{C|\nabla u_n|}{\|u_n\|_p^{p-1}} (|\nabla v_n| + |\nabla v|)\, dx \\
&\leqslant \varepsilon (\|\nabla v_n\|_p^p + \|\nabla v_n\|_p^{p-1} \|\nabla v\|_p) + RC(\|\nabla v_n\|_p + \|\nabla v\|_p) \frac{|\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}} \\
&\leqslant \frac{2\varepsilon |\lambda| (p-1) \|m\|_\infty}{C_0} + o(1) + RC \left( \frac{2|\lambda|(p-1)\|m\|_\infty}{C_0} + o(1) \right)^{1/p} \frac{|\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}
\end{aligned}$$

by (4.20), $\|v_n\|_p = 1$ and Hölder's inequality. Thus, by taking the limit superior in the above inequality, we can get

$$\limsup_{n\to\infty} \left| \frac{1}{\|u_n\|_p^{p-1}} \int_\Omega \tilde{a}(x, \nabla u_n) \nabla u_n \nabla (v_n - v)\, dx \right| \leqslant \frac{2\varepsilon |\lambda| (p-1) \|m\|_\infty}{C_0}$$

since $\|u_n\|_p \to \infty$ as $n \to \infty$. This implies (4.22) because $\varepsilon > 0$ is arbitrary. By taking the limit in the following and noting (4.22)

$$\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), v_n - v \rangle / \|u_n\|_p^{p-1} \\
&= \int_\Omega a_\infty |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v)\, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n}{\|u_n\|_p^{p-1}} \nabla (v_n - v)\, dx \\
&\quad - \lambda \int_\Omega m |v_n|^{p-2} v_n (v_n - v)\, dx - \int_\Omega \frac{h}{\|u_n\|_p^{p-1}} (v_n - v)\, dx,
\end{aligned}$$

we have the inequality (4.21) (note $\inf_\Omega a > 0$), whence $v_n \to v$ in $W^{1,p}(\Omega)$.

Finally, we shall show that $v$ is a solution for (4.19). So, we fix any $\varphi \in W^{1,p}(\Omega)$. Then, by considering $\varphi$ instead of $(v_n - v)$ in (4.24), we have the following inequality for every $\varepsilon > 0$:

$$\left| \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla\varphi\, dx \right| \leqslant \varepsilon \|\nabla v_n\|_p^{p-1} \|\nabla\varphi\|_p + \frac{CR\|\nabla\varphi\|_p |\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}$$
$$\leqslant \varepsilon \left( \frac{|\lambda|(p-1)\|m\|_\infty}{C_0} + o(1) \right)^{(p-1)/p} \|\nabla\varphi\|_p + \frac{CR\|\nabla\varphi\|_p |\Omega|^{(p-1)/p}}{\|u_n\|_p^{p-1}}.$$

This gives

$$\lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla\varphi\, dx = 0 \tag{4.25}$$

since $\varepsilon > 0$ is arbitrary. By taking the limit in

$$o(1) = \frac{\langle I'_\lambda(u_n), \varphi\rangle}{\|u_n\|_p^{p-1}},$$

we have

$$\int_\Omega a_\infty |\nabla v|^{p-2} \nabla v \nabla\varphi\, dx = \lambda \int_\Omega m|v|^{p-2} v\varphi\, dx$$

by (4.25), $v_n \to v$ in $W^{1,p}(\Omega)$ and $\|u_n\|_p \to \infty$ as $n \to \infty$. Because $\varphi$ is any function in $W^{1,p}(\Omega)$, our conclusion is shown.

Now, we state the result in the case of $(AH)$.

PROPOSITION 7. *Assume $(AH)$ and $\lambda$ is not an eigenvalue of*

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda m|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{4.26}$$

*where $a_\infty$ is the positive function as in $(AH)$. Then, $I_\lambda$ satisfies the Palais-Smale condition.*

*Proof.* Let $\{u_n\}$ be a Palais-Smale sequence of $I_\lambda$, namely,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \|I'_\lambda(u_n)\|_{W^*} \to 0 \quad \text{as } n \to \infty$$

for some $c \in \mathbb{R}$. It is sufficient to prove only the boundedness of $\|u_n\|_p$ by the same reason as in the proof of Proposition 6. So, by contradiction, we may suppose that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence. Set $v_n := u_n/\|u_n\|_p$. Then, it follows from Lemma 17 that $\{v_n\}$ has a subsequence strongly convergent to a non-trivial solution $v$ for (4.26) with $\|v\|_p = 1$. This is a contradiction because $\lambda$ is not an eigenvalue of (4.26).

REMARK 8. Concerning the existence of a solution, under the Dirichlet boundary condition also, we can similar results to the ones as in section 4 by using several constants corresponding to the Dirichlet problem.

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