Differential
Equations
& Applications
Volume 5, Number 1 (2013), 1–12
doi:10.7153/dea-05-01

COMPOSITION CONDITIONS FOR TWO–DIMENSIONAL POLYNOMIAL SYSTEMS

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(*Communicated by Vesna Županović*)

Abstract. Several papers published recently about the center composition condition of Abel differential equations and two-dimensional polynomial systems with homogeneous nonlinearities. We give necessary and sufficient conditions for the existence of composition centers for general two-dimensional polynomial systems.

1. Introduction

Consider the system

$$\begin{aligned} \dot{x} = \frac{dx}{dt} &= -y + \sum_2^n p_j(x,y), \\ \dot{y} = \frac{dy}{dt} &= x + \sum_2^n q_j(x,y) \end{aligned} \tag{1}$$

where, p_j and q_j are homogeneous polynomials in x and y of degree j. Recall that the center problem is to characterize the coefficients that imply the origin is a center. In polar coordinates, the system becomes

$$\frac{dr}{d\theta} = \frac{\sum_2^n r^j f_{j+1}(\theta)}{1 + \sum_2^n r^{j-1} g_{j+1}(\theta)} \tag{2}$$

with

$$f_{j+1}(\theta) = \cos\theta\, p_j(\cos\theta, \sin\theta) + \sin\theta\, q_j(\cos\theta, \sin\theta)$$
$$g_{j+1}(\theta) = \cos\theta\, q_j(\cos\theta, \sin\theta) - \sin\theta\, p_j(\cos\theta, \sin\theta).$$

In the case of homogeneous nonlinearities, the system can be transformed to Abel differential equation

$$\frac{dR}{d\theta} = A(\theta)R^3 + B(\theta)R^2, \tag{3}$$

where,

$$R = r^{n-1}(1 + r^{n-1}g(\theta))^{-1},$$

Mathematics subject classification (2010): 34C07, 34C05, 34C25, 37G15.
Keywords and phrases: two-dimensional polynomial systems, center problem, composition condition, Abel differential equation.

Paper DEA-05-01

and

$$A(\theta) = -(n-1)f(\theta)g(\theta), B(\theta) = (n-1)f(\theta) - g'(\theta).$$

The origin is a center for the two-dimensional system if and only if all solutions of the Abel equation starting near the origin are periodic with period 2π. In this case, we say that $R = 0$ is a center for the Abel equation. The origin is a center when the coefficients satisfy the following condition

$$A(\theta) = u'(\theta)A_1(u(\theta)), B(\theta) = u'(\theta)B_1(u(\theta)), \tag{4}$$

where u is a periodic function of period 2π, A_1, B_1 are continuous functions. This condition is called the *composition condition*. However, it was shown in [3] that the converse is not true; there are centers in which the coefficients are trigonometric polynomials and do not satisfy the composition condition. For the case of polynomial coefficients, the problem is still unsolved. The problem is considered in several articles and in particular its relations with the moments of polynomials and universal centers; see, for example, [2], [5], [6], [7], [8], [10], and [14]. In a recent paper [9], Cima, Gasull, and Manosas characterize all the centers in which the coefficients satisfy the composition condition for Abel differential equaltion. It came to our attention that a paper of Al'muhamedov [1] published in 1949 contains a characterization of composition centers for general polynomial systems and the results are not restricted to systems with homogeneous nonlinearities. A counterexample is presented in Section 3 to show that the result about the relation between stable centers and composition centers is not true. We give a similar result in this paper. In Section 2, we prove the results related to the composition condition. Section 3 contains the counterexample and the result related to stable centers. We use the same approach of [1]; the first four theorems are essentially given in [1]. We present complete and clear proofs to the results. Finally, we relate the stable centers to the result in [9].

2. Composition conditions

To distinguish a center from a focus, we look for a closed curve defined by the equation

$$\rho + \rho^2 u_1(\theta) + \rho^3 u_2(\theta) + \cdots = \varepsilon, \tag{5}$$

where, u_j are trigonometric polynomials and ε is a sufficiently small positive number. If $\frac{dr}{d\theta} - \frac{d\rho}{d\theta}$ has a constant sign at $r = \rho$, then the origin is a focus. Therefore, the origin is a center if and only if $\frac{dr}{d\theta} - \frac{d\rho}{d\theta} = 0$ at $r = \rho$. Now, we compute $\frac{dr}{d\theta} - \frac{d\rho}{d\theta}$ at $r = \rho$.

$$\frac{dr}{d\theta} - \frac{d\rho}{d\theta} = \frac{\sum_2^n r^j f_{j+1}}{1 + \sum_2^n r^{j-1} g_{j+1}} - \frac{\sum_1^\infty r^{j+1} u'_j}{\sum_1^\infty (j+1) r^j u_j}.$$

The condition $\frac{dr}{d\theta} - \frac{d\rho}{d\theta} = 0, r = \rho$ implies that

$$u'_k = f_{k+2} + \sum_2^k [(k-j+2) f_{j+1} u_{k-j+1} - g_{j+1} u'_{k-j+1}].$$

Therefore, the functions u_k are trigonometric polynomials if and only if the integrals of the right hand side in this recursive formula vanish for all $k \geqslant 1$. This proves the following result.

THEOREM 1. *The origin is a center for the system (1) if and only if*

$$\int_0^{2\pi}[f_{k+2}+\sum_2^k((k-j+2)f_{j+1}u_{k-j+1}-g_{j+1}u'_{k-j+1})]d\theta=0, k\geqslant 1.$$

The recursive formula in Theorem 1 is linear and the number of terms does not increase when k increases. The values of the integrals are the Poincare-Liapunov constants and were obtained in [4] using Liapunov functions. Theorem 1 is given in [1] with a mistake in the sign. The following corollary follows directly from Theorem 1.

COROLLARY 1. *If* (f_{m+1}, g_{m+1}) *is the first non-zero pair, then*

$$u'_k = 0, 1 \leqslant k \leqslant m-2,$$
$$u'_k = f_{k+2}, m-1 \leqslant k \leqslant 2m-3.$$

Now, we write the polynomials in (2) in the form

$$\begin{aligned} f_k(\theta) &= \sum_{j=0}^{n_k}(a_{kj}e^{ij\theta}+b_{kj}e^{-ij\theta}), \\ g_k(\theta) &= \sum_{j=0}^{n_k}(c_{kj}e^{ij\theta}+d_{kj}e^{-ij\theta}). \end{aligned} \tag{6}$$

With this form, integration and manipulating trigonometric polynomials become much easier. The integral of an exponential function is one term, and the integral of

$$\cos^k\theta\sin^{n-k}\theta$$

has n terms. In particular, the center conditions in Theorem 1 are simply the constant terms in the integrand. Since f_k is a homogeneous polynomial in $\cos\theta$ and $\sin\theta$ of degree k, $a_{kj} = b_{kj} = 0$ when k and j do not have the same parity.

The composition condition for the system (2) is given by

DEFINITION 1. The *Composition Condition* is satisfied if there exists a trigonometric polynomial $\lambda(\theta)$ such that

$$f_k(\theta) = \lambda'(\theta)\sum p_{kj}\lambda^j(\theta),\ \ g_k(\theta) = \sum q_{kj}\lambda^j(\theta), \tag{7}$$

for $3 \leqslant k \leqslant n+1$.

The second result is that the composition condition implies that the origin is a center.

THEOREM 2. *If the coefficients in equation* (2) *satisfy the composition condition then the origin is a center.*

It follows, inductively, that $u_j(\theta)$ are polynomials in $\lambda(\theta)$. Therefore, the conditions in Theorem 1 are satisfied. When the coefficients satisfy the composition condition, the center is called a *composition center.*

The next result is a generalization of Theorem 2.

THEOREM 3. *Consider the differential equation*

$$\frac{dr}{d\theta} = -\frac{F(r,\theta)}{G(r,\theta)},$$

where

$$F(r,\theta) = \sum q_{jl}\lambda' \mu^j \lambda^l + (1+\sum p_{km}\mu^k\lambda^m)\frac{\partial \mu}{\partial \theta},$$

$$G(r,\theta) = (1+\sum p_{km}\mu^k\lambda^m)\frac{\partial \mu}{\partial r}.$$

The function $\lambda(\theta)$ is a trigonometric polynomial and

$$\mu = r^h + \sum r^{h+k}\mu_k(\theta), h > 0.$$

The equation has a center at the origin.

Proof. With the transformation $r \mapsto \mu$, the equation becomes

$$\frac{d\mu}{d\theta} = \frac{\partial \mu}{\partial r}\frac{dr}{d\theta} + \frac{\partial \mu}{\partial \theta} = \frac{\partial \mu}{\partial r}(\frac{-F}{G}) + \frac{\partial \mu}{\partial \theta}.$$

Substituting the values of F and G and simplifying the fractions give

$$\frac{d\mu}{d\theta} = -\frac{\sum q_{jl}\lambda'\lambda^j\mu^l}{1+\sum p_{km}\lambda^k\mu^m}.$$

This is a differential equation in which the coefficients satisfy the composition conditions. Now the result follows from Theorem 2. □

Theorem 3 is given in [1] without a proof. The following theorem follows from Theorem 1 and Hilbert's basis theorem.

THEOREM 4. *The problem of distinguishing a focus from a center may always be solved by verifying a finite number of conditions.*

It is not clear how to relate the following result of [1] to polynomial systems. We give an alternative statement with a proof.

THEOREM 5. *Consider the differential equation*

$$\frac{dr}{d\theta} = \frac{r^2 f(\theta)}{1+rg(\theta)},$$

where $f(\theta)$ and $g(\theta)$ are trigonometric polynomials of degree n. By adding terms of higher degree to the functions f and g the origin may always be made into a center, and for this one requires not more than $(2n+1)n$ terms in each of them provided that $f(\theta)$ does not contain a constant term.

We present the following version.

THEOREM 5'. Consider the differential equation

$$\frac{dr}{d\theta} = \frac{r^n f(\theta)}{1+r^{n-1}g(\theta)}$$

where $f(\theta)$ and $g(\theta)$ are trigonometric polynomials of degree $n+1$. If $f(\theta)$ does not contain a constant term, then the differential equation

$$\frac{dr}{d\theta} = \frac{r^n(f(\theta)+g'(\theta)+k'(\theta))}{1+r^{n-1}(g(\theta)+\overline{f}(\theta)+k(\theta))}$$

has a composition center at the origin for any trigonometric polynomial $k(\theta)$. In the last equation $\overline{f}$ is the indefinite integral of f.

Proof. Since $\overline{f}$ does not contain a constant term, $\lambda = g+\overline{f}+k$ is a periodic function. The result follows from Theorem 2 with $\lambda = g+\overline{f}+k$. It should be mentioned that in the case of homogeneous nonlinearities, the change of variables $r \mapsto r^{n-1}$ reduces the equation to the form

$$\frac{dr}{d\theta} = \frac{r^2 f(\theta)}{1+rg(\theta)}.$$

This explains why $n=2$ in Theorem 5. □

3. Stable centers

In this section, we consider cases in which a center remains a center under certain changes of some or all of the coefficients in the forms (6). Such centers are called *stable* with respect to these coefficients. For example, the composition center is stable with respect to the coefficients p_{kj}, q_{kj} in the form (7). The next result relates composition centers and stable centers.

THEOREM 6. *Consider the system*

$$\frac{dr}{d\theta} = \frac{\sum_2^n r^{k-1} f_k(\theta)}{1+\sum_2^n r^{k-2} g_k(\theta)},$$

with

$$f_k(\theta) = \sum_{j=0}^{n_k} \varepsilon_{kj} f_{kj}(\theta),\ f_{kj}(\theta) = a_{kj}e^{ij\theta} + b_{kj}e^{-ij\theta},$$

$$g_k(\theta) = \sum_{j=0}^{n_k} \delta_{kj} g_{kj}(\theta),\ g_{kj}(\theta) = c_{kj}e^{ij\theta} + d_{kj}e^{-ij\theta}.$$

Assume that there exists k such that f_{k1} is not identically zero. The system has a center at the origin, for all ε_{kj} and δ_{kj}, if and only if the coefficients are of the form

$$f_{kj} = p_{kj}(\lambda_1^j e^{ij\theta} - \lambda_2^j e^{-ij\theta}),\ g_{kj} = q_{kj}(\lambda_1^j e^{ij\theta} + \lambda_2^j e^{-ij\theta}).$$

This center is a composition center. Moreover, all solutions are symmetric about a line through the origin.

To prove Theorem 6, we need the following lemmas.

LEMMA 1. *If the equation*

$$\frac{dr}{d\theta} = \frac{r^n f(\theta)}{1+\varepsilon r^{n-1} g(\theta)}$$

has a center at the origin for all small values of ε, *then*

$$\int_0^{2\pi} f(\theta)g(\theta)(\bar{f}(\theta))^m d\theta = 0, \;\; \bar{f}(\theta) = \int_0^{\theta} f(u)du,$$

for all non-negative integers m.

Proof. As we mentioned in the proof of Theorem 5', we may assume that $n=2$. Let $r(\varepsilon,\theta,c)$ be the solution that satisfies the initial condition $r(\varepsilon,0,c)=c$. We start with the equation

$$-\Big(\frac{1}{r(\varepsilon,\theta,c)}\Big)' = \frac{f(\theta)}{1+\varepsilon r(\varepsilon,\theta,c)g(\theta)}.$$

We integrate both sides over the interval $[0,2\pi]$. Since the origin is a center, we have

$$\int_0^{2\pi} \frac{f(\theta)}{1+\varepsilon r(\varepsilon,\theta,c)g(\theta)} d\theta = 0.$$

Now, we differentiate with respect to ε and then substitute $\varepsilon = 0$.

$$\int_0^{2\pi} f(\theta)g(\theta)r(0,\theta,c)d\theta = 0.$$

Integrating the differential equation when $\varepsilon = 0$ gives

$$r(0,\theta,c) = \frac{c}{1-c\bar{f}(\theta)}.$$

Therefore,

$$\int_0^{2\pi} f(\theta)g(\theta)\sum_0^{\infty}(c\bar{f}(\theta))^k d\theta = 0.$$

The result follows from the coefficients of the power series in c. □

LEMMA 2. *If the equation*

$$\frac{dr}{d\theta} = \frac{\varepsilon r^n f(\theta)}{1+r^{n-1}g(\theta)}$$

has a center at the origin for all small values of ε, *then*

$$\int_0^{2\pi} f(\theta)(g(\theta))^m d\theta = 0,$$

for all non-negative integers m.

Proof. Again, we may assume that $n=2$. Let $r(\varepsilon,\theta,c)$ be the solution that satisfies the initial condition $r(\varepsilon,0,c)=c$. We start with the equation

$$-\Big(\frac{1}{r(\varepsilon,\theta,c)}\Big)' = \frac{\varepsilon f(\theta)}{1+r(\varepsilon,\theta,c)g(\theta)}.$$

We integrate both sides over the interval $[0,2\pi]$. Since the origin is a center, we have

$$\int_0^{2\pi} \frac{\varepsilon f(\theta)}{1+r(\varepsilon,\theta,c)g(\theta)} d\theta = 0.$$

Now, we differentiate with respect to ε and then substitute $\varepsilon=0$.

$$\int_0^{2\pi} \frac{f(\theta)}{1+g(\theta)r(0,\theta,c)} d\theta = 0.$$

Integrating the differential equation when $\varepsilon=0$ gives

$$r(0,\theta,c)=c.$$

Therefore,

$$\int_0^{2\pi} \frac{f(\theta)}{1+cg(\theta)} d\theta = 0.$$

The result follows from the coefficients of the power series in c. □

LEMMA 3. *Consider the differential equation*

$$\frac{dr}{d\theta} = h(\theta)r^m + \varepsilon f(\theta)r^n, 2<m, 2<n, m\neq n.$$

If the equation has a center at the origin for all small ε, then

$$\int_0^{2\pi} f(\theta)(\bar{h}(\theta))^l d\theta = 0,$$

for all non-negative integers l.

Proof. Let $r(\varepsilon,\theta,c)$ be the solution that satisfies the initial conditions $r(\varepsilon,0,c)=c$. The method of proof is similar to that in Lemma 1. We start with

$$\frac{dr}{r^m} = (h(\theta)+\varepsilon r^{n-m} f(\theta))d\theta.$$

Now, we integrate over the interval $[0,2\pi]$ and use $r(\varepsilon,0,c)=r(\varepsilon,2\pi,c)$. This implies that

$$\int_0^{2\pi} [h(\theta)+\varepsilon r^{n-m} f(\theta)] d\theta = 0.$$

Now, we differentiate with respect to ε and then substitute $\varepsilon = 0$.

$$\int_0^{2\pi} r^{n-m}(\varepsilon,0,c)f(\theta)d\theta = 0.$$

But, $r(0,\theta,c) = [(1-m)\bar{h}+c^{1-m}]^{\frac{1}{1-m}}$. Substitute this solution in the previous equation to obtain

$$\int_0^{2\pi}\left[f(\theta)\left(\frac{c^{m-1}}{1-(m-1)c^{m-1}\bar{h}(\theta)}\right)^{\frac{n-m}{m-1}}\right]d\theta = 0.$$

Now the result follows from the expansion of this function as a power series around $c = 0$. □

LEMMA 4. *The polynomial*

$$\alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \cdots + \alpha^2\beta^{n-2} + \alpha\beta^{n-1} + \beta^n$$

with $\alpha\beta = c$ *can be written as a linear combination of the polynomials* $c^{j-1}(\alpha+\beta)^{n-j}$ *where* j *is an even non-negative integer.*

Proof. The the pair of terms $\alpha^n + \beta^n$ are obtained from $(\alpha+\beta)^n$; the remainder of the terms are added to the other terms. Since, the coefficients in the binomial expansion are symmetric, the next pair is a constant multiple of $\alpha^{n-1}\beta + \alpha\beta^{n-1} = c(\alpha^{n-2}+\beta^{n-2})$. Therefore, this pair is obtained from $(\alpha+\beta)^{n-2}$. We continue in this procedure to write all the terms in the required linear combination. □

LEMMA 5. *The terms in* f_k *has the form* $ae^{ij\theta} + \bar{a}e^{-ij\theta}$, *where* $\bar{a}$ *is the complex conjugate of* a. *Moreover, if* f_k *has a pair of terms* $ae^{i\theta} + \bar{a}e^{-i\theta}$ *then* g_k *has a pair of terms* $iae^{i\theta} - i\bar{a}e^{-i\theta}$.

Proof. Assume that $f_k(\theta) = ae^{ij\theta} + be^{-ij\theta}$. Since f_k is a real function, we have

$$ae^{ij\theta} + be^{-ij\theta} = \bar{a}e^{-ij\theta} + \bar{b}e^{ij\theta}.$$

Therefore, $a = \bar{b}$ and $b = \bar{a}$. This proves the first statement. Now, let $a = a_1 + a_2 i$ where a_1, a_2 are real numbers. This implies that

$$ae^{i\theta} + \bar{a}e^{-i\theta} = 2a_1\cos\theta - 2a_2\sin\theta,$$
$$iae^{i\theta} - i\bar{a}e^{-i\theta} = 2a_2\cos\theta - 2a_1\sin\theta.$$

Therefore, the functions p_k and q_k in the system (1) are given by $p_k = -2a_1, q_k = -2a_2$. Hence, the corresponding terms in g_k has the form $2a_2\cos\theta - 2a_1\sin\theta$. □

Proof. (Theorem 6.)
If the condition on the coefficients is satisfied then

$$f_k(\theta) = \sum_0^{n_k} p_{kj}(\lambda_1^j e^{ij\theta} - \lambda_2^j e^{-ij\theta}) = p_{kj}(\lambda_1 e^{i\theta} - \lambda_2 e^{-i\theta})\sum_0^{n_k}[(\lambda_1 e^{i\theta})^{n_k-l}(\lambda_2 e^{-i\theta})^l].$$

By Lemma 3, the last sum is a linear combination in the factors $(\lambda_1 e^{i\theta}+\lambda_2 e^{-i\theta})^l$. In applying Lemma 3, we have $c=\lambda_1 e^{i\theta}\lambda_2 e^{-i\theta}=\lambda_1\lambda_2$ which does not depend on θ. Therefore,

$$f_k(\theta)=(\lambda_1 e^{i\theta}-\lambda_2 e^{-i\theta})P(\lambda_1 e^{i\theta}+\lambda_2 e^{-i\theta}),$$

where P is a polynomial function. Now, the composition condition is satisfied with $\lambda=\lambda_1 e^{i\theta}+\lambda_2 e^{-i\theta}$.

Now, we prove the necessary part. The idea of proof is choosing different values of ε_{kj} and δ_{kj}. Each choice determines a pair of coefficients. We choose $\varepsilon_{kj}=\delta_{kj}=0$, except one pair. That is, we take

$$f_k(\theta)=\varepsilon_{kj}(a_{kj}e^{ij\theta}+b_{kj}e^{-ij\theta}), g_k(\theta)=\delta_{kj}(c_{kj}e^{ij\theta}-d_{kj}e^{-ij\theta}), (a_{kj},b_{kj})\neq(0,0).$$

By assumption, the origin is a center for values of ε_{kj} and δ_{kj}. The second center condition in Lemma 1 implies that

$$\int_0^{2\pi} f_k(\theta)g_k(\theta)d\theta=2\varepsilon_{kj}\delta_{kj}\pi(a_{kj}d_{kj}-b_{kj}c_{kj})=0.$$

Substituting $d_{kj}=\frac{b_{kj}c_{kj}}{a_{kj}}$, or $c_{kj}=\frac{a_{kj}d_{kj}}{b_{kj}}$ gives

$$g_{kj}(\theta)=\frac{\delta_{kj}c_{kj}}{a_{kj}}(a_{kj}e^{ij\theta}+b_{kj}e^{-ij\theta}),$$

if $a_{kj}\neq 0$, and

$$g_{kj}(\theta)=\frac{\delta_{kj}d_{kj}}{b_{kj}}(a_{kj}e^{ij\theta}+b_{kj}e^{-ij\theta}),$$

if $b_{kj}\neq 0$. This shows that any pair of functions, with the same j and k, in the coefficients are of the form given in the statement.

We choose a pair $a_{k1}\neq 0, b_{k1}\neq 0$ and call $\lambda_1=a_{k1},\lambda_2=b_{k1}$. If there are more than one pair then we choose the one with smallest k; we call this integer K. We define $\lambda(\theta)=\lambda_1 e^{i\theta}+\lambda_2 e^{-i\theta}$. Lemma 5 implies that if $f_{k1}\neq 0$ then $g_{k1}\neq 0$.

Now, we take

$$f_K(\theta)=\lambda_1 e^{ij\theta}-\lambda_2 e^{-ij\theta}, g_K(\theta)=c_{Kj}e^{i\theta}+b_{Kj}e^{-i\theta},$$

in the equation

$$\frac{dr}{d\theta}=\frac{\varepsilon_{Kj}r^{K-1}f_K(\theta)}{1+r^{K-2}g_K(\theta)}.$$

The center condition in Lemma 2 implies that

$$\int_0^{2\pi} f_K(\theta)(g_K(\theta))^m d\theta=0,$$

for all non-negative integers m. We take $m=j$. The center condition becomes $\lambda_1^j d_{Kj}-\lambda_2^j c_{Kj}=0$. This implies that

$$g_K(\theta)=\frac{\delta_{Kj}d_{Kj}}{\lambda_2^j}(\lambda_1^j e^{ij\theta}+\lambda_2^j e^{-ij\theta}).$$

In the case that the terms have different k's, we take

$$\frac{dr}{d\theta} = \varepsilon_{Kj} f_K(\theta) r^{K-1} + f_k(\theta) r^{k-1},$$

with

$$f_K(\theta) = \lambda_1 e^{i\theta} - \lambda_2 e^{-i\theta}, f_k(\theta) = a_{kj} e^{ij\theta} - b_{kj} e^{-ij\theta}, k > K.$$

The conditions in Lemma 3 imply that

$$\int_0^{2\pi} f_k(\theta)(\bar{f}_K(\theta))^m d\theta = 0.$$

In this case, we take $m = j$. The center condition becomes $a_{KJ}^j b_{kjj} - b_{KJ}^j a_{kjj} = 0$. Therefore,

$$f_k(\theta) = \frac{-b_{kj}\varepsilon_{KJ}}{\lambda_2^j}(\lambda_1^j e^{ij\theta} - \lambda_2^j e^{-ij\theta}).$$

Therefore, the composition conditions are satisfied with $\lambda = \lambda_1 e^{i\theta} + \lambda_2 e^{-i\theta}$.

It follows from the above steps that $r(\theta)$ is a function of $\lambda = \lambda_1 e^{i\theta} + \lambda_2 e^{-i\theta}$. If α is the angle defined by $\lambda_1 e^{i\alpha} = \lambda_2 e^{-i\alpha}$ then $\lambda(\alpha - \theta) = \lambda(\alpha + \theta)$. Therefore the solution $r(\theta)$ is symmetric about the line $\theta = \alpha$. □

It should be mentioned that Theorem 6 characterizes all composition centers of symmetry type. Theorem 6 is presented in [1] without the condition that f_k has a pair of terms of the form $ae^{i\theta} + be^{-i\theta}$. The following theorem gives a class of systems to show that the result is not true without this extra condition.

THEOREM 7. *The equation*

$$\frac{dr}{d\theta} = r^n f(\theta),$$

where

$$f(\theta) = \varepsilon_1(ae^{ji\theta} + \bar{a}e^{-ji\theta}) + \varepsilon_2(be^{(j+2)i\theta} + \bar{b}e^{-(j+2)i\theta}) + \varepsilon_3(ce^{(j+4)i\theta} + \bar{c}e^{-(j+4)i\theta}).$$

has a center at the origin for all $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and fixed coefficients a, b, c. Moreover, if either $\bar{c} = a$ and b^2 is not a real number, or $\bar{c} \neq a$ and b^2 is a real number, then terms of $f(\theta)$ do not satisfy the conclusion of Theorem 6.

Proof. The origin is a center because $\int_0^{2\pi} f(\theta) d\theta = 0$. Now, assume that there exists λ_1 such that

$$a = A\lambda_1^j, b = B\lambda_1^{j+2}, c = C\lambda_1^{j+4},$$

for some real constants A, B, C. This implies that

$$\lambda_1^2 = \frac{Ab}{Ba} = \frac{Bc}{Cb}.$$

Therefore, $ACb^2 = B^2 ac$. This contradicts the conditions on a, b, and c. □

The following interesting result is given recently in [9]. The result characterize all composition centers of Abel differential equation.

THEOREM 8. [9] *The Abel differential equation*

$$\frac{dr}{d\theta} = A(\theta)r^3 + B(\theta)r^2$$

with trigonometric polynomials coefficients has a composition center at the origin if and only if

$$\frac{dr}{d\theta} = (\varepsilon_1 A(\theta) + \varepsilon_2 B(\theta))r^3 + (\varepsilon_3 A(\theta) + \varepsilon_4 B(\theta))r^2$$

has a center at the origin for all $\varepsilon_k, k = 1,2,3,4$.

The following particular case of Theorem 6 gives a similar result for general two-dimensional polynomial systems.

COROLLARY 2. *Consider the differential equation*

$$\frac{dr}{d\theta} = r^2 f_1(\theta) + r^3 f_2(\theta) + \cdots + r^n f_{n-1}(\theta),$$

with

$$f_k(\theta) = \sum_{j=0}^{n_k} \varepsilon_{kj} f_{kj}(\theta), \; f_{kj}(\theta) = a_{kj} e^{ij\theta} + b_{kj} e^{-ij\theta}.$$

If the origin is a center for all ε_k *and there exists* K *such that* $f_{K1} \neq 0$*, then the composition condition is satisfied with* $\lambda = f_{K1}$*, and* $r = 0$ *is a composition center.*

REMARK 1. The conditions for composition centers in Corollary 9 can not be satisfied by a two-dimensional polynomial system (1). It follows from Lemma 5 that $g_{k1} \neq 0$ when $f_{k1} \neq 0$. This is the only result in this paper that deals with equation (2) and not with equation (1).

Acknowledgements. I would like to thank Dr. Vidya Swaminathan for a very fruitful discussion.

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(Received April 19, 2012)

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