Differential
Equations
& Applications
Volume 5, Number 1 (2013), 53–68
doi:10.7153/dea-05-04

POSITIVE PSEUDO–SYMMETRIC SOLUTIONS FOR A NONLOCAL p–LAPLACIAN BOUNDARY VALUE PROBLEM

L. X. TRUONG, P. D. PHUNG AND B. T. QUAN

(*Communicated by Johnny Henderson*)

Abstract. This paper is devoted to the study of the following nonlocal p-Laplacian functional differential equation

$$-\left(\phi_p(x'(t))\right)' = \frac{\lambda f(t,x(t),x'(t))}{\left(\int_0^1 f(s,x(s),x'(s))ds\right)^n}, \quad 0<t<1,$$

subject to multi point boundary conditions. We obtain some results on the existence of at least one (when $n \in \mathbb{Z}^+$) or triple (when $n = 0$) pseudo-symmetric positive solutions by using fixed-point theory in cone.

1. Introduction

Multi point boundary-value problems for second-order ordinary differential equations have many important and interesting applications, for details, please see [2, 4, 5, 6, 7, 8, 14, 15, 16] and the references therein.

In [3], Avery and Hederson introduced the definition of pseudo-symmetric function and discussed the existence of three positive pseudo-symmetric solutions for the problem

$$\begin{cases} (\phi_p(x'(t)))' + a(t)f(t,x(t)) = 0, \ t \in (0,1), \\ u(0) = 0, \ u(1) = u(\eta), \end{cases} \tag{1.1}$$

where $\phi_p(z) = |z|^{p-2}z$, $p > 1$ and $\eta \in (0,1)$. Next, by using the monotone iterative technique, Ma and Ge [11] proved the existence of positive pseudo-symmetric solutions for the problem (1.1). This technique is also used successfully by Sun and Ge [12] for the following problem with nonlinear term involving derivative

$$\begin{cases} (\phi_p(x'(t)))' + a(t)f(t,x(t),x'(t)) = 0, \ t \in (0,1), \\ u(0) = 0, \ u(1) = u(\eta). \end{cases} \tag{1.2}$$

Mathematics subject classification (2010): 34B07, 34B10, 34B18, 34B27.

Keywords and phrases: boundary value problem, pseudo-symmetric solutions, p-Laplacian, Leggett-Williams fixed point theorem, Guo-Krasnoselskii fixed point theorem.

This research is supported by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under Project 101.01-2012.12.

These results were considered more fully in [13].

The purpose of this work, motivated by the references [3, 11, 12, 13, 17], is to consider the existence of pseudo-symmetric positive solutions of the following nonlocal p-Laplacian boundary value problem

$$-\left(\phi_p(x'(t))\right)' = \frac{\lambda f(t,x(t),x'(t))}{\left(\int_0^1 f(s,x(s),x'(s))ds\right)^n}, \quad 0<t<1, \tag{1.3}$$

together with the boundary conditions

$$x(0) = \alpha_0 x'(0) + \sum_{i=1}^{m} \alpha_i x'(\eta_i), \tag{1.4}$$

and

$$\beta x(\eta_m) - \gamma x'(\eta_m) = \beta x(1) + \gamma x'(1), \tag{1.5}$$

where $m \geqslant 1$, $0<\eta_1<\eta_2<\cdots<\eta_m<1$; $\alpha_0>0$, $\alpha_i \geqslant 0$, $\forall i=1,2,\cdots,m$ and $\beta,\gamma \geqslant 0$, $\beta^2+\gamma^2 \neq 0$. The equations of the form (1.3) occur in many application models, see [1] and references therein. In the special case of $n=0$, we obtain that there exist at least triple positive pseudo-symmetric solutions to problem (1.3)-(1.5). When n is a non-negative integer number, the result on existence of at least one positive pseudo-symmetric solution is proved by using Guo-Krasnoselskii's fixed point theorem. The main difficulty arises from the presence of the nonlocal term. In our opinion, the monotone iterative technique is not suitable to be used.

2. Hypothesis and statement of results

Throughout, $E \equiv C^1([0,1];\mathbb{R})$ is the Banach space of all continuous function x from $[0,1]$ into $\mathbb{R}$ endowed with the sup-norm

$$\|x\|_1 = \max\left\{\max_{0\leqslant t\leqslant 1}|x(t)|, \max_{0\leqslant t\leqslant 1}|x'(t)|\right\}, \quad x\in E.$$

We recall that (see [3]) a function $x\in E$ is said to be pseudo-symmetric about η_m on $[0,1]$ if x is symmetric over the interval $[\eta_m,1]$, that is,

$$x(t) = x(1+\eta_m-t), \ \forall t\in[\eta_m,1].$$

By P we denote the cone in E defined by

$$P = \left\{ x\in E : \begin{cases} x(0)=\alpha_0 x'(0)+\sum_{i=1}^m \alpha_i x'(\eta_i), \\ x \text{ is nonnegative, concave and pseudo-symmetric about } \eta_m \text{ on } [0,1] \end{cases} \right\}.$$

In order to convenience to presentation later, we give here some notations needed

- $\sigma^* = (1+\eta_m)/2$

- $\mu = \max\{1, \sigma^* + \sum_{i=0}^{m}\sum_{i=0}^{m}\alpha_i\}$
- $\overline{\mu} = \eta_m + \sum_{i=0}^{m}\sum_{i=0}^{m}\alpha_i$
- $\mu_0 = (\eta_m \alpha_0)/(\mu\sigma^*)$
- $\widehat{m}(f,r) = \min\{f(t,z,w) : (t,z,w) \in [0,1] \times [\mu_0 r, r] \times [-r,r]\}$
- $m(f,r) = \min\{f(t,z,w) : (t,z,w) \in [0,1] \times [0,r] \times [-r,r]\}$
- $M(f,r) = \max\{f(t,z,w) : (t,z,w) \in [0,1] \times [0,r] \times [-r,r]\}$
- $\Omega_r = \{x \in P : \|x\|_1 < r\}$, $\partial\Omega_r = \{x \in P : \|x\|_1 = r\}$
- If δ is a nonnegative continuous concave function on P and r, R $(r < R)$ are two positive constants, we set

$$\Omega(\delta, r, R) = \{x \in P : \delta(x) \geqslant r \text{ and } \|x\|_1 \leqslant R\}.$$

Hypothesis

In the sequel, we alway use the following hypothesis

(H1) $n \in \mathbb{Z}^+, m \in \mathbb{Z}^+ \setminus \{0\}$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$; $\alpha_0 > 0$, $\alpha_i \geqslant 0$, $\forall i = 1, 2, \cdots, m$ and

$$\beta, \gamma \geqslant 0, \beta^2 + \gamma^2 \neq 0,$$

(H2) $f : [0,1] \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous function which satisfies the condtions:

(i) $f(t,z,w) > 0$, for all $(t,z,w) \in [0,1] \times [0,+\infty) \times \mathbb{R}$,

(ii) for every $(z,w) \in [0,+\infty) \times \mathbb{R}$, the function $f(\cdot,z,w)$ is pseudo-symmetric about η_m on $[0,1]$ which means that $f(t,z,w) = f(1+\eta_m - t, z, -w)$ for all $(t,z,w) \in [\eta_m, 1] \times [0,+\infty) \times \mathbb{R}$.

Main results

We can now state the main results of this paper. First, in the case of $n = 0$ we obtain the following theorem.

THEOREM 1. (Triple solutions) *Let $(H_1), (H_2)$ hold. Let $n = 0$. Suppose there exist positive constants a, b, c with $a < b < \frac{b}{\mu_0} < c$ and the positive constants A_0, B_0, C_0 such that*

$$\phi_p\left(\frac{\mu\sigma^*}{\overline{\mu}\eta_m}\right)\frac{\sigma^*}{\sigma^* - \eta_m} < \min\left\{\frac{A_0}{B_0}, \frac{C_0}{B_0}\right\}.$$

If the function f satisfies the following growth conditions

(A_1) $f(t,z,w) \leqslant \phi_p(c)/C_0$ *for* $(t,z,w) \in [0,1] \times [0,c] \times [-c,c]$;

(A_2) $f(t,z,w) < \phi_p(a)/A_0$ *for* $(t,z,w) \in [0,1] \times [0,a] \times [-a,a]$;

(A_3) $f(t,z,w) > \phi_p(b)/B_0$ *for* $(t,z,w) \in [\eta_m,1] \times [b,b/\mu_0] \times [-b/\mu_0, b/\mu_0]$,

then, for each λ *satisfies*

$$\frac{B_0}{\sigma^* - \eta_m}\phi_p\left(\frac{\sigma^*}{\overline{\mu}\eta_m}\right) \leqslant \lambda \leqslant \phi_p\left(\frac{1}{\mu}\right)\frac{1}{\sigma^*}\min\{A_0, C_0\}, \tag{2.1}$$

the boundary value problem (1.3)-(1.5) *has at least three positive pseudo-symmetric solutions* x_1, x_2, x_3 *satisfying*

$$\|x_1\|_1 \leqslant a,\ b < \min_{t\in[\eta_m,1]} x_2(t),\ a < \|x_3\|_1 \text{ with } \min_{t\in[\eta_m,1]} x_3(t) < b.$$

In general, for non-negative integer values of n, we establish an existence result of at least one pseudo-symmetric positive solution.

THEOREM 2. (one solution) *Let* (H_1) *and* (H_2) *hold. Assume that there are positive constants* a, A, b, B *such that*

(a) $\frac{A}{B}\left(\frac{\widehat{m}(f,a)}{M(f,b)}\right)^n > \frac{\sigma^*}{(\sigma^*-\eta_m)(1-\eta_m)^n}.\phi_p\left(\frac{\mu}{\overline{\mu}}\right)$,

(b) $f(t,z,w) \leqslant \frac{\phi_p(a)}{A}$, *for all* $(t,z,w) \in [0,1] \times [0,a] \times [-a,a]$,

(c) $f(t,z,w) \geqslant \frac{\phi_p(b)}{B}$, *for all* $(t,z,w) \in [0,1] \times [\mu_0 b, b] \times [-b,b]$.

Then, for every λ *satisfies*

$$\lambda_{\min} := \frac{B[M(f,b)]^n}{\sigma^* - \eta_m}\phi_p\left(\frac{1}{\overline{\mu}}\right) \leqslant \lambda \leqslant \frac{A[\widehat{m}(f,a)]^n(1-\eta_m)^n}{\sigma^*}\phi_p\left(\frac{1}{\mu}\right) := \lambda_{\max},$$

the problem (1.3)-(1.5) *has at least one positive pseudo-symetric solution.*

3. Preliminaries results

We start this section with some useful lemmas.

LEMMA 1. (see [3]) *For all* $x \in P$, *the following statements are true:*

(i) $\max_{t\in[0,1]} x(t) = x(\sigma^*)$,

(ii) $x(t) \geqslant \frac{x(\sigma^*)}{\sigma^*}\min\{t, 1+\eta_m - t\},\ t \in [0,1]$,

(iii) $x(t) \geqslant \frac{\eta_m}{\sigma^*}x(\sigma^*),\ t \in [\eta_m, 1]$.

LEMMA 2. *For all* $x \in P$, *the following statements are true:*

(i) $\max_{t\in[0,1]}|x'(t)| = x'(0)$

(ii) $\|x\|_1 \leqslant \mu x'(0)$

(iii) $x(t) \geqslant \mu_0 \|x\|_1,\ t \in [\eta_m, 1]$

here we recall that

$$\mu = \max\{1, \sigma^* + \textstyle\sum_{i=0}^m \alpha_i\} \text{ and } \mu_0 = \frac{\eta_m \alpha_0}{\mu \sigma^*}.$$

Proof. (i) By the definition of cone P, it is clear that $x'(t)$ is decreasing function on $[0,1]$. Further, it follows from (i) of the Lemma 1 that $x'(\sigma^*)$ has to be zero. So

$$x'(t) \geqslant 0, \forall t \in [0,\sigma^*] \ \text{ and } \ x'(t) \leqslant 0, \forall t \in [\sigma^*, 1],$$

which implies

$$\max_{t\in[0,1]} |x'(t)| = \max\{|x'(0)|, |x'(1)|\}.$$

On the other hand, because x is pseudo-symmetric about η_m we have

$$x'(0) \geqslant x'(\eta_m) = -x'(1).$$

Therefore $\max_{t\in[0,1]}|x'(t)| = |x'(0)| = x'(0)$.
(ii) For $x \in P$, we have

$$\begin{aligned} x(\sigma^*) &= x(0) + \int_0^{\sigma^*} x'(s)\,ds \\ &= \alpha_0 x'(0) + \sum_{i=1}^m \alpha_i x'(\eta_i) + \int_0^{\sigma^*} x'(s)\,ds \end{aligned} \tag{3.1}$$

which follows

$$\begin{aligned} x(t) \leqslant x(\sigma^*) &\leqslant \alpha_0 |x'(0)| + \sum_{i=1}^m \alpha_i |x'(\eta_i)| + \int_0^{\sigma^*} |x'(s)|\,ds \\ &\leqslant (\sigma^* + \textstyle\sum_{i=0}^m \alpha_i) \max_{0\leqslant t\leqslant 1} |x'(t)|,\ t\in[0,1]. \end{aligned}$$

Hence

$$\|x\|_1 = \max\left\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |x'(t)|\right\} \leqslant \mu |x'(0)|.$$

(iii) It follows from (3.1) and [Lemma 1, (ii)] that, for $t \in [\eta_m, 1]$,

$$x(t) \geqslant \frac{\eta_m}{\sigma^*} x(\sigma^*) \geqslant \frac{\eta_m \alpha_0}{\sigma^*} x'(0) \geqslant \frac{\eta_m \alpha_0}{\mu \sigma^*} \|x\|_1.$$

This ends our proofs.

Now we consider an existence result for auxiliary linear problem.

LEMMA 3. *Let $h:[0,1]\to\mathbb{R}$ be a continuous, nonnegative and pseudo-symmetric about η_m function, $h(t)\not\equiv 0$ on any subinterval of $[0,1]$. Then the boundary value problem*

$$\begin{cases} -\left(\phi_p\left(x'(t)\right)\right)' = h(t), \ \ 0<t<1, \\ x(0) = \alpha_0 x'(0) + \sum_{i=1}^m \alpha_i x'(\eta_i), \\ \beta x(\eta_m) - \gamma x'(\eta_m) = \beta x(1) + \gamma x'(1), \end{cases} \tag{3.2}$$

has a unique positive pseudo - symmetric solution u which is given by the integral representation formula

$$x(t) = Ah(t) := \begin{cases} \Lambda + \int_0^t \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds, 0\leqslant t\leqslant \sigma^*, \\ \widetilde{\Lambda} + \int_t^1 \phi_q\left(\int_{\sigma^*}^s h(r)\,dr\right) ds, \sigma^*\leqslant t\leqslant 1, \end{cases} \tag{3.3}$$

where

$$\Lambda = \alpha_0\phi_q\left(\int_0^{\sigma^*} h(r)\,dr\right) + \sum_{i=1}^m \alpha_i\phi_q\left(\int_{\eta_i}^{\sigma^*} h(r)\,dr\right)$$

and

$$\widetilde{\Lambda} = \Lambda + \int_0^{\eta_m} \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds.$$

Proof. First, it's not difficult to check that the function $x(t)$ given by (3.3) is a solution of the problem (3.2). Conversely, let $x\in E$ be a solution of (3.2). Then, by h is nonnegative, we deduce from $(3.2)_1$ that

$$\left(\phi_p\left(x'(t)\right)\right)' \leqslant 0, \ \forall t\in(0,1).$$

So $x'(t)$ is monotonically decreasing on $[0,1]$. We shall show that there exists $\sigma\in(\eta_m,1)$ such that $x'(\sigma)=0$ by using the boundary condition $(3.2)_3$. Indeed we consider the following cases.

Case 1 ($\gamma=0$): In this case our affirmation is evident by $x(1)=x(\eta_m)$

Case 2 ($\beta=0$): The boundary condition $(3.2)_3$ becomes $x'(1)+x'(\eta_m)=0$ which implies

$$x'(1)\,.x'(\eta_m)<0.$$

Hence by $x'(t)$ is continuous on $[\eta_m,1]$ we deduce that there exists $\sigma\in(\eta_m,1)$ such that $x'(\sigma)=0$.

Case 3 ($\beta,\gamma\neq 0$): Assume by contradiction that $x'(t)>0$, for all $t\in(\eta_m,1)$. This follows that $x(t)$ is strictly increasing on $[\eta_m,1]$. Hence

$$\beta x(\eta_m) - \gamma x'(\eta_m) < \beta x(1) - \gamma x'(1) < \beta x(1) + \gamma x'(1),$$

and we get a contradiction due to (1.5). Similarly, if $x'(t) < 0$, for all $t \in (\eta_m, 1)$ then $x(t)$ is strictly decreasing on $[\eta_m, 1]$ which leads to an other contradiction:

$$\beta x(\eta_m) - \gamma x'(\eta_m) > \beta x(1) + \gamma x'(\eta_m) > \beta x(1) + \gamma x'(1).$$

Next by integrating the equation (3.2) on $[s, \sigma]$ we get

$$\phi_p\left(x'(s)\right) = \phi_p\left(x'(\sigma)\right) + \int_s^\sigma h(r)\,dr,$$

which implies

$$\begin{aligned} x(t) &= x(0) + \int_0^t x'(s)\,ds \\ &= \alpha_0 x'(0) + \sum_{i=1}^m \alpha_i x'(\eta_i) + \int_0^t \phi_q\left(\int_s^\sigma h(r)\,dr\right) ds \\ &= \alpha_0 \phi_q\left(\int_0^\sigma h(r)\,dr\right) + \sum_{i=1}^m \alpha_i \phi_q\left(\int_{\eta_i}^\sigma h(r)\,dr\right) + \int_0^t \phi_q\left(\int_s^\sigma h(r)\,dr\right) ds, \end{aligned}$$

for all $t \in [0,1]$. On the other hand, as $\beta x(\eta_m) - \gamma x'(\eta_m) = \beta x(1) + \gamma x'(1)$, σ has to be a solution of the following equation

$$\Psi_1(\sigma) = \Psi_2(\sigma), \tag{3.4}$$

where

$$\Psi_1(\sigma) = \beta \int_{\eta_m}^1 \phi_q\left(\int_s^\sigma h(r)\,dr\right) ds,$$

$$\Psi_2(\sigma) = \gamma\left[\phi_q\left(\int_\sigma^1 h(r)\,dr\right) - \phi_q\left(\int_{\eta_m}^\sigma h(r)\,dr\right)\right].$$

By h is pseudo-symmetric about η_m and $\beta^2 + \gamma^2 \neq 0$, we can verify that $\sigma^* = \frac{1+\eta_m}{2}$ is the unique solution of (3.4). So

$$\sigma = \sigma^*$$

and we conclude that if $t \in [0, \sigma^*]$ then

$$\begin{aligned} x(t) &= \alpha_0 \phi_q\left(\int_0^{\sigma^*} h(r)\,dr\right) + \sum_{i=1}^m \alpha_i \phi_q\left(\int_{\eta_i}^{\sigma^*} h(r)\,dr\right) + \int_0^t \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds \\ &= \Lambda + \int_0^t \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds, \end{aligned}$$

and if $t \in [\sigma^*, 1]$ then

$$\begin{aligned} x(t) = \Lambda &+ \int_0^{\eta_m} \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds \\ &+ \int_t^1 \phi_q\left(\int_{\sigma^*}^s h(r)\,dr\right) ds + \int_{\eta_m}^1 \phi_q\left(\int_s^{\sigma^*} h(r)\,dr\right) ds \end{aligned}$$

$$= \Lambda + \int_0^{\eta_m} \phi_q \left(\int_s^{\sigma^*} h(r)\,dr \right) ds + \int_t^1 \phi_q \left(\int_{\sigma^*}^s h(r)\,dr \right) ds$$

thanks to the relation

$$\int_{\eta_m}^1 \phi_q \left(\int_s^{\sigma^*} h(r)\,dr \right) ds = 0$$

which is a consequence of pseudo-symmetric about η_m property of the function h.

Finally, if x_1 and x_2 are solutions of problem (3.2) then, by using the above arguments, we have $x_1(t) = x_2(t) = Ah(t), \forall t \in [0,1]$. This shows the uniqueness of solutions. The proof of this lemma is complete.

LEMMA 4. *$A : P \to P$ defined by (3.3) is a completely continuous operator.*

Proof. It follows from lemma 3 that $Ah \in P$, for all $h \in P$. On the other hand, a standard argument by using Arzelà - Ascoli theorem allows us to conclude that A is completely continuous.

Now, for each $x \in P$, we denote

$$F(x)(t) = \lambda f(t, x(t), x'(t)) \left(\int_0^1 f(s, x(s), x'(t))ds \right)^{-n}, t \in [0,1]. \tag{3.5}$$

From the assumption (H2), we deduce that the operator $F : P \to C([0,1];\mathbb{R})$ is continuous. Moreover $F(x)$ is positive on $[0,1]$, and pseudo-symmetric about η_m. So the operator

$$T \equiv A \circ F : P \to P$$

is also completely continuous by using lemma 4. It is also note that each nonzero fixed point of the operator T is a positive pseudo-symmetric solution of the problem (1.3)-(1.5).

Finally, in order to prove our main results we need to use the following fixed point theorems

THEOREM 3. (Leggett-Williams, see [10]) *Let K be a cone in the Banach space $(E, \|\cdot\|)$, $c > 0$ be a constant and define the set*

$$\overline{K}_c = \overline{\{x \in K : \|x\| < c\}}.$$

Let $T : \overline{K}_c \to \overline{K}_c$ be a completely continuous map. Suppose there exists a concave nonegative functional δ defined on K and there are numbers a, b, c with $0 < a < b < d \leqslant c$ such that

(a) $\{x \in S(\delta, b, d) : \delta(x) > b\} \neq \emptyset$ *and* $\delta(Tx) > b$, *for* $x \in S(\delta, b, d)$, *where*

$$S(\delta, r, R) = \{x \in P : \delta(x) \geqslant r \text{ and } \|x\| \leqslant R\},$$

(b) $\|Tx\| < a$, *for* $\|x\| \leqslant a$ *and*

(c) $\delta(Tx) > b$, *for* $x \in S(\delta, b, c)$ *with* $\|Tx\| > d$.

Then T *has at least three fixed points,* x_1, x_2 *and* x_3 *satisfying*

$$\|x_1\| < a, \ \ b < \delta(x_2) \text{ and } \|x_3\| > a, \text{ with } \delta(x_3) < b.$$

THEOREM 4. (Guo-Krasnoselskii, see [9]) *Let* E *be a Banach space and let* $K \subset E$ *be a cone in* E. *Assume that* Ω_1, Ω_2 *are open with* $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, *and let*

$$T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow K$$

be a completely continuous operator such that either

(i) $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_2$, *or*

(ii) $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_2$.

Then T *has a fixed point in* $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

4. Proofs and corollaries of main Theorems

First we will discuss the criterion for existence at least three positive pseudo-symmetric solutions of problem (1.3)-(1.5) by using Theorem 3.

Proof. [Proof of Theorem 1] We shall apply Theorem 3. For this we define the nonnegative continuous concave function on P by

$$\delta(x) = \min_{t \in [\eta_m, 1]} x(t).$$

Obviously, $\delta(x) \leqslant \|x\|$, for all $x \in P$. Let λ satisfy (2.1).

♦ First we prove that $T : \overline{\Omega}_c \to \overline{\Omega}_c$. In fact, for $x \in \overline{\Omega}_c$, we have $0 \leqslant x(t) \leqslant c$ and $|x'(t)| \leqslant c$, for all $t \in [0,1]$. So it follows from condition (A1) that

$$Fx(t) = \lambda f(t, x(t), x'(t)) \leqslant \frac{\lambda \phi_p(c)}{C_0},$$

for all $t \in [0,1]$. Using the lemma 2 and the definition of T we deduce

$$\begin{aligned}
\|Tx\|_1 &\leqslant \mu \left|(Tx)'(0)\right| \leqslant \mu \phi_q \left(\int_0^{\sigma^*} F(x)(\tau)\, d\tau\right) \\
&\leqslant \mu \phi_q(\lambda) \phi_q\left(\frac{\sigma^*}{C_0}\right) c \leqslant c.
\end{aligned}$$

Hence $Tx \in \overline{\Omega}_c$. By an analogous argument as above, it follows that $\|Tx\|_1 < a$, for $\|x\|_1 \leqslant a$, i.e. the condition (b) of theorem 3 is satisfied.

♦ To fulfill condition (a) of theorem 3, we consider the function

$$x_0(t) = \begin{cases} \kappa\left(t + \sum_{i=0}^{m} \alpha_i\right), & t \in [0, \eta_m], \\ \kappa\left(\overline{\mu} + \frac{1-\eta_m}{4}\right) - \frac{\kappa}{1-\eta_m}(t-\sigma^*)^2, & t \in [\eta_m, 1], \end{cases}$$

where κ is a constant satisfying

$$\frac{b}{\overline{\mu}} < \kappa < \frac{b}{\mu_0 \max\left\{1, \overline{\mu} + \frac{1-\eta_m}{4}\right\}}.$$

We can check that x_0 is a member of

$$\Omega\left(\delta, b, \frac{b}{\mu_0}\right) = \left\{x \in P : \delta(x) \geqslant b \text{ and } \|x\|_1 \leqslant \frac{b}{\mu_0}\right\}$$

without difficulty. Further

$$\delta(x_0) = \min_{t \in [\eta_m, 1]} x_0(t) = \kappa\overline{\mu} > b.$$

This shows that the set $\{x \in \Omega(\delta, b, b/\mu_0) : \delta(x) > b\}$ is non-empty. Next, let $x \in \Omega(\delta, b, b/\mu_0)$. Then

$$\begin{aligned} Tx(\sigma^*) \geqslant{}& \alpha_0 \phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)\, d\tau\right) + \sum_{i=1}^{m} \alpha_i \phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)\, d\tau\right) \\ &+ \int_0^{\eta_m} \phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)\, d\tau\right) ds, \end{aligned} \tag{4.1}$$

where

$$F(x)(t) = \lambda f(t, x(t), x'(t)) > \frac{\lambda \phi_p(b)}{B_0}, \forall t \in [\eta_m, \sigma^*]. \tag{4.2}$$

Combining (4.1) – (4.2) and (iii) of Lemma 1, we get

$$\begin{aligned} \delta(Tx) = \min\{Tx(t) : t \in [\eta_m, 1]\} &\geqslant \frac{\eta_m}{\sigma^*} Tx(\sigma^*) \\ &> \frac{\overline{\mu}\eta_m}{\sigma^*} \phi_q(\lambda)\phi_q\left(\frac{\sigma^* - \eta_m}{B_0}\right) b \geqslant b, \ \forall x \in \Omega(\delta, b, b/\mu_0). \end{aligned}$$

♦ Finally, we exhibit condition (c) of theorem 3 is also satisfied. To this end, let $x \in \Omega(\delta, b, c)$ with $\|Tx\|_1 > b/\mu_0$. An application of (iii) in Lemma 2 yields

$$\delta(Tx) = \min\{Tx(t) : t \in [\eta_m, 1]\} \geqslant \mu_0 \|Tx\|_1 > b.$$

The proof is finished by an application of theorem 3.

EXAMPLE 1. Consider the following problem

$$\begin{cases} -\left(\left|x'(t)\right|x'(t)\right)' = \lambda f(t,x(t),x'(t)),\ 0<t<1, \\ x(0)=x'(0),\ \ 2x\left(\frac{1}{3}\right)-x'\left(\frac{1}{3}\right)=2x(1)+x'(1), \end{cases}$$

where

$$f(t,z,w)=\left[1+\sin\left(\frac{3\pi}{4}t\right)\right]\rho(z,w),\ \text{for all } (t,z,w)\in[0,1]\times\mathbb{R}^{+}\times\mathbb{R},$$

with

$$\rho(z,w)=\begin{cases} \frac{1}{40}z^2+\frac{1}{40+\cos^2 w}, & (z,w)\in[0,1]\times[-10,10], \\ \left(\sqrt{30}-1/8\right)(z-1)+1/40+\frac{1}{40+\cos^2 w}, & (z,w)\in[1,6/5]\times[-10,10], \\ \sqrt{z}+\frac{1}{40+\cos^2 w}, & (z,w)\in[6/5,10]\times[-10,10]. \end{cases}$$

In this case we have

$$p=3,\ n=0,\ m=1,\ \alpha_0=1,\ \alpha_1=0,\eta_1=\frac{1}{3},$$

and so

$$\sigma^*=\frac{2}{3},\ \mu=\frac{5}{3},\ \overline{\mu}=\frac{4}{3},\ \mu_0=\frac{3}{10}.$$

Take $a=1,\ b=6/5, c=10$ and $A_0=48/5, B_0=19/25, C_0=\ 200/13$. Then we have $a<b<b/\mu_0<c$ and

- $\sigma^*\phi_p\left(\mu\sigma^*/\overline{\mu}\eta_m\right)\left(\sigma^*-\eta_m\right)=25/2<\min\left\{A_0/B_0, C_0/B_0\right\}\approx 12.632$
- $f(t,z,w)\leqslant 6.3746<6.5=\phi_p(c)/C_0$, for $(t,z,w)\in[0,1]\times[0,10]\times[-10,10]$,
- $f(t,z,w)<0.1<\phi_p(a)/A_0=5/48$, for $(t,z,w)\in[0,1]\times[0,1]\times[-1,1]$,
- $f(t,z,w)\geqslant 1.9117>\phi_p(b)/B_0\approx 1.8947$, for $(t,z,w)\in[1/3,1]\times[6/5,4]\times[-4,4]$.

This shows that the conditions of Theorem 1 hold and we can conclude that, for every $\lambda\in[5.13, 5.184]$, our problem has at least three positive pseudo-symmetric solutions x_1, x_2, x_3 such that

$$\|x_1\|_1\leqslant 1, \delta(x_2)>\frac{6}{5}, \delta(x_3)<\frac{6}{5}, \|x_3\|_1>1.$$

Proof. [Proof of Theorem 2] Let $\lambda\in(\lambda_{\min},\lambda_{\max})$. It is clear that Ω_a, Ω_b are open bounded subsets of E and

$$0\in\Omega_a,\Omega_b \text{ and } \left(\overline{\Omega}_b\subset\Omega_a \text{ or } \overline{\Omega}_a\subset\Omega_b\right).$$

For $x \in P \cap \partial\Omega_a$, we have

$$\begin{cases} 0 \leqslant x(t) \leqslant \max_{t\in[0,1]} |x(t)| \leqslant \|x\|_1 = a, \\ |x'(t)| \leqslant \max_{t\in[0,1]} |x'(t)| \leqslant \|x\|_1 = a, \end{cases}$$

for all $t \in [0,1]$. Further $x(t) \geqslant \mu_0 \|x\|_1 = \mu_0 a,\ \forall t \in [\eta_m, 1]$. It follows from (3.5) and the assumption (b) of Theorem 2 that

$$\begin{aligned} F(x)(t) &= \frac{\lambda f(t,x(t),x'(t))}{\left(\int_0^1 f(s,x(s),x'(s))ds\right)^n} \leqslant \frac{\lambda f(t,x(t),x'(t))}{\left(\int_{\eta_m}^1 f(s,x(s),x'(s))ds\right)^n} \\ &\leqslant \frac{\lambda \phi_p(a)}{A\left[\widehat{m}(f,a)\right]^n (1-\eta_m)^n}, \forall t \in [0,1]. \end{aligned}$$

By the definition of operator T and the lemma 2

$$\begin{aligned} \|Tx\|_1 &\leqslant \mu (Tx)'(0) \leqslant \mu \phi_q\left(\int_0^{\sigma^*} F(x)(\tau)d\tau\right) \\ &\leqslant a\mu\phi_q(\lambda)\phi_q\left(\frac{\sigma^*}{A\left[\widehat{m}(f,a)\right]^n (1-\eta_m)^n}\right) \\ &\leqslant a = \|x\|_1 \end{aligned}$$

Now, let $x \in P \cap \partial\Omega_b$. By Lemma 2 we have

$$\begin{cases} \mu_0 b \leqslant \mu_0 \|x\|_1 \leqslant x(t) \leqslant \|x\|_1 = b, \\ |x'(t)| \leqslant \|x\|_1 = b, \end{cases}$$

for all $t \in [\eta_m, 1]$. This implies from the assumption (c) that

$$F(x)(t) = \frac{\lambda f(t,x(t),x'(t))}{\left(\int_0^1 f(s,x(s),x'(s))ds\right)^n} \geqslant \frac{\lambda \phi_p(b)}{B\left[M(f,b)\right]^n}, \forall t \in [\eta_m, 1].$$

So using lemma 1 it follows

$$\begin{aligned} \|Tx\|_1 &\geqslant \max_{t\in[0,1]} |Tx(t)| = Tx(\sigma^*) \\ &\geqslant \alpha_0\phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)d\tau\right) + \sum_{i=1}^m \alpha_i\phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)d\tau\right) \\ &\quad + \int_0^{\eta_m} \phi_q\left(\int_{\eta_m}^{\sigma^*} F(x)(\tau)d\tau\right)ds \\ &\geqslant b\overline{\mu}\phi_q(\lambda)\phi_q\left(\frac{\sigma^*-\eta_m}{B\left[M(f,b)\right]^n}\right) \\ &\geqslant b = \|x\|_1. \end{aligned}$$

Applying theorem 4 we deduce that the operator T has a fixed point in $\Omega_a \backslash \overline{\Omega}_b$ or $\Omega_b \backslash \overline{\Omega}_a$. So our result is completely proved.

COROLLARY 1. *Let (H1) holds. Assume that*

♦ a, b, A, B *are four positive constants satisfy* $a > (\zeta_0 b)/\zeta_1$ *and*

$$\zeta_0 \frac{a}{b} < \frac{A}{B} < \zeta_1 \frac{a^2}{b^2},$$

where

$$\zeta_1 \in (0,1) \quad \textit{and} \quad \zeta_0 = \sqrt{\frac{\sigma^*}{(\sigma^* - \eta_m)(1-\eta_m)}.\phi_p\left(\frac{\mu}{\overline{\mu}}\right)},$$

♦ $\theta : [0,1] \to \mathbb{R}^*$ *is continuous, pseudo-symmetric about* η_m *on* $[0,1]$ *and satisfies*

$$0 < \zeta_1 = \min_{t\in[0,1]} \theta(t) < \max_{t\in[0,1]} \theta(t) = \overline{\theta} < 1,$$

♦ $\varphi : [0,a] \to \mathbb{R}^*$ *is a continuous function such that*

$$\min_{z\in[\mu_0 b, a]} \varphi(z) \geqslant \frac{b^2}{B\zeta_1}, \ \max_{z\in[0,a]} \varphi(z) \leqslant \frac{a^2}{A}.$$

Set

$$f(t,z,w) = \theta(t)\varphi(z) + \frac{a^2(1-\overline{\theta})}{A}\left(1 - \sin^2 \frac{\pi w}{a}\right).$$

Then, for every λ *satisfying the condition*

$$\frac{BM(f,b)}{\sigma^* - \eta_m}\phi_p\left(\frac{1}{\overline{\mu}}\right) \leqslant \lambda \leqslant \frac{A\widehat{m}(f,a)(1-\eta_m)}{\sigma^*}\phi_p\left(\frac{1}{\mu}\right),$$

the boundary value problem

$$\begin{cases} -(|x'(t)|x'(t))' = \frac{\lambda f(t,x(t),x'(t))}{\int_0^1 f(t,x(t),x'(t))dt}, \ 0 < t < 1, \\ x(0) = \alpha_0 x'(0) + \sum_{i=1}^m \alpha_i x'(\eta_i), \\ \beta x(\eta_m) - \gamma x'(\eta_m) = \beta x(1) + \gamma x'(1), \end{cases}$$

has at least one positive pseudo-symmetric solution.

Proof. It is clear that f is continuous on $[0,1] \times (0,+\infty) \times \mathbb{R}$ and satisfies the condition (H2). On the other hand we have

$$M(f,b) \leqslant \overline{\theta}.\frac{a^2}{A} + \frac{a^2(1-\overline{\theta})}{A} \leqslant \frac{a^2}{A},$$

and

$$\widehat{m}_a(f,a) \geqslant \zeta_1. \min_{z\in[\mu_0 b, a]} \varphi(z) \geqslant \frac{b^2}{B}.$$

So, we can obtain the following estimates

$$\frac{A\widehat{m}(f,a)}{BM_b(f,b)} \geqslant \frac{A^2b^2}{B^2a^2} > \zeta^2 = \frac{\sigma^*}{(\sigma^*-\eta_m)(1-\eta_m)}.\phi_p\left(\frac{\mu}{\overline{\mu}}\right),$$

$$f(t,z,w) \leqslant \frac{a^2}{A}, \forall (t,z,w) \in [0,1]\times[0,a]\times[-a,a],$$

$$f(t,z,w) \geqslant \frac{b^2}{B}, \forall (t,z,w) \in [0,1]\times[\mu_0 b,b]\times[-b,b],$$

which means that the conditions of Theorem 2 are satisfied. Therefore, our conclude is proved.

EXAMPLE 2. Let

- $\eta_1 = 1/4,\ \eta_2 = 2/3,\ \alpha_0 = 2,\ \alpha_1 = \frac{1}{2},\ \alpha_2 = 1, \zeta_1 = 5/18$
- $a = 30, b = 2, A = 122, B = 2$
- $f(t,z,w) = \left[\frac{35}{36} - \left(t-\frac{5}{6}\right)^2\right]\left(\frac{1}{250}z + \frac{36}{5}\right) + \frac{25}{122}\left(1-\sin^2\frac{\pi w}{15}\right)$

In this case we have

$$\begin{cases} \sigma^* = (1+\eta_m)/2 = 5/6,\ \sum_{i=0}^m \alpha_i = 7/2, \\ \mu = \max\{1, \sigma^* + \sum_{i=0}^m \alpha_i\} = 13/3, \\ \overline{\mu} = \eta_m + \sum_{i=0}^m \alpha_i = 25/6, \\ \mu_0 = \frac{\eta_m \alpha_0}{\mu \sigma^*} = 24/65, \\ \zeta_0 = \sqrt{\frac{\sigma^*}{(\sigma^*-\eta_m)(1-\eta_m)}.\phi_p\left(\frac{\mu}{\overline{\mu}}\right)} = \frac{26}{25}\sqrt{15}. \end{cases}$$

On the other hand

$$M(f,b) = \max\{f(t,z,w) : (t,z,w) \in [0,1]\times[0,2]\times[-2,2]\} = \frac{395\,977}{54\,900},$$

and

$$\widehat{m}(f,a) = \min\left\{f(t,z,w) : (t,z,w) \in [0,1]\times\left[\frac{144}{13},30\right]\times[-30,30]\right\} = \frac{654}{325}.$$

It is not difficulty to check that the conditions of corollary 1 are satisfied. Consequently the following boundary value problem

$$-\left(\left|x'(t)\right|x'(t)\right)' = \frac{\lambda f(t,x(t),x'(t))}{\int_0^1 f(t,x(t),x'(t))\,dt},\ 0<t<1,$$

$$x(0) = 2x'(0) + \frac{1}{2}x'\left(\frac{1}{4}\right) + x'\left(\frac{2}{3}\right),$$

$$x\left(\frac{2}{3}\right) - 2x'\left(\frac{2}{3}\right) = x(1) + 2x'(1),$$

has at least one positive pseudo-symmetric solution for every λ satisfying

$$4.9854 \approx \frac{4751724}{953125} \leqslant \lambda \leqslant \frac{1436184}{274625} \approx 5.2296.$$

Acknowledgements. The authors wish to express their sincere thanks to the referees for the suggestions and valuable comments.

REFERENCES

[1] M. R. S. AMMI, D. F. M. TORRES, *Existence of positive solutions for nonlocal p-Laplacian thermistor problems on time scales*, J. Inequalities in Pure and Apll. Math., **8** (2007), 1–10.

[2] R.I. AVERY, J. HENDERSON, *Three symmetric positive solutions for a second order boundary value problem*, Appl. Math. Lett., **13** (2000), 1–7.

[3] R. AVERY, J. HENDERSON, *Existence of three positive pseudo-symmetric solutions for a one dimensional p-Laplacian operator*, J. Math. Anal. Appl., **277** (2003), 395–404.

[4] R.P. AGARWAL, H. L¨U, D. O'REGAN, *Eigenvalues and the one-dimensional p-Laplacian*, J. Math. Anal. Appl., **266** (2002), 383–340.

[5] Y. GUO, W. GE, *Three positive solutions for the one-dimensional p-Laplacian*, J. Math. Anal. Appl., **286** (2003), 491–508.

[6] Y. GUO, W. GE, *Positive solutions for three point boundary value problems with dependence on the first order derivative*, J. Math. Anal. Appl., **290** (2004), 291–301.

[7] V.A. IL'IN, E.I. MOISEEV, *Nonlocal boundary value problem of the rst kind for a Sturm-Liouville operator*, Differ. Equations, **23** (1987), 979–987.

[8] V.A. IL'IN, E.I. MOISEEV, *Nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in the differential and difference treatment*, Differ. Equations, **23** (1987), 1198–1207.

[9] M. A. KRASNOSEL'SKII, *Positive solution of Operator Equations*, Noordho, Groningen, 1964.

[10] R.W. LEGGETT, L.R. WILLIAMS, *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J., **28** (1979), 673–688.

[11] D. MA, W. GE, *Existence and iteration of positive pseudo-symmetric solutions for a three-point second-order p-Laplacian BVP*, Appl. Math. Lett., **20** (2007), 1244–1249.

[12] B. SUN, W. GE, *Successive iteration and positive pseudo-symmetric solutions for a three-point second-order p-Laplacian boundary value problems*, Appl. Math. Comput., **188** (2007), 1772–1779.

[13] Y. H. SU, W. WU, X. YANG, *Existence Theory for Pseudo-Symmetric Solution to p-Laplacian Differential Equations Involving Derivative*, Abstract and Applied Analysis, **2011** (2011), Article ID 182831, 19 pages.

[14] L. X. TRUONG, L. T. P. NGOC, N. T. LONG, *Positive solutions for an m-point boundary value problem*, Electron. J. Differential Eqns., **111** (2008), 1–11.

[15] L.X. TRUONG, P.D. PHUNG, *Existence of positive solutions for a multi-point four-order boundary-value problem*, Electron. J. Differential Eqns., **119** (2011), 1–10.

[16] J.Y. WANG, *The existence of positive solutions for the one-dimensional p-laplacian*, Proc. Amer. Math. Soc., **125** (1997), 2275–2283.

[17] Y. YANG, *Existence of positive pseudo-symmetric solutions for one dimensional p-Laplacian boundary value problem*, Electron. J. Differential Eqns., **70** (2007), 1–6.

(Received September 11, 2012)

L. X. Truong
Department of Mathematics and Statistics
University of Economics HoChiMinh City
59C, Nguyen Dinh Chieu Str, District 3, HoChiMinh city
Vietnam
e-mail: lxuantruong@gmail.com

P. D. Phung
Nguyen Tat Thanh University
300A, Nguyen Tat Thanh Str, District 4
HoChiMinh city
Vietnam
e-mail: pdphung@ntt.edu.vn

B. T. Quan
Dong Nai University
4 Le Quy Don Str
District Tan Hiep, Bien Hoa city
Vietnam
e-mail: buithequan2002@yahoo.com