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MATCHING METHOD FOR NODAL SOLUTIONS OF MULTI–POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study the nonlinear boundary value problem consisting of the equation $y'' + w(t)f(y) = 0$ on $[a,b]$ and two multi-point boundary conditions. We establish the existence of various nodal solutions of this problem by matching the solutions of two boundary value problems, each of which involves one separated boundary condition and one multi-point boundary condition, at some point in (a,b). We also obtain conditions for this problem not to have certain types of nodal solutions.

1. Introduction

We study the nonlinear boundary value problem (BVP) consisting of the equation

$$y'' + w(t)f(y) = 0, \quad t \in (a,b), \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $a < b$; and the multi-point boundary condition (BC)

$$y(a) - \sum_{j=1}^{l} h_j y(\xi_j) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0. \tag{1.2}$$

Throughout this paper and without further mention we assume the following:

(H1) $w \in C^1[a,b]$ such that $w(t) > 0$ on $[a,b]$;

(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, $f(-y) = -f(y)$, and f is locally Lipschitz on $(-\infty,0) \cup (0,\infty)$;

(H3) there exist extended real numbers $f_0, f_\infty \in [0,\infty]$ such that

$$f_0 = \lim_{y\to 0} f(y)/y \quad \text{and} \quad f_\infty = \lim_{|y|\to\infty} f(y)/y;$$

(H4) $a < \eta_1 < \ldots < \eta_m < b$ and $k_i \in \mathbb{R}$ for $i = 1,\ldots,m$;

(H5) $a < \xi_1 < \ldots < \xi_l < b$ and $h_j \in \mathbb{R}$ for $j = 1,\ldots,l$.

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The existence of solutions, especially positive solutions, of BVPs with multi-point BCs have been studied extensively, see [2, 5, 8, 9, 10, 18, 19, 30] and the references therein. In this paper, we study the existence of nodal solutions, i.e., solutions with a specific zero-counting property in (a,b), of the multi-point BVP (1.1), (1.2). Great progress has been made to the study of such solutions for nonlinear BVPs consisting of Eq. (1.1) (and more general forms of equations) and two-point separated BCs, see [11, 12, 14, 21, 24, 25, 26]. The existence of nodal solutions of BVPs with nonlocal BCs has also received a lot of attention in research. We refer the reader to [1, 3, 4, 6, 11, 13, 20, 22, 23, 27, 28, 29] for some recent work on this topic. In particular, many researchers have been working on the existence of nodal solutions of the BVP consisting of Eq. (1.1) and the separated–multi-point BC

$$\begin{aligned} &\cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, \quad \alpha \in [0,\pi), \\ &y(b) - \textstyle\sum_{i=1}^{m} k_i y(\eta_i) = 0, \end{aligned} \tag{1.3}$$

where $a,b \in \mathbb{R}$ with $a<b$. However, due to the complexity of BC (1.3), the majority of the results are only for a special case of BVP (1.1), (1.3). In fact, Ma [22], Ma and O'Regan [23], Rynne [27], Xu [28], and Xu, Sun, and O'Regan [29] studied the special case of BVP (1.1), (1.3) with $w \equiv 1$, $\alpha = 0$, and $[a,b] = [0,1]$, i.e., the BVP consisting of the equation

$$-y'' = f(y),\ t \in (0,1), \tag{1.4}$$

and the BC

$$y(0) = 0, \quad y(1) - \sum_{i=1}^{m} k_i y(\eta_i) = 0. \tag{1.5}$$

The main approach was to use the Rabinowitz global bifurcation method to establish the existence of nodal solutions of BVP (1.4), (1.5) by relating it to the eigenvalues of the corresponding linear Sturm-Liouville problem (SLP) with the multi-point BC (1.5). By extending and improving the work in Ma and O'Regan [23], Rynne [27] showed that the associated SLP consisting of the equation $y'' + \lambda y = 0$ and BC (1.5) has a strictly increasing sequence of simple eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with eigenfunctions $\phi_n(t) = \sin(\sqrt{\lambda_n} t)$. Let $\mathbb{N}_0 = \{0,1,2,\dots\}$. The following is a brief sketch of the results in [27] on the existence of nodal solutions of BVP (1.4), (1.5).

PROPOSITION 1.1. *Let $f \in C^1(\mathbb{R})$ and $k_i > 0$ for $i = 1,\dots,m$ such that* $\sum_{i=1}^{m} k_i < 1$.

(a) *Assume $f_0, f_\infty < \infty$, and $(\lambda_n - f_0)(\lambda_n - f_\infty) < 0$ for some $n \in \mathbb{N}_0$. Then BVP (1.4), (1.5) has solutions $y^\pm$ whose derivatives have exactly $n+1$ zeros in $(0,1)$ such that $\pm y^\pm(t) > 0$ in a right-neighborhood of 0.*

(b) *Assume $f_\infty = \infty$ and $\lambda_n > f_0$ for some $n \in \mathbb{N}_0$. Then for any $i \geqslant n$, BVP (1.4), (1.5) has solutions $y^\pm$ whose derivatives have exactly $i+1$ zeros in $(0,1)$ such that $\pm y^\pm(t) > 0$ in a right-neighborhood of 0.*

The establishment of these results relies heavily on the direct computations of the eigenvalues and eigenfunctions of the SLP associated with BVP (1.4), (1.5), and hence cannot be extended to the general BVP (1.1), (1.3) with a variable w and general BC parameter α by the same approach. The difficulty lies in the fact that the existence of such eigenvalues is to be established and their algebraic multiplicities are proved to be 1.

Kong, Kong, and Wong [13] studied BVP (1.1), (1.3) in a different way: they obtained conditions for the existence of nodal solutions by comparing f_0 and f_∞ with the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of the SLP consisting of the equation

$$-y'' = \lambda w(t)y,\ t \in (a,b), \tag{1.6}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, \quad \alpha \in [0,\pi), \\ &y'(b) = 0. \end{aligned} \tag{1.7}$$

The results in [13] are good because they work with a variable w and a general BC parameter α, and f_0, f_∞ are allowed to be 0 and ∞. Moreover, the eigenvalues of SLP (1.6), (1.7) are guaranteed to exist, easy to compute numerically, and are algebraically simple. The ideas in [13] have been applied in [3, 4, 11] to deal with other BVPs with one separated BC and one multi-point or integral BC. However, we note that the shooting method, which was used in [13] to deal with nodal solutions, fails to work alone on BVPs with double multi-point BC (1.2).

Recently, Genoud and Rynne [6] discussed the double multi-point BVP (1.1), (1.2). By establishing the existence of eigenvalues of the corresponding linear SLP (1.6), (1.2) and using the Rabinowitz global bifurcation theorem, they obtained results on the existence of nodal solutions. This work is significant since it made the first progress in the existence of nodal solutions of double multi-point BVPs. However, their results were derived under certain assumptions which can be roughly stated as follows:

(a) For sufficiently small $\delta \in (0,1)$

$$\sum_{i=1}^{m} |k_i| < \delta \quad \text{and} \quad \sum_{j=1}^{l} |h_j| < \delta, \tag{1.8}$$

(b) the function f satisfies that $f(x)/x \in C^1(\mathbb{R})$ and $0 < f_0, f_\infty < \infty$.

It is required that the δ in Assumption (a) be small, but it fails to determine how small this δ should be. Actually, the implicit function theorem, which was used in the proofs to guarantee the existence of δ, does not provide its magnitude. Therefore, although the work in [6] is of theoretical importance, it is practically difficult in implementation. Moreover, the restrictions on f given in Assumption (b) exclude the possibility for f to be superlinear or sublinear.

In this paper, we will further develop the methods used in [13] for BVPs with separated–multi-point BCs to BVPs with double multi-point BCs. More specifically, we will show that the nodal solutions for BVPs with the separated–multi-point BCs

$$y'(c) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0$$

and

$$y(a)-\sum_{j=1}^{l}h_jy(\xi_j)=0, \quad y'(d)=0,$$

respectively, will meet at some $c=d\in(a,b)$ and hence produce nodal solutions for BVPs (1.1), (1.2). Our results are under explicit conditions and f is allowed to be superlinear and sublinear. We will also obtain conditions for the nonexistence of certain types of nodal solutions.

This paper is structured as follows: we present the main results of the paper in Section 2 and then give the proofs in Section 3 after several technical lemmas are established.

2. Main results

We aim to study solutions of BVP (1.1), (1.2) which fall into certain classes defined as follows.

DEFINITION 2.1. Let $n\in\mathbb{N}_0:=\{0,1,2,\dots\}$. Then a solution y of BVP (1.1), (1.2) is said to belong to a class $\mathscr{T}_n^{\gamma}$ for $\gamma\in\{+,-\}$ if

(i) y and y' have only simple zeros in $[a,b]$,

(ii) y' has exactly $n+1$ zeros in (a,b),

(iii) $\gamma y(t)\geqslant 0$ in a right-neighborhood of a.

REMARK 2.1. One can easily see that for $y\in\mathscr{T}_n^{\gamma}$ with $n\in\mathbb{N}_0$ and $\gamma\in\{+,-\}$, y may have n, $n+1$, or $n+2$ zeros in (a,b).

To establish criteria for BVP (1.1), (1.2) to have various nodal solutions, we need to use the eigenvalues of the SLP consisting of the equation

$$y''+\lambda w(t)y=0, \quad t\in(a,b), \tag{2.1}$$

and the two-point BC

$$y'(a)=y'(b)=0. \tag{2.2}$$

It is well-known that the spectrum of SLP (2.1), (2.2) consists of an infinite number of real simple eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ satisfying that

$$0=\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots, \quad \text{and} \quad \lambda_n\to\infty;$$

and any eigenfunction associated with λ_n has exactly n zeros in (a,b) for $n\in\mathbb{N}_0$, see [31, Theorem 4.3.2].

Let $F(y)=\int_0^y f(\xi)\,d\xi$ for $y\in\mathbb{R}$ and denote $w'_{\pm}(t):=\max\{\pm w'(t),0\}$ along with

$$\gamma_j^+=\int_a^{\xi_j}\frac{w'_+(t)}{w(t)}dt,\ j=1,\dots,l,\ \text{ and }\ \gamma_i^-=\int_{\eta_i}^{b}\frac{w'_-(t)}{w(t)}dt,\ i=1,\dots,m.$$

By (H2), F is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty,0]$. Let F_+^{-1} and F_-^{-1} be the inverses of F on $[0,\infty)$ and $(-\infty,0]$, respectively. Clearly, since f is an odd function on $\mathbb{R}$, then $F_+^{-1} = -F_-^{-1}$. Thus we may define

$$F^{-1} := F_+^{-1} = -F_-^{-1}.$$

THEOREM 2.1. *Let $n \in \mathbb{N}_0$. Assume either (i) $f_0 \leqslant \lambda_{\lfloor n/2 \rfloor}$ and $f_\infty = \infty$, or (ii) $f_\infty \leqslant \lambda_{\lfloor n/2 \rfloor}$ and $f_0 = \infty$, where $\lfloor n/2 \rfloor$ is the integer part of $n/2$. Suppose that for any $r > 0$,*

$$\sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{re^{\gamma_i^-}}{w(\eta_i)}\right) < F^{-1}\left(\frac{r}{w(b)}\right) \tag{2.3}$$

and

$$\sum_{j=1}^{l} |h_j| F^{-1}\left(\frac{re^{\gamma_j^+}}{w(a+b-\xi_j)}\right) < F^{-1}\left(\frac{r}{w(a)}\right) \tag{2.4}$$

hold. Then BVP (1.1), (1.2) has a solution $y_n^\gamma \in \mathscr{T}_n^\gamma$ for $\gamma \in \{+,-\}$.

Note that when $n = 0$ or 1, the assumptions of Theorem 2.1 imply that either $f_0 = 0$ or $f_\infty = 0$.

REMARK 2.2. (a) We comment that (2.3) implies that

$$\sum_{i=1}^{m} |k_i| < 1. \tag{2.5}$$

In fact, since $w'_-(t) \geqslant -w'(t)$,

$$\gamma_i^- \geqslant \int_{\eta_i}^{b} \frac{-w'(t)}{w(t)} dt = \ln \frac{w(\eta_i)}{w(b)}.$$

Hence

$$\sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{re^{\gamma_i^-}}{w(\eta_i)}\right) \geqslant \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{r}{w(b)}\right).$$

Then (2.5) follows from (2.3). Similarly, (2.4) implies that

$$\sum_{j=1}^{l} |h_j| < 1. \tag{2.6}$$

On the other hand, when $w(t) \equiv 1$, (2.3) reduces to (2.5), and (2.4) reduces to (2.6).

(b) If $f(y) = |y|^{q-1}y$ for $q > 0$, then (2.3) reduces to

$$\sum_{i=1}^{m} |k_i| \left(\frac{w(b)e^{\gamma_i^-}}{w(\eta_i)}\right)^{1/(q+1)} < 1;$$

and, (2.4) reduces to

$$\sum_{j=1}^{l} |h_j| \left(\frac{w(b)e^{\gamma_j^+}}{w(a+b-\xi_j)} \right)^{1/(q+1)} < 1.$$

As a consequence of Theorem 2.1 we have the corollary below.

COROLLARY 2.1. *Assume inequalities (2.3) and (2.4) hold and*

$$\text{either } f_0 = 0 \text{ and } f_\infty = \infty, \text{ or } f_\infty = 0 \text{ and } f_0 = \infty.$$

Then

(a) *BVP (1.1), (1.2) has positive and negative solutions in* $\mathscr{T}_0^\gamma$ *for* $\gamma \in \{+,-\}$ *if* $k_i \geqslant 0$ *for* $i = 1,\dots,m$ *and* $h_j \geqslant 0$ *for* $j = 1,\dots,l$.

(b) *BVP (1.1), (1.2) has solutions in* $\mathscr{T}_0^\gamma$ *for* $\gamma \in \{+,-\}$ *with exactly one zero in* (a,b) *if either* $k_i \geqslant 0$ *for* $i = 1,\dots,m$ *and* $h_j \leqslant 0$ *for* $j = 1\dots,l$ *such that* $\sum_{j=1}^{l} h_j < 0$*; or* $h_j \geqslant 0$ *for* $j = 1,\dots,l$ *and* $k_i \leqslant 0$ *for* $i = 1,\dots,m$ *such that* $\sum_{i=1}^{m} k_i < 0$.

(c) *BVP (1.1), (1.2) has solutions in* $\mathscr{T}_0^\gamma$ *for* $\gamma \in \{+,-\}$ *with exactly two zeros in* (a,b) *if* $k_i \leqslant 0$ *for* $i = 1,\dots,m$ *such that* $\sum_{i=1}^{m} k_i < 0$ *and* $h_j \leqslant 0$ *for* $j = 1\dots,l$ *such that* $\sum_{j=1}^{l} h_j < 0$.

Let $\{\zeta_n^1\}_{n=0}^\infty$ and $\{\zeta_n^2\}_{n=0}^\infty$ be the eigenvalues of SLPs consisting of the equation

$$y'' + \zeta w(t) y = 0, \quad t \in (a,b), \tag{2.7}$$

and the BC

$$y(a) = 0, \quad y'(b) = 0 \tag{2.8}$$

and

$$y'(a) = 0, \quad y(b) = 0, \tag{2.9}$$

respectively. Then the following is about the nonexistence of certain types of nodal solutions of BVP (1.1), (1.2).

THEOREM 2.2. *(i) Assume* $f(y)/y < \zeta_n^1$ *for some* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP (1.1), (1.2) has no solution in* $\mathscr{T}_i^\gamma$ *for all* $i \geqslant n+1$ *and* $\gamma \in \{+,-\}$*;*

Assume $f(y)/y > \zeta_{n+1}^1$ *for some* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP (1.1), (1.2) has no solution in* $\mathscr{T}_i^\gamma$ *for all* $i \leqslant n$ *and* $\gamma \in \{+,-\}$.

(ii) Assume $f(y)/y < \zeta_n^2$ *for some* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP (1.1), (1.2) has no solution in* $\mathscr{T}_i^\gamma$ *for all* $i \geqslant n+1$ *and* $\gamma \in \{+,-\}$*;*

Assume $f(y)/y > \zeta_{n+1}^2$ *for some* $n \in \mathbb{N}_0$ *and all* $y \neq 0$. *Then BVP (1.1), (1.2) has no solution in* $\mathscr{T}_i^\gamma$ *for all* $i \leqslant n$ *and* $\gamma \in \{+,-\}$.

3. Proofs of the main results

In order to prove Theorems 2.1-2.2, we first consider the BVPs consisting of Eq. (1.1) and one of the BCs

$$y'(c) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0 \tag{3.1}$$

and

$$y(a) - \sum_{j=1}^{l} h_j y(\xi_j) = 0, \quad y'(d) = 0, \tag{3.2}$$

where $c \in [a,b)$ and $d \in (a,b]$ are arbitrary. We classify the solutions of the above BVPs into the following classes, as extensions of the class defined in Definition 2.1.

DEFINITION 3.1. Let $n \in \mathbb{N}_0$.
(a) For any $c \in [a,b)$, a solution y of BVP (1.1), (3.1) is said to belong to class $\mathscr{T}_n^{\gamma}[c,b]$ for $\gamma \in \{+,-\}$ if

(i) y and y' have only simple zeros in $[c,b]$,
(ii) y' has exactly n zeros in (c,b),
(iii) $\gamma y(c) > 0$.

(b) For any $d \in (a,b]$, a solution y of BVP (1.1), (3.2) is said to belong to class $\mathscr{T}_n^{\gamma}[a,d]$ for $\gamma \in \{+,-\}$ if

(i) y and y' have only simple zeros in $[a,d]$,
(ii) y' has exactly n zeros in (a,d),
(iii) $\gamma y(d) > 0$.

For any $c \in [a,b)$ and $d \in (a,b]$, we let $\{\mu_n(c)\}_{n=0}^{\infty}$ and $\{\nu_n(d)\}_{n=0}^{\infty}$ be the eigenvalues of the SLPs consisting of Eq. (2.1) and the two-point BCs

$$y'(c) = 0, \; y'(b) = 0 \tag{3.3}$$

and

$$y'(a) = 0, \; y'(d) = 0, \tag{3.4}$$

respectively. It is well-known that $\{\mu_n(c)\}_{n=0}^{\infty}$ and $\{\nu_n(d)\}_{n=0}^{\infty}$ satisfy that

$$0 = \mu_0(c) < \mu_1(c) < \cdots \mu_n(c) < \cdots, \text{ and } \mu_n(c) \to \infty,$$

and

$$0 = \nu_0(d) < \nu_1(d) < \cdots \nu_n(d) < \cdots, \text{ and } \nu_n(d) \to \infty;$$

and any eigenfunction associated with $\mu_n(c)$ or $\nu_n(d)$ has exactly n simple zeros in (c,b) or (a,d), respectively, for $n \in \mathbb{N}_0$, see [31, Theorem 4.3.2].

From [26], it follows that any initial value problem (IVP) associated with Eq. (1.1) has a unique solution which exists on $[a,b]$. As a result, the solution depends continuously on the initial condition (IC) and parameters. Let $c \in [a,b)$. For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of the IVP consisting of the Eq. (1.1) and the initial conditions

$$y(c) = \gamma\rho \quad \text{and} \quad y'(c) = 0, \tag{3.5}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$, ie, $\theta(t,\rho)$ is a continuous function on $[a,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/y'(t,\rho) \quad \text{and} \quad \theta(c,\rho) = \pi/2.$$

By the continuous dependence of solutions on parameters, we have that $\theta(t,\rho)$ is continuous in ρ on $[0,\infty)$ for any $t \in [a,b]$. We note that the following two lemmas are minor extensions of Lemmas 4.1, 4.2, 4.4, and 4.5 in [14]. Here, we assume $c \in [a,b)$.

LEMMA 3.1. *(i) Assume $f_0 \leqslant \mu_n(c)$ for some $n \in \mathbb{N}_0$. Then for any $\varepsilon > 0$, there exists $\rho_* > 0$ such that $\theta(b,\rho) \leqslant n\pi + \pi/2 + \varepsilon$ for all $\rho \in (0,\rho_*]$.*

(ii) Assume $\mu_n(c) \leqslant f_\infty$ for some $n \in \mathbb{N}_0$. Then for any $\varepsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \geqslant n\pi + \pi/2 - \varepsilon$ for all $\rho \in [\rho^*,\infty)$.

LEMMA 3.2. *(i) Assume $f_\infty \leqslant \mu_n(c)$ for some $n \in \mathbb{N}_0$. Then for any $\varepsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \leqslant n\pi + \pi/2 + \varepsilon$ for all $\rho \in [\rho^*,\infty)$.*

(ii) Assume $\mu_n(c) \leqslant f_0$ for some $n \in \mathbb{N}_0$. Then for any $\varepsilon > 0$, there exists $\rho_* > 0$ such that $\theta(b,\rho) \geqslant n\pi + \pi/2 - \varepsilon$ for all $\rho \in (0,\rho_*]$.

Based on Lemmas 3.1 and 3.2, we establish the following result which is an improvement of [13, Theorem 2.1].

LEMMA 3.3. *Assume either (i) $f_0 \leqslant \mu_n(c)$ and $\mu_{n+1}(c) < f_\infty$, or (ii) $f_\infty \leqslant \mu_n(c)$ and $\mu_{n+1}(c) < f_0$, for some $n \in \mathbb{N}_0$. Suppose that (2.3) holds for any $r > 0$. Then BVP (1.1), (3.1) has a solution $y_n^\gamma \in \mathcal{T}_n^\gamma[c,b]$ for $\gamma \in \{+,-\}$.*

Note that the assumption $f_0 = \mu_n(c)$ or $f_\infty = \mu_n(c)$ is allowed in Lemma 3.3, but not in Theorem 2.1 in [13].

Proof of Lemma 3.3. Consider the case when $f_0 \leqslant \mu_n(c)$ and $\mu_{n+1}(c) < f_\infty$. Without loss of generality, assume $\gamma = +$. The case for when $\gamma = -$ is done similarly. Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying (3.5) with $\gamma = +$ and $\theta(t,\rho)$ its Prüfer angle. By Lemma 3.1, for any small $\varepsilon > 0$, there exists $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) \leqslant n\pi + \pi/2 + \varepsilon \quad \text{for all} \quad \rho \in (0,\rho_*]$$

and

$$\theta(b,\rho) \geqslant (n+1)\pi + \pi/2 - \varepsilon \quad \text{for all} \quad \rho \in [\rho^*,\infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exists $\rho_* \leqslant \rho_n < \rho_{n+1} \leqslant \rho^*$ such that

$$\theta(b,\rho_n) = n\pi + \pi/2 + \varepsilon \quad \text{and} \quad \theta(b,\rho_{n+1}) = (n+1)\pi + \pi/2 - \varepsilon, \tag{3.6}$$

and

$$\theta(b,\rho_n) < \theta(b,\rho) < \theta(b,\rho_{n+1}) \ \text{ for } \ \rho_n < \rho < \rho_{n+1}. \tag{3.7}$$

Defining an energy function for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2}[y'(t,\rho)]^2 + w(t)F(y(t,\rho)) \quad \text{for } t \in [a,b] \text{ and } \rho > 0.$$

Then by Eq. (1.1)

$$E'(t,\rho) = w'(t)F(y(t,\rho)) \geqslant -\frac{w'_-(t)}{w(t)}E(t,\rho).$$

Thus we have that for $i = 1,\dots,m$,

$$\ln\frac{E(b,\rho)}{E(\eta_i,\rho)} = \int_{\eta_i}^{b} \frac{E'(t,\rho)}{E(t,\rho)}dt \geqslant -\int_{\eta_i}^{b} \frac{w'_-(t)}{w(t)}dt = -\gamma_i^-.$$

Hence

$$E(\eta_i,\rho) \leqslant e^{\gamma_i^-} E(b,\rho) \ \text{ for } \ i = 1,2,\dots,m. \tag{3.8}$$

Note that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$,

$$E(\eta_i,\rho) \geqslant w(\eta_i)F(y(\eta_i,\rho)) \ \text{ for } \ i = 1,2,\dots,m. \tag{3.9}$$

It is seen from (3.6) that as $\varepsilon \to 0$

$$y'(b,\rho) = o(1) \quad \text{ and } \quad |y(b,\rho)| = \rho + o(1)$$

and hence

$$E(b,\rho) = w(b)F(y(b,\rho)) + o(1) = w(b)F(y(b,\rho))[1+o(1)].$$

Since F has a continuous inverse F^{-1}, it follows that for $\rho = \rho_n$ and $\rho = \rho_{n+1}$

$$|y(b,\rho)| = F^{-1}\left(\frac{E(b,\rho)}{w(b)}\right)(1+o(1)) \ \text{ as } \ \varepsilon \to 0; \tag{3.10}$$

and it follows from (3.9) that for $i = 1,\dots,m$,

$$y(\eta_i,\rho) \leqslant F^{-1}\left(\frac{E(\eta_i,\rho)}{w(\eta_i)}\right) \ \text{ if } \ y(\eta_i,\rho) \geqslant 0$$

and

$$-y(\eta_i,\rho) \leqslant F^{-1}\left(\frac{E(\eta_i,\rho)}{w(\eta_i)}\right) \ \text{ if } \ y(\eta_i,\rho) \leqslant 0.$$

Hence

$$|y(\eta_i,\rho)| \leqslant F^{-1}\left(\frac{E(\eta_i,\rho)}{w(\eta_i)}\right). \tag{3.11}$$

Define

$$\Gamma(\rho) = y(b,\rho) - \sum_{i=1}^{m} k_i y(\eta_i,\rho). \tag{3.12}$$

Let $n = 2k$ with $k \in \mathbb{N}_0$. Since $y(b,\rho_{2k}) > 0$ and $y(b,\rho_{2k+1}) < 0$, by (3.10), (3.11), (3.8), and (2.3) we have for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}
\Gamma(\rho_{2k}) &= y(b,\rho_{2k}) - \sum_{i=1}^{m} k_i y(\eta_i,\rho_{2k}) \\
&\geqslant y(b,\rho_{2k}) - \sum_{i=1}^{m} |k_i||y(\eta_i,\rho_{2k})| \\
&\geqslant F^{-1}\left(\frac{E(b,\rho_{2k})}{w(b)}\right)(1+o(1)) - \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{E(\eta_i,\rho_{2k})}{w(\eta_i)}\right) \\
&\geqslant F^{-1}\left(\frac{E(b,\rho_{2k})}{w(b)}\right) - \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{e^{\gamma_i^-} E(b,\rho_{2k})}{w(\eta_i)}\right) + o(1) > 0
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\rho_{2k+1}) &= y(b,\rho_{2k+1}) - \sum_{i=1}^{m} k_i y(\eta_i,\rho_{2k+1}) \\
&\leqslant y(b,\rho_{2k+1}) + \sum_{i=1}^{m} |k_i||y(\eta_i,\rho_{2k+1})| \\
&\leqslant -F^{-1}\left(\frac{E(b,\rho_{2k+1})}{w(b)}\right)(1+o(1)) + \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{E(\eta_i,\rho_{2k+1})}{w(\eta_i)}\right) \\
&\leqslant -F^{-1}\left(\frac{E(b,\rho_{2k+1})}{w(b)}\right) + \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{e^{\gamma_i^-} E(b,\rho_{2k+1})}{w(\eta_i)}\right) + o(1) < 0.
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k},\rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k+1$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1},\rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, since $\varepsilon > 0$ in (3.7) we see that

$$n\pi + \pi/2 < \theta(b,\bar{\rho}) < (n+1)\pi + \pi/2.$$

Since

$$\theta'(t,\rho) = \cos^2\theta(t,\rho) + w(t)\frac{f(y(t,\rho))y(t,\rho)}{r^2(t,\rho)},$$

where $r = (y^2 + y'^2)^{1/2}$, we have that $\theta(\cdot,\rho)$ is strictly increasing on $[c,b]$. We note that $y(t) = 0$ if and only if $\theta(t,\rho) = 0 \pmod{\pi}$ and $y'(t) = 0$ if and only if $\theta(t,\rho) = \pi/2 \pmod{\pi}$. Thus, y' has exactly n zeros in (c,b) and y has exactly one zero strictly between any two consecutive zeros of y'. Initial condition (3.5) implies that $y(t,\bar{\rho})>0$ in a right-neighborhood of c. Therefore, $y(t,\bar{\rho})\in\mathscr{T}_n^+[c,b]$.

The proof for the case when $f_\infty\leqslant\mu_n(c)$ and $\mu_{n+1}(c)<f_0$ is essentially the same as above except that the discussion is based on Lemma 3.2 instead of Lemma 3.1. $\square$

The next lemma follows from Lemma 3.3 for the case where $d\in(a,b]$.

LEMMA 3.4. *Assume (i) $f_0\leqslant\nu_n(d)$ and $\nu_{n+1}(d)<f_\infty$, or (ii) $f_\infty\leqslant\nu_n(d)$ and $\nu_{n+1}(d)<f_0$, for some $n\in\mathbb{N}_0$. Suppose that (2.4) holds for any $r>0$. Then BVP (1.1), (3.2) has a solution $y_n^\gamma\in\mathscr{T}_n^\gamma[a,d]$ for $\gamma\in\{+,-\}$.*

Proof. Consider the following transformation: $t=a+b-\tau$, $d=a+b-c$. Then BVP (1.1), (3.2) becomes the problem consisting of the equation

$$\frac{d^2y}{d\tau^2}+w(a+b-\tau)f(y)=0,\quad \tau\in(a,b), \tag{3.13}$$

and BC

$$\frac{dy}{d\tau}(c)=0,\quad y(b)-\sum_{j=1}^{l}h_jy(a+b-\xi_j)=0. \tag{3.14}$$

Clearly, $c\in[a,b)$, $a\leqslant a+b-\xi_j<b$ for $j=1,2,\dots,l$ and

$$\begin{aligned}\int_{a+b-\xi_j}^{b}\frac{[w(a+b-\tau)]'_-}{w(a+b-\tau)}d\tau&=\int_{a+b-\xi_j}^{b}\frac{[-w'(a+b-\tau)]_-}{w(a+b-\tau)}d\tau\\&=\int_{a+b-\xi_j}^{b}\frac{w'_+(a+b-\tau)}{w(a+b-\tau)}d\tau=\int_a^{\xi_j}\frac{w'_+(t)}{w(t)}dt=\gamma_j^+.\end{aligned}$$

Hence inequality (2.4) implies that inequality (2.3) holds for the transformed BVP (3.13), (3.14). Also note that $\{\nu_n\}_{n=0}^\infty$ are eigenvalues of the SLP involving the equation

$$\frac{d^2y}{d\tau^2}+\lambda w(a+b-\tau)y=0,\quad \tau\in(a,b),$$

and BC (3.3). Thus the conclusion follows from Lemma 3.3.

The Lemmas below play critical roles in the proof of Theorem 2.1.

LEMMA 3.5. *Assume (2.3) holds for any $r>0$ and $c\in[a,b)$. Let $\{\mu_i(c)\}_{i=0}^\infty$ be the eigenvalues of SLP (2.1), (3.3). Let $i\in\mathbb{N}_0$.*

(i) *Suppose $f_0\leqslant\mu_i(c)$ and $f_\infty=\infty$ and let $y_i(t;c)\in\mathscr{T}_i^+[c,b]$ be the solution of BVP (1.1), (3.1) given by Lemma 3.3. Then $\lim\limits_{c\to b^-}y_i(c;c)=\infty$.*

(ii) *Suppose $f_\infty\leqslant\mu_i(c)$ and $f_0=\infty$ and let $y_i(t;c)\in\mathscr{T}_i^+[c,b]$ be the solution of BVP (1.1), (3.1) given by Lemma 3.3. Then $\lim\limits_{c\to b^-}y_i(c;c)=0$.*

Proof. (i) Assume the contrary. Then there exists a sequence $\{c_k\}_{k=1}^{\infty} \subset [a,b)$ such that $c_k \to b^-$ and $y_i(c_k;c_k) \to l$ for some $l \in [0,\infty)$.

(a) Assume first $l \in (0,\infty)$. Let $\bar{y}(t)$ be the solution of Eq. (1.1) satisfying the IC

$$\bar{y}(b) = l \text{ and } \bar{y}'(b) = 0. \tag{3.15}$$

Note that for $k \in \mathbb{N}$

$$y_i(c_k;c_k) \to \bar{y}(b) \text{ as } c_k \to b^-$$

and

$$y_i'(c_k;c_k) = \bar{y}'(b) = 0.$$

By the continuous dependence of solutions of IVPs on the ICs and parameters, we have

$$\lim_{k\to\infty} y_i(t;c_k) = \bar{y}(t) \text{ uniformly for all } t \in [a,b].$$

Since for each k, $y_i(t;c_k)$ satisfies

$$y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0, \tag{3.16}$$

then $\bar{y}(t)$ satisfies (3.16). Define an energy function for $\bar{y}(t)$ by

$$E(t) = \frac{1}{2}[\bar{y}'(t)]^2 + w(t)F(\bar{y}(t)), \quad t \in [a,b]. \tag{3.17}$$

It follows that (3.8) holds with $E(\cdot,\rho)$ replaced by $E(\cdot)$ and so does (3.11). Additionally, with $\bar{y}'(b) = 0$ we have

$$E(b) = w(b)F(\bar{y}(b)),$$

and so

$$|\bar{y}(b)| = F^{-1}\left(\frac{E(b)}{w(b)}\right). \tag{3.18}$$

Since $\bar{y}(b) = l > 0$, by (3.11), (3.18), (3.8), and (2.3) we have

$$\begin{aligned}
\bar{y}(b) - \sum_{i=1}^{m} k_i \bar{y}(\eta_i) &\geqslant |\bar{y}(b)| - \sum_{i=1}^{m} |k_i||\bar{y}(\eta_i)| \\
&\geqslant F^{-1}\left(\frac{E(b)}{w(b)}\right) - \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{E(\eta_i)}{w(\eta_i)}\right) \\
&\geqslant F^{-1}\left(\frac{E(b)}{w(b)}\right) - \sum_{i=1}^{m} |k_i| F^{-1}\left(\frac{e^{\bar{\gamma_i}} E(b)}{w(\eta_i)}\right) > 0.
\end{aligned}$$

However, this contradicts that $\bar{y}(t)$ satisfies (3.16).

(b) Now assume $l = 0$. Since $y_i(c_k;c_k) \neq 0$, let $z_i(t;c_k) = y_i(t;c_k)/y_i(c_k,c_k)$. It follows that $z_i(t;c_k)$ is a solution of

$$z'' + w(t)g_k(z)z = 0,$$

where

$$g_k(z) := \begin{cases} \frac{f(y_i(c_k;c_k)z)}{y_i(c_k;c_k)z}, & \text{for } z \neq 0, \\ f_0, & \text{for } z = 0; \end{cases} \tag{3.19}$$

and $g_k(z)$ is a continuous function for $z \in \mathbb{R}$ since $f_0 < \infty$. Note that as $k \to \infty$, $g_k(z) \to f_0$. Also note that

$$z_i(c_k;c_k) = 1 \text{ and } z_i'(c_k,c_k) = 0.$$

Let $\bar{z}(t)$ be the solution of the IVP

$$z'' + f_0 w(t) z = 0, \quad \bar{z}(b) = 1, \ \bar{z}'(b) = 0. \tag{3.20}$$

By the continuous dependence of solutions of IVPs on parameters, we see that

$$\lim_{k\to\infty} z_i(t;c_k) = \bar{z}(t) \text{ uniformly for all } t \in [a,b].$$

Since $y_i(t;c_k)$ satisfies (3.16) for each k, then $z_i(t;c_k)$ satisfies (3.16) for each k and so does $\bar{z}(t)$.
If $f_0 = 0$ then $\bar{z}(t) \equiv 1$. It follows from (3.16) that $\sum_{i=1}^m k_i = 1$ which implies that $\sum_{i=1}^m |k_i| \geqslant 1$. This contradicts (2.5) and hence contradicts (2.3) by Remark 2.2.
If $f_0 > 0$, define an energy function for $\bar{z}(t)$ by

$$E(t) = \frac{1}{2}[\bar{z}'(t)]^2 + \frac{f_0}{2} w(t)[z(t)]^2, \quad t \in [a,b]. \tag{3.21}$$

Then

$$E'(t) = \frac{f_0}{2} w'(t)[\bar{z}(t)]^2 \geqslant -\frac{w'_-(t)}{w(t)} E(t).$$

So we have,

$$\ln\left(\frac{E(b)}{E(\eta_i)}\right) = \int_{\eta_i}^{b} \frac{E'(t)}{E(t)} dt \geqslant -\gamma_i^-.$$

So,

$$E(\eta_i) \leqslant e^{\gamma_i^-} E(b), \text{ for } i = 1,2,\ldots,m.$$

Additionally,

$$E(\eta_i) \geqslant \frac{f_0}{2} w(\eta_i)[\bar{z}(\eta_i)]^2, \text{ for } i = 1,2,\ldots,m,$$

and

$$E(b) = \frac{f_0}{2} w(\eta_i)[\bar{z}(b)]^2.$$

Hence,

$$|\bar{z}(\eta_i)| \leqslant \sqrt{\frac{2E(\eta_i)}{f_0 w(\eta_i)}} \text{ and } |\bar{z}(b)| = \sqrt{\frac{2E(b)}{f_0 w(b)}}.$$

From the assumption that (3.16) holds, we have

$$\bar{z}(b)-\sum_{i=1}^{m}k_i z(\eta_i) \geqslant |\bar{z}(b)|-\sum_{i=1}^{m}|k_i||\bar{z}(\eta_i)| \tag{3.22}$$

$$\geqslant \sqrt{\frac{2E(b)}{f_0 w(b)}}-\sum_{i=1}^{m}|k_i|\sqrt{\frac{2E(\eta_i)}{f_0 w(\eta_i)}} \tag{3.23}$$

$$\geqslant \sqrt{\frac{2E(b)}{f_0}}\left(\frac{1}{w(b)}-\sum_{i=1}^{m}|k_i|\frac{e^{\gamma_i^-/2}}{\sqrt{w(\eta_i)}}\right)>0, \tag{3.24}$$

contradicting $\bar{z}(t)$ satisfying (3.16).

(ii) Assume the contrary. Then there exists $\{c_k\}_{k=1}^{\infty}\subset[a,b)$ such that $c_k\to b^-$ and $y_0(c_k;c_k)\to l$ for $l\in(0,\infty]$.

(a) Assume $l\in(0,\infty)$. Then the argument follows similarly to that in part (i), (a) above and is omitted.

(b) Assume $l=\infty$. Since $f_\infty<\infty$, then by replacing f_0 by f_∞, the argument follows similarly to that in part (i), (b) above and is omitted.

The next lemma for BVP (1.1), (3.2) is a parallel result to Lemma 3.5 with a similar proof.

LEMMA 3.6. *Assume* (2.4) *holds for any* $r>0$ *and* $d\in(a,b]$. *Let* $\{\nu_j(d)\}_{j=0}^{\infty}$ *be the eigenvalues of SLP* (2.1), (3.4). *Let* $j\in\mathbb{N}_0$.

(i) *Suppose* $f_0\leqslant\nu_j(d)$ *and* $f_\infty=\infty$ *and let* $y_j(t;d)\in\mathscr{T}_j^+[a,d]$ *be the solution of BVP* (1.1), (3.2) *given by Lemma* 3.4. *Then* $\lim_{d\to a^+}y_j(d;d)=\infty$.

(ii) *Suppose* $f_\infty\leqslant\nu_j(d)$ *and* $f_0=\infty$ *and let* $y_j(t;d)\in\mathscr{T}_j^+[a,d]$ *be the solution of BVP* (1.1), (3.2) *given by Lemma* 3.4. *Then* $\lim_{d\to a^+}y_j(d;d)=0$.

REMARK 3.1. Lemmas 3.5 and 3.6 are for the existence of nodal solutions for BVPs (1.1), (3.1) and (1.1), (3.2) in the classes $\mathscr{T}_i^\gamma[c,b]$ and $\mathscr{T}_j^\gamma[a,d]$, respectively, with $\gamma=+$. Parallel results hold for $\gamma=-$.

REMARK 3.2. (a) For $n\in\mathbb{N}_0$ and $c\in[a,b)$, Lemma 3.3 establishes the existence of a solution $y_n(t;c)$ of BVP (1.1), (3.1) in $\mathscr{T}_n^+[c,b]$. However, the uniqueness of such solutions are not guaranteed. We claim that for each $n\in\mathbb{N}_0$, there is at least one continuous curve Λ_n^c in the ρ-c plane which satisfies that

(i) for each $(\rho,c)\in\Lambda_n^c$, $c\in[a,b)$ and $\rho=y_n(c;c)$;

(ii) for each $c\in[a,b)$, there is at least one point $(\rho,c)\in\Lambda_n^c$.

This is shown as follows:

Note that the solution y of Eq. (1.1) used to define the function Γ in (3.12) satisfies the IC (3.5) and as a result, y and Γ have continuous dependence on the initial point c. To emphasize such dependence, we rewrite (3.12) as

$$\Gamma(\rho,c) = y_c(b,\rho) - \sum_{i=1}^{m} k_i y_c(\eta_i,\rho).$$

Then Γ is a continuous function of (ρ,c). Since for any $n \in \mathbb{N}_0$ and $c \in [a,b)$, $y_n(c;c)$ is a root of $\Gamma(\rho,c)$, then $\rho = y_n(c;c)$ if and only if (ρ,c) is on the intersection set I of the continuous surfaces $z = \Gamma(\rho,c)$ and $z = 0$. From the proof of Lemma 3.3 we see that when $n = 2k$, $\Gamma(\rho_{2k}) > 0$ and $\Gamma(\rho_{2k+1}) < 0$ for all $c \in [a,b)$. Therefore, the intersection set I must contain a continuous curve in the ρ-c plane which starts at $c = a$ and ends up with $c = b$. Similarly for the case when $n = 2k+1$.

(b) For $n \in \mathbb{N}_0$ and $d \in (a,b]$, Lemma 3.4 establishes the existence of a solution $y_n(t;d)$ of BVP (1.1), (3.2) in $\mathscr{T}_n^+[a,d]$. With the same argument as above, for each $n \in \mathbb{N}_0$, there is at least one continuous curve Λ_n^d in the ρ-d plane which satisfies that

(i) for each $(\rho,d) \in \Lambda_n^d$, $d \in (a,b]$ and $\rho = y_n(d;d)$;

(ii) for each $d \in (a,b]$, there is at least one point $(\rho,d) \in \Lambda_n^d$.

Now we prove our main result, Theorem 2.1.

Proof of Theorem 2.1. Without loss of generality, we consider the case where $\gamma = +$, $f_0 \leqslant \lambda_{\lfloor n/2 \rfloor}$ and $f_\infty = \infty$. The other cases can be proved similarly. For any $c \in [a,b)$ and $d \in (a,b]$, let $\mu_n(c)$ be the n-th eigenvalue of SLP (2.1), (3.3) and $\nu_n(d)$ the n-th eigenvalue of SLP (2.1), (3.4). We note that $\mu_n(a)$ and $\nu_n(b)$ are the n-th eigenvalues for SLP (2.1), (2.2), and hence $\lambda_n = \mu_n(a) = \nu_n(b)$.

For $n \in \mathbb{N}_0$, let $i = \lfloor n/2 \rfloor$, $j = n - i$. Clearly $j \geqslant i$. From [17, Theorem 4.1] and [16, Theorem 2.2] we see that for $i, j \geqslant 1$, $\mu_i(c)$ is strictly increasing and $\lim_{c \to b^-} \mu_i(c) = \infty$, and $\nu_j(d)$ is strictly decreasing and $\lim_{d \to a^+} \nu_j(d) = \infty$. We note that $\mu_0(c) = \nu_0(d) = 0$ for any $c \in [a,b)$ and $d \in (a,b]$. It follows from the assumptions that for any $c \in [a,b)$ and $d \in (a,b]$

$$f_0 \leqslant \mu_i(a) \leqslant \mu_i(c) \;\text{ and }\; \mu_{i+1}(a) < \mu_{i+1}(c) < f_\infty,$$

and

$$f_0 \leqslant \nu_j(b) \leqslant \nu_j(d) \;\text{ and }\; \nu_{j+1}(b) < \nu_{j+1}(d) < f_\infty.$$

Since (2.3) and (2.4) hold, by Lemmas 3.3, 3.4 we have that BVPs (1.1), (3.1) and (1.1), (3.2) have solutions $y_i^{[1]} \in \mathscr{T}_i^+[c,b]$ and $y_j^{[2]} \in \mathscr{T}_j^+[a,d]$, respectively. Therefore, by Lemma 3.5, (i) and Lemma 3.6, (i)

$$\lim_{c \to b^-} y_i^{[1]}(c;c) = \infty \;\text{ and }\; \lim_{d \to a^+} y_j^{[2]}(d;d) = \infty.$$

Let $\rho_i^{[1]}(c) = y_i^{[1]}(c;c)$ such that $(\rho_i^{[1]}(c),c)$ is on the continuous curve Λ_i^c and $\rho_j^{[2]}(d) = y_j^{[2]}(d;d)$ such that $(\rho_j^{[2]}(d),d)$ is on the continuous curve Λ_j^d, as defined in

Remark 3.2. Note that $y_i^{[1]}(a;a), y_j^{[2]}(b;b) \in (0,\infty)$. By the continuity of the curves Λ_i^c and Λ_j^d, there exists $c^* = d^* \in (a,b)$ such that $y_i^{[1]}(c^*;c^*) = y_j^{[2]}(d^*;d^*)$. Also note that

$$(y_i^{[1]})'(c^*,c^*) = 0 \text{ and } (y_j^{[2]})'(d^*,d^*) = 0.$$

By the uniqueness of solutions of IVPs, we have

$$y_i^{[1]}(t,c^*) \equiv y_j^{[2]}(t,d^*) \text{ for } t \in [a,b].$$

We denote

$$y_n(t) = y_i^{[1]}(t,c^*) = y_j^{[2]}(t,d^*) \text{ on } [a,b].$$

Thus, we have that $y_n \in \mathscr{T}_i^+[c^*,b] \cap \mathscr{T}_j^+[a,d^*]$. Considering that $y_n'(c^*) = 0$, we see that y_n' has $n+1$ zeros in (a,b). It is easy to see $-y_n$ is also a solution of BVP (1.1), (1.2) since f is an odd function. Thus $-y_n'$ has $n+1$ zeros in (a,b). Clearly, condition (iii) in Definition 2.1 is satisfied by one of y_n and $-y_n$ for $\gamma = +$ and $\gamma = -$, respectively. Therefore, one of y_n and $-y_n$ is in $\mathscr{T}_n^+$ and the other is in $\mathscr{T}_n^-$. □

The following Lemma plays an important role in the proof of Corollary 2.1.

LEMMA 3.7. *(i) Assume either (a) $f_0 \leqslant \mu_0(c)$ and $\mu_1(c) < f_\infty$ or (b) $f_\infty \leqslant \mu_0(c)$ and $\mu_1(c) < f_0$ and suppose (2.3) holds for any $r > 0$. Then the solutions $y_0^\gamma \in \mathscr{T}_0^\gamma[c,b]$ of BVP (1.1), (3.1) obtained from Lemma 3.3 are positive and negative, respectively for $\gamma = \{+,-\}$.*

(ii) Assume either (a) $f_0 \leqslant \nu_0(d)$ and $\nu_1(d) < f_\infty$ or (b) $f_\infty \leqslant \nu_0(d)$ and $\nu_1(d) < f_0$ and suppose (2.4) holds for any $r > 0$. Then the solutions $y_0^\gamma \in \mathscr{T}_0^\gamma[a,d]$ of BVP (1.1), (3.2) obtained from Lemma 3.4 are positive and negative, respectively for $\gamma = \{+,-\}$.

Proof of Corollary 2.1. (a) Without loss of generality, we consider the case where $f_0 = 0$ and $f_\infty = \infty$ and $\gamma = +$. The other cases can be proved similarly. Under the assumptions, from Theorem 2.1, BVP (1.1), (1.2) has a solution $y_0 := y_0^\gamma \in \mathscr{T}_0^\gamma$. We claim that $y_0(t) > 0$ on (a,b). In fact, from the proof of Theorem 2.1,

$$y_0(t) = y_0^{[1]}(t;c^*) = y_0^{[2]}(t;d^*) \text{ for some } c^* = d^* \in (a,b),$$

where $y_0^{[1]} \in \mathscr{T}_0^+[c^*,b]$ and $y_0^{[2]} \in \mathscr{T}_0^+[a,d^*]$. By Lemma 3.7,

$$y_0^{[1]}(t) > 0 \text{ on } [c^*,b] \text{ and } y_0^{[2]}(t) > 0 \text{ on } [a,d^*].$$

Therefore, $y_0(t) > 0$ on $[a,b]$.
The proofs for parts (b) and (c) are similar and hence are omitted. □

The following are needed in the proof of Theorem 2.2 on nonexistence of solutions of BVP (1.1), (1.2). For $\alpha \in [0,\pi)$, let $\{\zeta_n^1(\alpha)\}_{n=0}^\infty$ denote the eigenvalues of the SLP consisting of Eq. (2.7) and the BC

$$\begin{cases} \cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, & \alpha \in [0,\pi), \\ y'(b) = 0. \end{cases}$$

We note that for $n \in \mathbb{N}_0$, $\zeta_n^1(0) = \zeta_n^1$, where ζ_n^1 is the n-th eigenvalue of SLP (2.7), (2.8). From [15, Lemma 3.32] and [17, Theorem 4.2], $\zeta_n^1(\alpha)$ is continuous and $\zeta_n^1(\alpha)$ is strictly decreasing in α on $[0,\pi)$; moreover,

$$\lim_{\alpha\to\pi^-} \zeta_0^1(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi^-} \zeta_n^1(\alpha) = \zeta_{n-1}^1(0). \tag{3.25}$$

Consider the BVP consisting of Eq. (1.1) and the BCs

$$\begin{cases} \cos\alpha\, y(a) - \sin\alpha\, y'(a) = 0, & \alpha \in [0,\pi), \\ y(b) - \sum\limits_{i=1}^{m} k_i y(\eta_i) = 0. \end{cases} \tag{3.26}$$

The following result from [13, Theorem 2.2] plays a key role in the proof of Theorem 2.2.

LEMMA 3.8. *(i) Assume $f(y)/y < \zeta_n^1(\alpha)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP* (1.1)*,* (3.26) *has no solution with the derivative having $i+1$ zeros on (a,b) if $\alpha \in [0,\pi/2)$, and has no solution with the derivative having i zeros on (a,b) if $\alpha \in [\pi/2,\pi)$, for all $i \geqslant n$.*

(ii) Assume $f(y)/y > \zeta_{n+1}^1(\alpha)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1)*,* (3.26) *has no solution with the derivative having $i+1$ zeros on (a,b) if $\alpha \in [0,\pi/2)$, and has no solution with the derivative having i zeros on (a,b) if $\alpha \in [\pi/2,\pi)$, for all $i \leqslant n$.*

Proof of Theorem 2.2. (i) By contradiction, suppose BVP (1.1), (1.2) has a solution $y \in \mathscr{T}_i^\gamma$ for some $i \geqslant n+1$, $\gamma \in \{+,-\}$. Then there exists $\alpha^* \in [0,\pi)$ such that

$$\cos\alpha^*\, y(a) - \sin\alpha^*\, y'(a) = 0.$$

This means that $y(t)$ is also a solution of BVP (1.1), (3.26) for $\alpha = \alpha^*$. From our assumptions, along with (3.25) and the fact that $\zeta_n(\alpha)$ strictly decreasing in α on $[0,\pi)$, we have that for any $\alpha \in [0,\pi)$

$$f(y)/y < \zeta_n^1 = \zeta_n^1(0) < \zeta_{n+1}^1(\alpha).$$

By Lemma 3.8, (i), BVP (1.1), (3.26) has no solution with the derivative having i or $i+1$ zeros, depending on α^*, on (a,b) for all $i \geqslant n+1$. We have reached a contradiction to $y \in \mathscr{T}_i^\gamma$.

The proof of the second part of Theorem 2.2, (i) is similar to above except that Lemma 3.8, (ii) instead of Lemma 3.8, (i), is used.

(ii) The proof is similar to part (i) and is omitted. □

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