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QUASILINEAR ELLIPTIC PROBLEM WITH HARDY POTENTIAL AND A REACTION–ABSORBTION TERM

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Abstract. We consider the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u \pm u^q = \lambda \dfrac{u^{p-1}}{|x|^p} + h & \text{in } \Omega, \\ u \geqslant 0 \text{ and } u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, $1 < p < N, \Omega \subset \mathbb{R}^N$ is a bounded regular domain such that $0 \in \Omega$, $q > p-1$ and h is a nonnegative measurable function with suitable hypotheses. The main goal of this paper is to analyze the interaction between the Hardy potential, and the term u^q, in order to get existence and non existence of positive solution. We can summarize our main results, in the two following points:

(i) If u^q appears as a reaction term, then we show the existence of a critical exponent $q_+(\lambda)$, such that for $q > q_+$, the considered problem has no positive distributional solution. If $q < q_+$ we find solutions under suitable hypothesis on h.

(ii) If u^q appears as an absorption term, then there exists q_* such that if $q > q_*$, the problem under consideration has a positive solution for all $\lambda > 0$ and for all $h \in L^1(\Omega)$. The optimality of q_* is proved in the sense that if $q < q_*$, then nonexistence holds if $\lambda > \Lambda_{N,p}$.

1. Introduction and preliminaries results

In this paper we study existence and nonexistence of positive solutions to the problem

$$(P_\pm) \begin{cases} -\Delta_p u \pm u^q = \lambda \dfrac{u^{p-1}}{|x|^p} + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < N$, $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin, $q > p-1$ and h is a nonnegative measurable function, with suitable hypotheses.

Problem $(P_\pm)$ is related to the classical Hardy-Sobolev inequality

$$\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p}\, dx \leqslant \int_{\mathbb{R}^N} |\nabla \phi|^p\, dx \quad \text{for all } \phi \in \mathscr{C}_0^\infty(\mathbb{R}^N),$$

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where $\Lambda_{N,p} = (\frac{N-p}{p})^p$ is optimal and not achieved, we refer to [14] for more details about this constant.

In the case where u^q appears as a reaction term (problem (P_-)), then for $\lambda > \Lambda_{N,p}$, a strong local nonexistence result is obtained in [4].

The case $p=2$ and $\lambda \leqslant \Lambda_{N,2}$ was studied in [12], the authors prove the existence of a critical exponent $q_+(\lambda)$ such that existence holds if and only if $q < q_+$.

If $p \neq 2$ and $q \leqslant p^* - 1$, the problem is widely studied in the literature, we refer to [2] where the authors got the exact behavior of the solution near the origin and studied also the case where $\Omega = \mathbb{R}^N$.

In the case where u^q appears as an absorption term, then if $\lambda = 0$, the existence and uniqueness of "entropy" solution is obtained in [11]. If $\lambda \leqslant \Lambda_{N,p}$ and $h \in L^{\frac{p^*}{p^*-1}}(\Omega)$, then existence holds in the Sobolev space $W_0^{1,p}(\Omega)$ using variational arguments. The authors proved that for all $h \in L^1(\Omega)$, there exists at least one distributional solution. The regularity of the solution is obtained according to the one of h and the value of q.

If $\lambda > 0$, the situation is a quite different, in the case where $q=0$ and $p=2$, then an integrability condition on h near the origin is needed to insure the existence of a distributional solution; see [5] for a complete discussion about this case.

The problem (P_-) can also be seen, as the stationary case associated to the parabolic problem:

$$\begin{cases} u_t - \Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + u^q + f, \; u \geqslant 0 \text{ in } \Omega \times (0,T), \\ u(x,t) = 0 \;\text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \;\; x \in \Omega, \end{cases}$$

which will be studied in a forthcoming paper [3]. Notice that for the semilinear case, some related results were obtained in [7].

Since we are considering solution with data in L^1, then we need to use a week concept of solutions. More precisely we have the next definitions.

DEFINITION 1. We say that u is a nonnegative distributional solution to problem $(P_\pm)$ if

$$|\nabla u|^{p-1} \in L^1_{loc}(\Omega) \quad \text{and} \quad \lambda \frac{u^{p-1}}{|x|^p},\; u^q,\; h \in L^1_{loc}(\Omega),$$

and for all $\phi \in \mathscr{C}_0^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle dx = \int_\Omega \Big(\lambda \frac{u^{p-1}}{|x|^p} \pm u^q + h \Big) \phi dx. \tag{1.1}$$

In the case where $\lambda \dfrac{u^{p-1}}{|x|^p}, u^q, h \in L^1(\Omega)$ we can use the concept of entropy solution.

For $k > 0$, define

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leqslant k; \\ k\frac{s}{|s|}, & \text{if } |s| > k; \end{cases}$$

then we have the following definitions.

DEFINITION 2. Let u be a measurable function, we say that $u \in \mathscr{T}_0^{1,p}(\Omega)$ if $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$. Let $F \in L^1(\Omega)$, then $u \in \mathscr{T}_0^{1,p}(\Omega)$ is an entropy solution to

$$\begin{cases} -\Delta_p u = F \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

if for all $k > 0$ and all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla (T_k(u-v)) \rangle dx = \int_\Omega F\, T_k(u-v) dx. \tag{1.3}$$

Hence we say that u is an entropy solution to problem $(P\pm)$ if

$$\lambda \frac{u^{p-1}}{|x|^p}, u^q, h \in L^1(\Omega)$$

and the above definition holds with

$$F(x) \equiv \lambda \frac{u^{p-1}}{|x|^p} \pm u^q + h.$$

From the results of [10], we know that if u is an entropy solution, then $|\nabla u|^{p-1} \in L^s(\Omega)$ for all $s < \frac{N}{N-1}$. Hence, we conclude that if u is an entropy solution, then u is also a distributional solution.

We recall the following existence result obtained in [10].

THEOREM 1. *Assume that $1 < p$ and $F \in L^1(\Omega)$. Let $\{f_n\}_n \subset L^\infty(\Omega)$ be such that $f_n \to F$ strongly in $L^1(\Omega)$. Consider $u_n \in W_0^{1,p}(\Omega)$, the unique solution to problem*

$$\begin{cases} -\Delta_p u_n = f_n \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases}$$

then there exits $u \in \mathscr{T}_0^{1,p}(\Omega)$ such that u is the unique entropy solution of (1.2), $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p}(\Omega)$, $u_n^{p-1} \to u^{p-1}$ strongly in $L^\sigma(\Omega)$ for all $\sigma < \frac{N}{N-2}$ and $|\nabla u_n|^{p-1} \to |\nabla u|^{p-1}$ strongly in $L^s(\Omega)$ for all $s < \frac{N}{N-1}$.

The paper is organized as follows. In Section 2 we deal with the problem (P_-). In Subsection 2.1 we prove the existence of a critical exponent $q_+(\lambda)$ such that a strong non existence result holds if $q > q_+(\lambda)$. As a consequence we prove some complete Blow-up results for approximated problems.

The case $q < q_+(\lambda)$ is treated in Subsection 2.2, then, under suitable hypothesis on h, problem (P_-) has a positive solution. This prove the optimality of $q_+(\lambda)$.

The case of absorption term is considered in Section 3, we find an exponent q_* such that if $q > q_*$, then problem (P_+) has an entropy solution for all $\lambda > 0$ and $h \in L^1(\Omega)$.

Notice that, without the absorption term u^q, existence holds if and only if $\lambda \leqslant \Lambda_{N,p}$ with strong condition on h. Thus this show the strong effect of the absorption term u^q in order to break down any resonant effect of the reaction term $\lambda \frac{u^{p-1}}{|x|^p}$.

The optimality of q_* is proved by showing that if $q < q_+$, then for $\lambda > \Lambda_{N,p}$, problem (P_+) has no positive solution. Some extensions are given at the end of the section.

2. Problem with reaction term

In this section we consider the next problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + u^q + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ containing the origin, $1 < p < N$ and $q > p-1$.

First, let us consider the equation.

$$-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p}. \tag{2.2}$$

By setting $w(x) = C|x|^{-\alpha}$, there results

$$\frac{-1}{r^{N-1}}\left(\left|w'\right|^{p-2} w' r^{N-1}\right)' = \lambda w^{p-1} r^{-p}.$$

Hence, we get the next algebraic equation

$$D(\alpha) \equiv (p-1)\,\alpha^p - (N-p)\,\alpha^{p-1} + \lambda = 0 \tag{2.3}$$

Under the hypothesis $\lambda < \Lambda_{N,p} \equiv \left((N-p)/p\right)^p$, equation (2.3) posses exactly two solutions $\alpha_1 < (N-p)/p < \alpha_2$ (see the computation details in [2]).

Since $\lambda > 0$, then using the strong maximum principle and a suitable comparison function we can show that $u(x) \to \infty$ as $|x| \to 0$. The next result give a more precise information about the behavior of any supersolution of (2.1) near the origin, the proof can be seen in [2].

LEMMA 1. *Assume that $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative supersolution to problem (2.2), then if $u \not\equiv 0$, there exist a positive constant C and a small ball $B_\eta(0) \subset\subset \Omega$ such that*

$$u \geqslant C|x|^{-\alpha_1} \text{ in } B_\eta(0) \tag{2.4}$$

where α_1 is defined above.

2.1. Nonexistence results: the optimal exponent.

Assume that $\lambda < \Lambda_{N,p}$, we look now for radial solutions to the elliptic equation

$$-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p} + w^q,\ x \in \mathbb{R}^N. \tag{2.5}$$

Then by setting $w(x) = |x|^{-\alpha}$, it follows that

$$((p-1)\alpha^p - (N-p)\alpha^{p-1} - \lambda)r^{-\alpha(p-1)-p} = r^{-\alpha q} \tag{2.6}$$

so by identification, one have that

$$\alpha = \frac{p}{1+q-p},\ q > p-1, \quad \text{and} \quad \alpha < \frac{N-p}{p-1}.$$

Since

$$\frac{w^{p-1}}{|x|^p} \in L^1_{loc}(\Omega) \quad \text{and} \quad w^q \in L^1_{loc}(\Omega),$$

then by the result of Lemma 1 we obtain that $\alpha_1 < \alpha < \alpha_2$ which is equivalent to

$$\frac{p}{\alpha_2} + p - 1 < q < \frac{p}{\alpha_1} + p - 1, \tag{2.7}$$

We set

$$q_+(\lambda) \equiv \frac{p}{\alpha_1} + p - 1 \text{ and } q_-(\lambda) \equiv \frac{p}{\alpha_2} + p - 1, \tag{2.8}$$

then

$$p-1 < q_-(\lambda) < p^* - 1 < q_+(\lambda).$$

with $p^* = Np/(N-p)$.

Since we are considering an equation with right hand side in L^1, then we will use the concept of entropy solutions given in Definition 2

We are now able to prove the next nonexistence result.

THEOREM 2. *Assume that $q > q_+(\lambda) \equiv (p-1) + p/\alpha_1$. Then for all $\lambda > 0$, the problem (2.1) has no positive entropy solution.*

To prove this theorem we need the following well known inequality [8].

THEOREM 3. (Picone inequality) *Let $v \in W_0^{1,p}(\Omega)$ be such that $-\Delta_p v \gneqq 0$ is a bounded Radon measure $v \gneq 0$, then for all $u \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^p dx \geqslant \int_\Omega \frac{|u|^p}{v^{p-1}}(-\Delta_p v) dx.$$

Proof of Theorem 2.

If $\lambda > \Lambda_{N,p}$, then the nonexistence result is obtained in [4]. Let us consider the case $\lambda \leqslant \Lambda_{N,p}$.

We argue by contradiction. Let u be an entropy solution to (2.1), then using an approximation argument as in [4], we get the existence of a minimal entropy solution obtained as a limit of approximation problems. We note u the minimal solution. Let $\varphi \in \mathscr{C}_0^\infty(B_r(0))$, then using Picone inequality of Theorem 3 to u, it follows that

$$\int_{B_r(0)} |\nabla\varphi|^p\,dx \geqslant \int_{B_r(0)} \frac{-\Delta_p u}{u^{p-1}}|\varphi|^p\,dx \geqslant \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p}dx + \int_{B_r(0)} u^{q-p+1}|\varphi|^p\,dx$$
$$\geqslant \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p}dx + \int_{B_r(0)} \frac{|\varphi|^p}{|x|^{\alpha_1(q-p+1)}}dx.$$

If $q > q_+(\lambda)$, then $\alpha_1(q-p+1) > p$, thus we get a contradiction with the Hardy inequality, hence non existence holds.

REMARK 1. Since the arguments used in the proof of the nonexistence result, are local, then we conclude that problem (2.1) has no non-trivial supersolution in the sense that $\lambda \frac{u^{p-1}}{|x|^p} + u^q \in L^1_{loc}(\Omega)$ in any domain containing the origin.

THEOREM 4. *Assume that* $g : \mathbb{R} \to [0,\infty)$ *is a continuous function such that* $g(s) > 0$ *if* $s > 0$ *and*

$$\liminf_{s\to\infty} \frac{g(s)}{s^q} = c > 0 \textit{ for some } q > q_+(\lambda).$$

Then we have:

(i) *if* $g(0) = 0$*, then the unique entropy solution to problem*

$$-\Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + g(u), \textit{ in } \Omega, u_{|\partial\Omega} = 0, \tag{2.9}$$

is $u = 0$*;*

(ii) *if* $g(0) > 0$ *than problem (2.9), does not admit any entropy positive solution.*

As a consequence we get the next blow-up result.

THEOREM 5. *Fix* $q > q_+(\lambda)$ *and* $\lambda < \Lambda_{N,p}$. *Define*

$$a_n(x) = \min\{n, \frac{1}{|x|^p}\} \quad \textit{and} \quad D_n(s) = \min\{n, s^q\},\ s \geqslant 0.$$

Let $\{h_n\}_n \subset L^\infty(\Omega)$ *be such that* $h_n \geqslant 0$ *and* $h_n \uparrow h \in L^1(\Omega)$. *Let* u_n *be the minimal solution to problem*

$$\begin{cases} -\Delta_p u_n = \lambda a_n(x) u_n^{p-1} + g_n(u_n) + h_n \textit{ in } \Omega, \\ u_n \geqslant 0 \textit{ in } \Omega, \\ u_n = 0 \textit{ on } \partial\Omega. \end{cases} \tag{2.10}$$

Then $u_n(x) \to \infty$ *as* $n \to \infty$ *uniformly in* $x \in \Omega$.

We first recall the following Lemma proved in [4], which will be useful in the proof of Theorem 5.

LEMMA 2. *Let u be the unique positive energy solution to problem*

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega, \\ u \geqslant 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.11}$$

where $f \in L^\infty(\Omega)$ and $f \geqslant 0$. Then for all ball $B_r \subset \Omega$ such that $\overline{B}_{4r} \subset \Omega$, there exist a positive constant $c = c(r,N,p)$ such that,

$$\frac{u^{p-1}(x)}{(d(x,\partial\Omega))^{p-1}} \geqslant c\int_{B_{2r}} f(y)dy \text{ for all } x \in \Omega. \tag{2.12}$$

Proof of Theorem 5.

Since $\lambda < \Lambda_{N,p}$, then we get easily the existence of a minimal solution $u_n \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ to (2.10). Using the fact that $\lambda a_n(x)u_n^{p-1} + g_n(u_n) + h_n$ is increasing in n, then we conclude that $\{u_n\}$ is an increasing sequence in n.

Assume by contradiction that there exist $x_0 \in \Omega$, such that $\sup_n u_n(x_0) = c_0 < \infty$, then using Lemma (2) for a ball satisfying $0 \in B_{2r} \subset \Omega$, we obtain that

$$\int_{B_r} (\lambda a_n(x)u_n^{p-1} + g_n(u_n) + h_n)dx \leqslant c_1 \text{ for all } n.$$

Since $F_n(x) := \lambda a_n(x)u_n^{p-1} + g_n(u_n) + h_n$ is increasing we obtain that $F_n \to w$, $n \to \infty$ in $L^1(B_{2r})$ to some $w \geqslant 0$. Starting from $v_0 = 0$, we define the sequence,

$$\begin{cases} -\Delta_p v_{n+1} = \lambda a_{n+1}(x)v_n^{p-1} + g_n(v_n) + h_n \text{ in } B_r, \\ v_{n+1}|_{\partial B_r} = 0. \end{cases} \tag{2.13}$$

Since $\lambda a_n(x)s^{p-1} + g_n(s) + h_n$ is increasing in n, by comparison we obtain that $v_n \leqslant v_{n+1}$.

Claim: $v_n \leqslant u_n$ in $B_r(0)$ for all $n \in \mathbb{N}$.

We prove the claim by induction. We have $-\Delta_p v_1 = h_1 \leqslant -\Delta_p u_1$ and since $u_1|_{\partial B_r} > 0 = v_1|_{\partial B_r}$ we conclude that $v_1 \leqslant u_1$. Suppose $v_n \leqslant u_n$. Recall that $u_n \leqslant u_{n+1}$, then using the fact that

$$\lambda a_{n+1}(x)v_n^{p-1} + g_n(v_n) + h_n \leqslant a_{n+1}(x)u_{n+1}^{p-1} + g_n(u_{n+1}) + h_{n+1},$$

we conclude that

$$-\Delta_p v_{n+1} \leqslant -\Delta_p u_{n+1} \text{ and } u_{n+1} \geqslant v_{n+1} \text{ on } \partial B_r.$$

Thus $v_{n+1} \leqslant u_{n+1}$ and the claim follows.

Moreover we have

$$\int_{B_r} |\nabla T_k(v_n)|^p dx \leqslant k\int_{B_r} \Big(a_{n+1}(x)u_{n+1}^{p-1} + g_n(u_{n+1}) + h_{n+1}\Big)dx \leqslant ck,$$

Therefore we reach that $T_k(v_n)$ is bounded in $W_0^{1,p}(B_r)$ for all $k>0$. Thus $T_k(v_n) \rightharpoonup T_k(v)$ weakly in $W_0^{1,p}(B_r)$

Since $\{T_k(v_n)\}_n$ is increasing in n, then using a simple variation of the compactness argument of [10] we can prove that and then $T_k(v_n) \to T_k(v)$ strongly in $W_0^{1,p}(B_r)$. Hence, v is an entropy solution to problem

$$\begin{cases} -\Delta_p v = \lambda \dfrac{v^{p-1}}{|x|^p} + v^q + h \text{ in } B_r, \\ v \geqslant 0 \text{ in } B_r \text{ and } v \neq 0, \\ v|_{\partial B_r} = 0 \end{cases} \tag{2.14}$$

with $q > q_+(\lambda)$. This is a contradiction with the nonexistence result of Theorem (2).

Hence for all $x_0 \in \Omega$, $u_n(x_0) \to \infty$ as $n \to \infty$ and the proof is complete.

2.2. Existence result for $q < q_+(\lambda)$.

To show the optimality of the exponent $q_+(\lambda)$ we will prove the next existence result.

THEOREM 6. *Assume that $\lambda \leqslant \Lambda_{N,p}$ and $q < q_+(\lambda)$, then:*
(i) *if $q < p^* - 1$ and $\lambda < \Lambda_{N,p}$, then for $h \equiv 0$, problem* (2.1) *has a positive solution $u \in W_0^{1,p}(\Omega)$;*
(ii) *if $q < p^* - 1$ and $\lambda = \Lambda_{N,p}$, then for $h \equiv 0$, problem* (2.1) *has a positive solution $u \in W_0^{1,s}(\Omega)$ for all $s < p$;*
(iii) *if $p^* - 1 \leqslant q < q_+(\lambda)$ and $\lambda < \Lambda_{N,p}$, then there exists a positive constant c such that if $h(x) \leqslant c/|x|^p$, then problem* (2.1) *has an entropy positive solution u such that $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$.*

Proof. We divide the proof in several steps.

The first case: $q < p^* - 1$ and $\lambda < \Lambda_{N,p}$.

In this case problem (2.1) has a variational structure in the space $W_0^{1,p}(\Omega)$, then we can find a solution as a critical points of the functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^p}\,dx - \frac{1}{q+1}\int_\Omega u_+^{q+1}\,dx.$$

By a direct application of the Mountain-Pass theorem [9], we reach the existence of positive solution as a mountain pass point.

The second case $q < p^* - 1$ and $\lambda = \Lambda_{N,p}$.

To get the existence result in this case we use the following improved Hardy-Sobolev inequality obtained in [1], for any $s < p$, there exists a positive constant $C \equiv C(N,p,s,\Omega)$ such that

$$\int_\Omega \left(|\nabla u|^p - \Lambda_{N,p}\frac{|u|^p}{|x|^p}\right)dx \geqslant C\|u\|^p_{W_0^{1,s}(\Omega)} \text{ for all } u \in \mathscr{C}_0^\infty(\Omega). \tag{2.15}$$

Let now $\{\lambda_n\}_n$ be a strictly increasing sequence of positive constants, such that $\lambda_n \uparrow \Lambda_{N,p}$ as $n \to \infty$. Using the result of the first case, we reach that the problem

$$-\Delta_p u_n = \lambda_n \frac{u_n^{p-1}}{|x|^p} + u_n^q, \text{ in } \Omega, u_n \in W_0^{1,p}(\Omega). \tag{2.16}$$

has a positive solution u_n obtained using the Mountain-Pass Theorem [9]. Notice that

$$J_{\lambda_n}(u_n) = (\frac{1}{p} - \frac{1}{q+1})\Big(\int_\Omega |\nabla u_n|^p\,dx - \frac{\lambda}{p}\int_\Omega \frac{|u_n|^p}{|x|^p}\,dx\Big) \equiv C_n$$

where

$$C_n = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda_n}(\gamma(t))$$

with

$$\Gamma = \{\gamma \in \mathscr{C}([0,1], W_0^{1,p}(\Omega)),\ \gamma(0) = 0,\ \gamma(1) = v\}$$

and $v \in W_0^{1,p}(\Omega)$ is such that

$$\frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \frac{1}{q+1}\int_\Omega v_+^{q+1}\,dx << 0.$$

It is clear that $J_{\lambda_n}(v) << 0$ uniformly for $\lambda_n \in [0, \Lambda_{N,p}]$. If $\gamma(t) = tv$, then $\gamma \in \Gamma$, and

$$C_n \leqslant \max_{t\in[0,1]} J_{\lambda_n}(tv) \leqslant A,$$

where

$$A = \max_{t\in[0,1]}\Big(\frac{t^p}{p}\int_\Omega |\nabla v|^p\,dx - \frac{t^{q+1}}{q+1}\int_\Omega v_+^{q+1}\,dx\Big).$$

Hence we conclude that

$$\Big(\frac{1}{p} - \frac{1}{q+1}\Big)\Big(\int_\Omega |\nabla u_n|^p\,dx - \frac{\lambda}{p}\int_\Omega \frac{|u_n|^p}{|x|^p}\,dx\Big) \leqslant A.$$

Now using the improved Hardy-Sobolev inequality stated in (2.15), it follows that $\|u_n\|^p_{W_0^{1,s}(\Omega)} \leqslant C$ for all $s < p$ and for all $n \geqslant 1$.

Since $q+1 < p^*$, we get the existence of $1 < s_0 < p$, such that

$$q+1 < s_0^* \equiv \frac{s_0 N}{N - s_0}.$$

Fix s_0 to get the above estimate, then $\|u_n\|^p_{W_0^{1,s_0}(\Omega)} \leqslant C$. In the same way and using (2.15), we get the existence of a positive constant a such that $J_{\lambda_n}(u_n) \geqslant a$. This follows using the fact that $a^p - Ca^{q+1} > 0$ if a is small enough.

Hence we get the existence of $u_0 \in W_0^{1,s_0}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,s_0}(\Omega)$ and $u_n \to u$ strongly in $L^{q+1}(\Omega)$.

Since

$$J_{\lambda_n}(u_n) = \Big(\frac{1}{p} - \frac{1}{q+1}\Big)\int_\Omega (u_n)_+^{q+1}\,dx \to \Big(\frac{1}{p} - \frac{1}{q+1}\Big)\int_\Omega (u_0)_+^{q+1}\,dx,$$

then $u_0 \gneqq 0$ and u_0 solves

$$-\Delta_p u = \Lambda_{N,p}\frac{u^{p-1}}{|x|^p} + u^q \ \text{ in } \Omega,\ u_n \in W_0^{1,s_0}(\Omega)$$

at least in the distributional sense. Il is clear that, by the above computation, $u_0 \in W_0^{1,s}(\Omega)$ for all $s < p$. Hence the existence result follows.

The third case $q_-(\lambda) \leqslant q < q_+(\lambda)$ *and* $\lambda < \Lambda_{N,p}$.

Recall that $p-1 < q_-(\lambda) < p^* - 1 < q_+(\lambda)$. Let $R > 0$ be such that $\Omega \subset\subset B_R(0)$, then using a dilatation argument, without loss of generality one can put $R = 1$. Assume that $h(x) \leqslant c/|x|^p$, where $c > 0$ will be chosen later.

By a continuity argument, we get the existence of $\lambda_1 > \lambda$ such that $q_-(\lambda_1) < q < q_+(\lambda_1)$. Define

$$w(x) = |x|^{-\alpha} - 1 \ \text{ for } x \in B_1(0),\ \text{with } \alpha = \frac{p}{q-(p-1)},$$

then

$$\frac{w^{p-1}}{|x|^p} + w^q \in L^1(B_1(0))$$

and w solves

$$-\Delta_p w = \lambda_1 \frac{(w+1)^{p-1}}{|x|^p} + (w+1)^q \text{ in } \mathscr{D}'(B_1(0)).$$

Using the fact that $\lambda < \lambda_1$ we get the existence of a positive constant $c_1 > 0$ such that

$$\lambda_1 \frac{(w+1)^{p-1}}{|x|^p} \geqslant \lambda \frac{w^{p-1}}{|x|^p} + \frac{c_1}{|x|^p}.$$

Choosing $c \leqslant c_1$, then we obtain a supersolution to problem (2.1).

Let w_0 be the unique solution to the problem

$$\begin{cases} -\Delta_p w_0 = h \text{ in } \Omega, \\ w_0 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.17}$$

it is clear that w_0 is subsolution to problem (2.1) with $w_0 \leqslant w$. Thus using a monotonicity argument we get the existence result.

3. Problem with absorption term: breaking of resonance

In this section we deal with the existence of a nonnegative solutions to the problem

$$\begin{cases} -\Delta_p u + u^q = \lambda \dfrac{u^{p-1}}{|x|^p} + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

THEOREM 7. *Assume that* $q > q_* \equiv \frac{N(p-1)}{N-p}$*, then for all* $\lambda > 0$ *and for all* $h \in L^1(\Omega)$*, problem* (3.1) *has a minimal positive entropy solution.*

We will also need the following comparison principle, proved in [4]

THEOREM 8. (Comparison Principle) *Assume that* $1 < p$ *and let* f *be a nonnegative continuous function such that* $f(x,s)/s^{p-1}$ *is decreasing for* $s > 0$*. Suppose that* $u, v \in W_0^{1,p}(\Omega)$ *are such that*

$$\begin{cases} -\Delta_p u \geqslant f(x,u), & u > 0 \text{ in } \Omega, \\ -\Delta_p v \leqslant f(x,v), & v > 0 \text{ in } \Omega. \end{cases} \tag{3.2}$$

Then $u \geqslant v$ *in* Ω*.*

Proof of Theorem 7. Let

$$h_n \equiv T_n(h) \quad \text{and} \quad a_n(x) = \frac{1}{|x|^p + \frac{1}{n}}.$$

Using the sub-supersolution argument, we get the existence of a unique positive solution to problem

$$-\Delta_p w_n + w_n^q = h_n, w_n \in W_0^{1,p}(\Omega).$$

Notice that the positivity of w_n follows from the strong maximum principle obtained in [16], moreover the uniqueness is obtained using the comparison principle of Theorem 8. We claim that the approximated problem

$$\begin{cases} -\Delta_p u_n + u_n^q = \lambda a_n(x) T_n(u_n^{p-1}) + h_n, \\ u_n \in W_0^{1,p}(\Omega), \quad u_n \geqslant 0 \end{cases} \tag{3.3}$$

has a unique positive solution u_n, such that $u_n \leqslant u_{n+1}$ for all $n \geqslant 1$. Let us begin by showing the existence. Define v_n as the unique positive solution to problem

$$-\Delta_p v_n = n\lambda a_n(x) + h_n, v_n \in W_0^{1,p}(\Omega),$$

then v_n is a supersolution to problem (3.3). Since $\underline{u} \equiv 0$ is a subsolution to (3.3), then using an iteration argument, we get the existence of a solution u_n such that $u_n \leqslant v_n$. The positivity of u_n follows using the result of [16]. To get the uniqueness we use Lemma 8. Let $D_n(x,s) \equiv \lambda a_n(x) T_n(s^{p-1}) + h_n(x) - s^q$, $s \geqslant 0$, it is clear that, for $s > 0$,

$D_n(x,s)/s^{p-1}$ is a decreasing function, then the uniqueness result follows. Using the fact that $D_n(x,s) \leqslant D_{n+1}(x,s)$, it results that u_{n+1} is a supersolution to (3.3), thus $u_n \leqslant u_{n+1}$ and the claim follows.

Let $k>0$ fixed, using $T_k(u_n)$ as a test function in (3.1) we get

$$\int_\Omega |\nabla T_k u_n|^p\, dx + \int_\Omega u_n^q T_k u_n\, dx$$
$$= \lambda \int_\Omega a_n(x)\, T_n(u_n^{p-1}) T_k(u_n) dx + \int_\Omega h_n(x) T_k u_n\, dx.$$

Using Hölder inequality we reach that

$$\lambda \int_\Omega a_n(x)\, T_n(u_n^{p-1}) dx \leqslant \lambda \Big(\int_\Omega u_n^q dx\Big)^{\frac{p-1}{q}} \Big(\int_\Omega \frac{1}{|x|^{\frac{pq}{q-(p-1)}}} dx\Big)^{\frac{q-(p-1)}{q}}.$$

Recall that $q > \frac{N(p-1)}{N-p}$, then $\frac{pq}{q-(p-1)} < N$, hence

$$\lambda \int_\Omega a_n(x)\, T_n(u_n^{p-1}) T_k(u_n) dx \leqslant C\lambda \Big(\int_\Omega u_n^q dx\Big)^{\frac{p-1}{q}}.$$

Thus

$$\int_\Omega |\nabla T_k u_n|^p\, dx + \int_\Omega u_n^q T_k u_n\, dx \leqslant Ck\lambda \Big(\int_\Omega u_n^q dx\Big)^{\frac{p-1}{q}} + k||h||_{L^1}.$$

Notice that

$$\int_\Omega u_n^q T_k u_n\, dx \geqslant \int_\Omega u_n^q\, dx - C(k).$$

Hence

$$\int_\Omega |\nabla T_k u_n|^p\, dx + \int_\Omega u_n^q\, dx \leqslant C(k, \lambda, ||h||_{L^1}).$$

Therefore we conclude that

$$\int_\Omega u_n^q dx \leqslant C \text{ uniformly in } n,$$
$$\int_\Omega a_n(x)\, T_n(u_n^{p-1})\, dx \leqslant C \text{ uniformly in } n.$$

Using the monotonicity of the sequence $\{u_n\}_n$ we get the existence of a measurable function u such that

$$u_n^q \uparrow u \quad \text{and} \quad a_n(x)\, T_n(u_n^{p-1}) \uparrow \frac{u^{p-1}}{|x|^p} \quad \text{strongly in } L^1(\Omega).$$

Setting $f_n = a_n(x)\, T_n(u_n^{p-1}) - u_n^q$, then $f_n \to f \equiv \frac{u^{p-1}}{|x|^p} - u^q$ strongly in $L^1(\Omega)$. Thus following the arguments of [10], we reach that u is an entropy solution to (3.1). It is not difficult to show that if v is another positive entropy solution to (3.1), then $v \geqslant u_n$ for all $n \geqslant 0$, thus $v \geqslant u$.

To show the optimality of the condition imposed in the Theorem 7 we prove the next non existence result.

THEOREM 9. *Assume that $q < q_*$. If $\lambda > \Lambda_{N,p}$, then problem (3.1) has no very weak positive supersolution in the sense that $u^q, u^{p-1}/|x|^p \in L^1_{loc}(\Omega)$ and*

$$\int \Big((-\Delta_p u)\phi + |u|^q \phi\Big) dx \geqslant \lambda \int \frac{u^{p-1}\phi}{|x|^p} dx + \int h(x)\phi\, dx, \text{ for all } \phi \in \mathscr{C}_0^\infty(\Omega).$$

Proof. Without loss of generality we can assume that $h \in L^\infty(\Omega)$. We argue by contradiction. Suppose that for some $\lambda > \Lambda_{N,p}$, problem (3.1) has a nonnegative very weak supersolution u in the sense defined above. Let Ω_1 be a regular domain such that $\Omega_1 \subset\subset \Omega$, then u is a supersolution to problem (3.1) in Ω_1. Thus using an iteration argument we get the existence of u_1, the minimal entropy positive solution to (3.1) in Ω_1. It is clear that $u_1^{p-1} \in L^s(\Omega_1)$ for all $s < N/(N-p)$.

Let $B_\eta(0) \subset\subset \Omega_1$ where η is a small constant to be chosen later. Consider $\phi \in \mathscr{C}_0^\infty(B_\eta(0))$, since $u_1 > 0$ in $B_\eta(0)$, then using Picone inequality it follows that

$$\int_{B_\eta(0)} |\nabla\phi|^p dx \geqslant \lambda \int_{B_\eta(0)} \frac{|\phi|^p}{|x|^p} dx - \int_{B_\eta(0)} u_1^{q-(p-1)} |\phi|^p. \tag{3.4}$$

Since $q < q_*$, then

$$(q-(p-1))\frac{p^*}{p^*-p} < \frac{N(p-1)}{N-p},$$

thus using Hölder and Sobolev inequality inequalities we obtain that

$$\begin{aligned}\int_{B_\eta(0)} u^{q-(p-1)}|\phi|^p dx &\leqslant \Big(\int_{B_\eta(0)} |\phi|^{p^*} dx\Big)^{\frac{p}{p^*}} \Big(\int_{B_\eta(0)} u^{(q-(p-1))\frac{p^*}{p^*-p}} dx\Big)^{\frac{p^*-p}{p}} \\ &\leqslant S^{-1} \int_{B_\eta(0)} |\nabla\phi|^p dx \Big(\int_{B_\eta(0)} u^{(q-(p-1))\frac{p^*}{p^*-p}} dx\Big)^{\frac{p^*-p}{p}}.\end{aligned}$$

Using the fact that

$$(q-(p-1))\frac{p^*}{p^*-p} < \frac{N(p-1)}{N-p},$$

there result that

$$\int_{\Omega_1} u^{(q-(p-1))\frac{p^*}{p^*-p}} dx < \infty. \tag{3.5}$$

We claim that

$$\lim_{\eta\to 0} \int_{B_\eta(0)} u^{(q-(p-1))\frac{p^*}{p^*-p}} dx = 0. \tag{3.6}$$

To prove (3.6), we set

$$k_\eta(x) \equiv u^{(q-(p-1))\frac{p^*}{p^*-p}} \chi_{B_\eta(0)},$$

then

$$k_\eta \leqslant u^{(q-(p-1))\frac{p^*}{p^*-p}} \quad \text{and} \quad ||k_\eta||_{L^1(\Omega_1)} \leqslant ||u^{(q-(p-1))\frac{p^*}{p^*-p}}||_{L^1(\Omega_1)}$$

for all $\eta << 1$.

Using the fact that $k_\eta \to 0$ a.e. in Ω_1, then by the dominated convergence Theorem we reach that

$$k_\eta \to 0 \text{ strongly in } L^1.$$

Hence the claim follows.

Since $\lambda > \Lambda_{N,p}$, we get the existence of $\varepsilon > 0$ such that choosing η small enough we obtain that

$$\frac{\lambda}{1+S^{-1}\Big(\int_{B_\eta(0)} u^{(q-(p-1))\frac{p^*}{p^*-p}}dx\Big)^{\frac{p^*-p}{p}}} \geqslant \Lambda_{N,p}+\varepsilon.$$

Hence, back to (3.4) we obtain

$$\int_{B_\eta(0)} |\nabla\phi|^p dx \geqslant (\Lambda_{N,p}+\varepsilon)\int_{B_\eta(0)} \frac{|\phi|^p}{|x|^p}dx$$

a contradiction with Hardy inequality.

REMARK 2. Fix $q > p-1$ and let g be a measurable function such that $g \geqslant 0$ and $g^{\frac{q}{q-(p-1)}} \in L^1(\Omega)$, then using the same arguments as in the proof of Theorem 7 we can prove that problem

$$\begin{cases} -\Delta_p u + u^q = \lambda g(x)u + h(x) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.7}$$

has an entropy positive solution for all $\lambda > 0$ and for all $h \in L^1(\Omega)$. In this case we say that g is an *admissible weight* related to the problem (3.7).

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