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THREE POSITIVE SOLUTIONS OF STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish the results on the existence of three positive solutions to Sturm-Liouville boundary value problems of the singular fractional differential equation with the nonlinearity depending on $D^{\mu}_{0+}u$

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t),D^{\mu}_{0+}u(t))=0, \quad t\in(0,1), 1<\alpha<2,\\ a\lim_{t\to 0} I^{2-\alpha}_{0+}u(t)-b\ \lim_{t\to 0}\ \left[I^{2-\alpha}_{0+}u(t)\right]'=\int_0^1 g(s,u(s),D^{\mu}_{0+}u(s))ds,\\ c\,D^{\mu}_{0+}u(1)+du(1)=\int_0^1 h(s,u(s),D^{\mu}_{0+}u(s))ds. \end{cases}$$

Our analysis relies on the well known five functional fixed point theorems. An example is given to illustrate the efficiency of the main theorems.

1. Introduction

There have been many papers concerned with the existence of positive solutions of boundary value problems for fractional differential equations see [1, 3, 4, 6, 7, 8, 10, 13, 11, 12, 15, 17, 19, 20, 22, 23, 24]. The interpretation and applications of the fractional differential equations can be found in [9, 18, 16].

In [23, 24], the authors consider the existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary-value problem

$$\begin{cases} \mathbf{D}^{\alpha}_{\mathbf{0}+}u(t)=f(t,u(t)), \quad 0<t<1\\ u(0)+u'(0)=0,\\ u(1)+u'(1)=0, \end{cases} \tag{1.1}$$

where $1<\alpha\leqslant 2$ is a real number, and $\mathbf{D}^{\alpha}_{0+}$ is the Caputo's fractional derivative, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. By means of a fixed-point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

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In [6, 8], the authors study the existence of solutions of the following more generalized boundary value problem

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}u(t)=f(t,u(t)), \quad t\in[0,T],\\ u(0)+u'(0)=\int_0^T g(s,y)ds,\\ u(T)+u'(T)=\int_0^T h(s,y)ds, \end{cases} \tag{1.2}$$

where $T>0$, $1<\alpha\leqslant 2$ is a real number, and $\mathbf{D}_{0+}^{\alpha}$ is the Caputo's fractional derivative, and $f,g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous.

In [2, 21], boundary value problems for fractional differential equations with the the Caputo fractional derivative ${}^{C}D_{0+}^{\alpha}$ of order α were discussed where the boundary conditions

$$x(0)=\lambda_1 x(T)+\mu_1,\ x'(0)=\lambda_2 x'(T)+\mu_2,\ x(0)=x''(0)=x(1)-\beta x(\eta)=0$$

with $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$, $T>0$, $\eta\in(0,1)$, $0\leqslant\beta\eta<1$, were involved.

For the fractional differential equation $D_{0+}^{\alpha}u(t)=f(t)$ with Riemann-Liouville fractional derivative of order $\alpha\in(1,2)$, it is well known from [7] that

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2},$$

where $c_1,c_2\in\mathbb{R}$. One sees that u is not continuous at $t=0$ if $c_2\neq 0$. Hence the boundary condition $u(0)+u'(0)=\int_0^T g(s,y)ds$ doesn't make any sense.

In [14], Liu studied the solvability of the following boundary value problem for the nonlinear fractional differential equation with the nonlinearity depending on $D_{0+}^{\alpha}u$

$$\begin{cases} D_{0+}^{\beta}[\rho(t)\Phi(D_{0+}^{\alpha}u(t))]+q(t)f(t,u(t),D_{0+}^{\alpha}u(t))=0, \quad t\in(0,1),\\ \lim_{t\to 0}t^{1-\alpha}u(t)+\sum_{i=1}^m a_iu(\xi_i)=\int_0^1 g(s,u(s),D_{0+}^{\alpha}u(s))ds,\\ \lim_{t\to 0}\Phi^{-1}(t^{1-\beta}\rho(t))D_{0+}^{\alpha}u(t)+\sum_{i=1}^m b_iD_{0+}^{\alpha}u(\xi_i)=\int_0^1 h(s,u(s),D_{0+}^{\alpha}u(s))ds \end{cases} \tag{1.3}$$

is discussed. Here $0<\alpha,\beta\leqslant 1$, D_{0+}^{α} (or D_{0+}^{β}) is the Riemann-Liouville fractional derivative of order α (or β), $\Phi(s)=|s|^{p-2}s$ with $s>1$ is called one-dimensional $p-$Laplacian, its inverse function is $\Phi^{-1}(s)=|s|^{q-2}s$ with $\frac{1}{p}+\frac{1}{q}=1$, $0<\xi_1<\xi_2<\cdots<\xi_m\leqslant 1$, $a_i,b_i(i=1,2,\cdots,m)$ are nonnegative numbers, $\rho\in C^0(0,1)$ is positive and satisfies that there exists $\sigma_1\in\mathbb{R}$ such that $\sigma_1(q-1)<\alpha$ and $t^{1-\beta}\rho(t)\geqslant t^{\sigma_1}$, $t\in(0,1)$, q defined on $(0,1)$ is nonnegative and satisfies that there exists $\sigma_2>-\beta$ such that $I_{0+}^{\beta}q\in C^0[0,1]$, $q(t)\leqslant t^{\sigma_2}$, $t\in(0,1)$, f,g,h defined on $[0,1]\times\mathbb{R}\times\mathbb{R}$ are nonnegative Caratheodory functions.

As far as we know, there has been no paper concerned with the existence of *three positive solutions* of Sturm-Liouville boundary value problems for fractional differential equations with *Riemann-Liouville fractional derivative.*

Motivated by the reason mentioned above, in this paper, we discuss the existence positive solutions to the Sturm-Liouville boundary value problems of the nonlinear fractional differential equation with the nonlinearity depending on $D_{0+}^{\alpha-1}u$

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0+}^{\mu}u(t))=0, \quad t\in(0,1), 1<\alpha<2,\\ a\lim_{t\to 0}I_{0+}^{2-\alpha}u(t)-b\lim_{t\to 0}\left[I_{0+}^{2-\alpha}u(t)\right]'=\int_0^1 g(s,u(s),D_{0+}^{\mu}u(s))ds,\\ c\,D_{0+}^{\mu}u(1)+du(1)=\int_0^1 h(s,u(s),D_{0+}^{\mu}u(s))ds. \end{cases} \tag{1.4}$$

where $a,b,c,d\in[0,\infty)$, D_{0+}^{α} (or D_{0+}^{μ}) is the Riemann-Liouville fractional derivative of order α (or μ), $\mu\in(0,\alpha-1)$ and f,g,h defined on $(0,1)\times[0,\infty)\times\mathbb{R}$ are nonnegative Caratheodory functions that may be singular at $t=0$ and $t=1$. We obtain the results on the existence of at least three positive solutions of BVP(1.4). An example is given to illustrate the efficiency of the main theorem.

The remainder of this paper is as follows: in section 2, we present preliminary results. In section 3, the main theorems and their proof are given. In section 4, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [3, 11, 4, 15, 10].

DEFINITION 1. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $g:(0,\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

DEFINITION 2. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $g:(0,\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\leqslant n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

DEFINITION 3. A function $F:(0,1)\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is called a Caratheodory function if $F(t,t^{\alpha-2}\bullet,t^{\mu+2-\alpha}\bullet):\mathbb{R}^2\to\mathbb{R}$ is continuous for each $t\in(0,1)$ and for each $r>0$ there exists $\phi_r\in L^1(0,1)$ such that $|F(t,t^{\alpha-2}u,t^{\alpha-\mu-2}v)|\leqslant\phi_r(t)$ for all $t\in(0,1)$ and $|u|,|v|\leqslant r$.

As usual, let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax\in P$ and $x+y\in P$ for all $x,y\in P$ and $a\geqslant 0$; $x\in X$ and

$-x \in X$ imply $x = 0$. A map $\psi : P \to [0,\infty)$ is a nonnegative continuous concave (or convex) functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx+(1-t)y) \geqslant \text{ (or } \leqslant \text{) } t\psi(x)+(1-t)\psi(y) \text{ for all } x,y \in P, t \in [0,1].$$

An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let $c_1,c_2,c_3,c_4,c_5 > 0$ be positive constants, α_1,α_2 be two nonnegative continuous concave functionals on the cone P, β_1,β_2,β_3 be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$\begin{aligned}
&P_{c_5} = \{x \in P : ||x|| < c_5\},\\
&P(\beta_1,\alpha_1;c_2,c_5) = \{x \in P : \alpha_1(x) \geqslant c_2,\ \beta_1(x) \leqslant c_5\},\\
&P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5) = \{x \in P : \alpha_1(x) \geqslant c_2,\ \beta_3(x) \leqslant c_4,\ \beta_1(x) \leqslant c_5\},\\
&Q(\beta_1,\beta_2;,c_1,c_5) = \{x \in P : \beta_2(x) \leqslant c_1,\ \beta_1(x) \leqslant c_5\},\\
&Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5) = \{x \in P : \alpha_2(x) \geqslant c_3,\ \beta_2(x) \leqslant c_1,\ \beta_1(x) \leqslant c_5\}.
\end{aligned}$$

LEMMA 1. ([[5]:Five functionals fixed point theorem) *] Let X be a real Banach space, P be a cone in X. α_1,α_2 be two nonnegative continuous concave functionals on the cone P, β_1,β_2,β_3 be three nonnegative continuous convex functionals on the cone P. There exist positive numbers c_1,c_2,c_3,c_4,c_5 with $c_1 < c_2$. Then T has at least three fixed points y_1, y_2 and y_3 such that*

$$\beta_2(y_1) < c_1,\ \alpha_1(y_2) > c_2,\ \beta_2(y_3) > c_1,\ \alpha_1(y_3) < c_2$$

if
$(B1)$ *$T : X \to X$ is a completely continuous operator;*
$(B2)$ *there exist constant $M > 0$ such that*

$$\alpha_1(x) \leqslant \beta_2(x),\ ||x|| \leqslant M\beta_1(x) \text{ for all } x \in P;$$

$(B3)$ *it holds that*
(i) $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;
(ii) $\{y \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5) | \alpha_1(x) > c_2\} \neq \emptyset$ *and*

$$\alpha_1(Tx) > c_2 \text{ for every } x \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5);$$

(iii) $\{y \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5) | \beta_2(x) < c_1\} \neq \emptyset$ *and*

$$\beta_2(Tx) < c_1 \text{ for every } x \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5);$$

(iv) $\alpha_1(Ty) > c_2$ *for* $y \in P(\beta_1,\alpha_1;c_2,c_5)$ *with* $\beta_3(Ty) > c_4$;
(v) $\beta_2(Tx) < c_1$ *for each* $x \in Q(\beta_1,\beta_2;c_1,c_5)$ *with* $\alpha_2(Tx) < c_3$.

LEMMA 2. ([11]) *Let $n-1 < \alpha \leqslant n$, $u \in C^0(0,\infty)\bigcap L^1(0,\infty)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},$$

where $C_i \in \mathbb{R}$, $i = 1,2,\ldots n$.

It is easy to know for $\alpha \geqslant 0,\ \mu > -1$ that [6]

$$I_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha},\ D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

Let

$$\delta = bc\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)} + bd + ac\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)} + ad\frac{1}{\alpha-1}.$$

LEMMA 3. *Suppose that $\delta \neq 0$. Given $h \in L^1(0,1)$ and $k,l \in \mathbb{R}$, the unique solution of*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0, 0 < t < 1, \\ a \lim_{t\to 0} I_{0^+}^{2-\alpha} u(t) - b\ \lim_{t\to 0}\ \left[I_{0^+}^{2-\alpha} u(t)\right]' = k, \\ c\, D_{0^+}^{\mu} u(1) + du(1) = l, \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{at^{\alpha-1} + (\alpha-1)bt^{\alpha-2}}{(\alpha-1)\delta} l + \frac{\frac{ct^{\alpha-2}}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1} - t\right) + \frac{dt^{\alpha-2}}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta} k, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{act^{\alpha-1}}{(\alpha-1)\delta} + \frac{bct^{\alpha-2}}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\ \qquad + \left(\frac{adt^{\alpha-1}}{(\alpha-1)\delta} + \frac{bdt^{\alpha-2}}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},\ s \leqslant t, \\ \left(\frac{act^{\alpha-1}}{(\alpha-1)\delta} + \frac{bct^{\alpha-2}}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\ \qquad + \left(\frac{adt^{\alpha-1}}{(\alpha-1)\delta} + \frac{bdt^{\alpha-2}}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},\ t \leqslant s. \end{cases} \tag{2.3}$$

Proof. We may apply Lemma 2 to reduce BVP(2.1) to an equivalent integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},\ t \in (0,1)$$

for some $c_i \in \mathbb{R},\ \ i = 1,2.$ We get

$$I_{0^+}^{2-\alpha} u(t) = -\int_0^t (t-s)h(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1),$$
$$\left[I_{0^+}^{2-\alpha} u(t)\right]' = -\int_0^t h(s)ds + c_1\Gamma(\alpha)$$

and

$$D^{\mu} u(t) = -\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} h(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)} t^{\alpha-\mu-1} + c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)} t^{\alpha-\mu-2}.$$

From the boundary conditions in (2.1), we get $a\Gamma(\alpha-1)c_2 - b\Gamma(\alpha)c_1 = k$, and

$$\left(c\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+d\right)c_2+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)c_1$$
$$=c\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds+d\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+l.$$

It follows that

$$c_1=\frac{ac\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds+ad\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+al+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{ak}{b\Gamma(\alpha)}}{bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu-1)}+bd\frac{\Gamma(\alpha)}{\Gamma(\alpha-1)}+ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+ad}$$
$$-\frac{k}{b\Gamma(\alpha)},$$

$$c_2=\frac{bc\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds+bd\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+bl+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{k}{\Gamma(\alpha)}}{bc\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+bd+ac\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}+ad\frac{1}{\alpha-1}}.$$

Therefore, the unique solution of BVP(2.1) is

$$\begin{aligned}
u(t)=&-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\frac{1}{\delta(\alpha-1)}\left(ac\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds\right.\\
&\left.+ad\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+al+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{ak}{b\Gamma(\alpha)}\right)t^{\alpha-1}\\
&-\frac{k}{b\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\delta}\left(bc\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds\right.\\
&\left.+bd\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+bl+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{k}{\Gamma(\alpha)}\right)t^{\alpha-2}\\
=&\int_0^1G(t,s)h(s)ds+\frac{at^{\alpha-1}+(\alpha-1)bt^{\alpha-2}}{(\alpha-1)\delta}l\\
&+\frac{1}{(\alpha-1)\delta}\left(\frac{ct^{\alpha-2}}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{dt^{\alpha-2}}{\Gamma(\alpha-1)}(1-t)\right)k.
\end{aligned}$$

Here G is defined by (2.3). Reciprocally, let u satisfy (2.2). Then

$$a\lim_{t\to 0}I_{0^+}^{2-\alpha}u(t)-b\lim_{t\to 0}\left[I_{0^+}^{2-\alpha}u(t)\right]'=k,\ \ c\,D_{0^+}^{\mu}u(1)+du(1)=l,$$

furthermore, we have $D_0^{\alpha}u(t)=-h(t)$. The proof is complete.

REMARK 1. It is easy to see that if u is a solution of BVP(2.1), then

$t^{\mu+2-\alpha}D^{\mu}u(t)$

$$
\begin{aligned}
&= -t^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t + c_2\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}\\
&= -t^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds\\
&\quad + \frac{\Gamma(\alpha)t}{\Gamma(\alpha-\mu)}\frac{ac\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + ad\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds}{(\alpha-1)\delta}\\
&\quad + \frac{\Gamma(\alpha)t}{\Gamma(\alpha-\mu)}\left[\frac{al+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{ak}{b\Gamma(\alpha)}}{(\alpha-1)\delta} - \frac{k}{b\Gamma(\alpha)}\right]\\
&\quad + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}\frac{bc\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds + bd\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds}{\delta}\\
&\quad + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}\frac{bl+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{k}{\Gamma(\alpha)}}{\delta}\\
&= t^{\mu+2-\alpha}\int_0^1 H(t,s)h(s)ds + \frac{bd\Gamma(\alpha-\mu-1)\left(\frac{1}{\alpha-1}-t\right)+bc\Gamma(\alpha-1)(1-t)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}k\\
&\quad + \left(\frac{b}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)} + \frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}t\right)l,
\end{aligned}
$$

where

$$
t^{\mu+2-\alpha}H(t,s) = \begin{cases}
-\frac{t^{\mu+2-\alpha}(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+bd(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu-1)} \\
\quad + \frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}t+ad(1-s)^{\alpha-1}t}{\delta(\alpha-1)\Gamma(\alpha-\mu)}, & 0\leqslant s\leqslant t\leqslant 1,\\
\frac{bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+bd(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu-1)} \\
\quad + \frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}t+ad(1-s)^{\alpha-1}t}{\delta(\alpha-1)\Gamma(\alpha-\mu)}, & 0\leqslant t\leqslant s\leqslant 1.
\end{cases} \tag{2.4}
$$

REMARK 2. One sees that

$$
\begin{aligned}
t^{\mu+2-\alpha}|H(t,s)| \leqslant{}& \frac{[ac+bc(\alpha-\mu-1)]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+\delta(\alpha-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)}(1-s)^{\alpha-\mu-1}\\
&+\frac{ad+bd(\alpha-\mu-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)}(1-s)^{\alpha-1}.
\end{aligned}
$$

Hence

$$
t^{\mu+2-\alpha}|D^\mu u(t)| \leqslant \Pi\left(\int_0^1\left((1-s)^{\alpha-\mu-1}+(1-s)^{\alpha-1}\right)h(s)ds+k+l\right),
$$

where

$$\Pi=\max\left\{\frac{[ac+bc(\alpha-\mu-1)]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+\delta(\alpha-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)},\right.$$
$$\frac{ad+bd(\alpha-\mu-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)}\times$$
$$\left.\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha)}{b\delta(\alpha-1)\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)},\ \frac{b}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}\right\}.$$

REMARK 3. It is easy to see that

$$\begin{aligned}
&\sup_{t\in(0,1]} t^{\mu+2-\alpha}|D^{\mu}u(t)|\\
&\quad\geqslant \lim_{t\to 0} t^{\mu+2-\alpha}|D^{\mu}u(t)|\\
&\quad\geqslant \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}\frac{bc\int_0^1\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}h(s)ds+bd\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds}{\delta}\\
&\qquad+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}\frac{bl+\left(c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+d\right)\frac{k}{\Gamma(\alpha)}}{\delta}\\
&\quad=\frac{\max\left\{bc\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)},\ \frac{bd}{\alpha-1}\right\}}{\delta\Gamma(\alpha-\mu-1)}\int_0^1\left((1-s)^{\alpha-\mu-1}+(1-s)^{\alpha-1}\right)h(s)ds\\
&\qquad+\frac{b\Gamma(\alpha-1)}{\delta\Gamma(\alpha-\mu-1)}l+\frac{\Gamma(\alpha-1)\left(c\frac{1}{\Gamma(\alpha-\mu)}+d\frac{1}{\Gamma(\alpha)}\right)}{\delta\Gamma(\alpha-\mu-1)}k\\
&\quad\geqslant\Lambda\left(\int_0^1\left((1-s)^{\alpha-\mu-1}+(1-s)^{\alpha-1}\right)h(s)ds+k+l\right),
\end{aligned}$$

where

$$\Lambda=\min\left\{\frac{\max\left\{bc\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)},\ \frac{bd}{\alpha-1}\right\}}{\delta\Gamma(\alpha-\mu-1)}\ \frac{b\Gamma(\alpha-1)}{\delta\Gamma(\alpha-\mu-1)},\right.$$
$$\left.\frac{\Gamma(\alpha-1)\left(c\frac{1}{\Gamma(\alpha-\mu)}+d\frac{1}{\Gamma(\alpha)}\right)}{\delta\Gamma(\alpha-\mu-1)}\right\}.$$

LEMMA 4. *Suppose that* $\delta>0$. *Then*

$$t^{2-\alpha}G(t,s)\leqslant\left(\frac{ac}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}+\left(\frac{ad}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.5}$$

and

$$G(t,s)\geqslant\frac{bc(\alpha-1-(\alpha-\mu-1)t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}+\frac{bd(\alpha-1)(1-t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.6}$$

hold for all $t\in(0,1),\ s\in[0,1]$.

Proof. One sees from (2.3) that

$$
\begin{aligned}
t^{2-\alpha}G(t,s) = & -\frac{t^{2-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
\leqslant & \left(\frac{ac}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \left(\frac{ad}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}
$$

On the other hand, we have from (2.3) that

$$
\begin{aligned}
t^{2-\alpha}G(t,s) = & -\frac{t^{2-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
\geqslant & -\frac{t^{2-\alpha}t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
= & \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}-t\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
= & \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \left(\frac{adt}{(\alpha-1)\delta}+\frac{bd}{\delta}-\frac{t}{\Gamma(\alpha)}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
= & \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} \\
& + \frac{bd(\alpha-1)(1-t)-ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t-bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu-1)}t}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
= & \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{bd(\alpha-1)(1-t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
& + \frac{-ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t-bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu-1)}t}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
\geqslant & \left(\frac{act}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{bd(\alpha-1)(1-t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}t-bc\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu-1)}t}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha)}\\
=&\frac{bc(\alpha-1-(\alpha-\mu-1)t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}+\frac{bd(\alpha-1)(1-t)}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\\
\geqslant&\, 0.
\end{aligned}
$$

Then both (2.5) and (2.6) hold. The proof is completed.

Choose $0<\mu_1<\mu_2<1$. Let

$$
\begin{aligned}
\Lambda_1 &= \min\left\{\frac{bc(\alpha-1-(\alpha-\mu-1)\mu_2)}{(\alpha-1)\delta}\frac{1}{\Gamma(\alpha-\mu)},\ \frac{bd(\alpha-1)(1-\mu_2)}{(\alpha-1)\delta}\frac{1}{\Gamma(\alpha)}\right\},\\
\Pi_1 &= \max\left\{\left(\frac{ac}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{1}{\Gamma(\alpha-\mu)},\ \left(\frac{ad}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{1}{\Gamma(\alpha)}\right\},\\
\Lambda_2 &= \min\left\{\frac{a\mu_1+(\alpha-1)b}{(\alpha-1)\delta},\ \frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-\mu_2\right)+\frac{d}{\Gamma(\alpha-1)}(1-\mu_2)}{(\alpha-1)\delta}\right\},\\
\Pi_2 &= \max\left\{\frac{a+(\alpha-1)b}{(\alpha-1)\delta},\ \frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-\mu_1\right)+\frac{d}{\Gamma(\alpha-1)}(1-\mu_1)}{(\alpha-1)\delta}\right\},\\
\sigma &= \frac{\min\{\Lambda_1,\ \Lambda_2\}}{\max\{\Pi_1,\ \Pi_2\}}.
\end{aligned}
$$

LEMMA 5. *Suppose that $\delta>0$ and $h\in L^1(0,1)$ is nonnegative. Then the unique solution of BVP(2.1) satisfies*

$$
\min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}u(t)\geqslant\sigma\sup_{t\in(0,1]} t^{2-\alpha}u(t). \tag{2.7}
$$

Proof. It follows from Lemma 3 that

$$
\begin{aligned}
t^{2-\alpha}u(t) = \int_0^1 t^{2-\alpha}G(t,s)h(s)ds+\frac{at+(\alpha-1)b}{(\alpha-1)\delta}l&\\
+\frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{d}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta}k&.
\end{aligned}
$$

Lemma 4 implies that

$$
\begin{aligned}
\min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}u(t) &\geqslant \Lambda_1\int_0^1(1-s)^{\alpha-\mu-1}h(s)ds+\Lambda_1\int_0^1(1-s)^{\alpha-1}h(s)ds+\Lambda_2k+\Lambda_2l,\\
\sup_{t\in(0,1]} t^{2-\alpha}u(t) &\leqslant \Pi_1\int_0^1(1-s)^{\alpha-\mu-1}h(s)ds+\Pi_1\int_0^1(1-s)^{\alpha-1}h(s)ds+\Pi_2k+\Pi_2l.
\end{aligned}
$$

It is easy to see that

$$\min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}u(t) \geqslant \frac{\min\{\Lambda_1,\ \Lambda_2\}}{\max\{\Pi_1,\ \Pi_2\}} \sup_{t\in(0,1]} t^{2-\alpha}u(t). \tag{2.8}$$

The proof is complete.

REMARK 4. If u is a solution of BVP(2.1), then Remark 3 and the proof of Lemma 5 imply that

$$\sup_{t\in(0,1]} t^{2-\alpha}u(t) \leqslant \frac{\max\{\Pi_1,\ \Pi_2\}}{\Pi} \sup_{t\in(0,1]} t^{\alpha-\mu-1}|D_{0+}^{\mu}u(t)|.$$

For our construction, we let

$$X = \left\{ x:(0,1]\to\mathbb{R} \begin{array}{c} x\in C(0,1], \\ D_{0+}^{\mu}x\in C^0(0,1], \\ \text{there exist the limits} \\ \lim_{t\to 0} t^{2-\alpha}x(t), \\ \lim_{t\to 0} t^{\mu+2-\alpha}D_{0+}^{\mu}x(t) \end{array} \right\}.$$

For $x\in X$, let

$$\|x\| = \max\left\{ \sup_{0<t\leqslant 1} t^{2-\alpha}|x(t)|,\ \sup_{0<t\leqslant 1} t^{\mu+2-\alpha}|D_{0+}^{\mu}x(t)| \right\}.$$

Then X is a Banach space. We seek solutions of BVP(1.4) that lie in the cone

$$P = \left\{ u\in X:\ \min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}u(t) \geqslant \sigma \sup_{t\in(0,1]} t^{2-\alpha}|u(t)|,\ u(t)\geqslant 0, 0<t\leqslant 1 \right\}.$$

Define the operator T on P, by

$$\begin{aligned} Tu(t) =& \int_0^1 G(t,s)f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &+\frac{at^{\alpha-1}+(\alpha-1)bt^{\alpha-2}}{(\alpha-1)\delta}\int_0^1 h(s,u(s),D_{0+}^{\mu}u(s))ds \\ &+\frac{\frac{ct^{\alpha-2}}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{dt^{\alpha-2}}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta}\int_0^1 g(s,u(s),D_{0+}^{\mu}u(s))ds. \end{aligned}$$

LEMMA 6. *Suppose that $\delta>0$ and f,g,h are Caratheodory functions. Then T is completely continuous.*

Proof. For $u\in P\subset X$, we have $u(t)\geqslant 0$ for all $t\in(0,1]$ and

$$\|u\| = \max\left\{ \sup_{0<t\leqslant 1} t^{2-\alpha}u(t),\ \sup_{0<t\leqslant 1} t^{\mu+2-\alpha}|D_{0+}^{\mu}u(t)| \right\} = r<\infty.$$

Hence there exists $\phi_r \in L^1(0,1)$ such that

$$0 \leqslant f(s,u(s),D_{0^+}^{\alpha-1}u(t)),\ g(t,u(t),D_{0^+}^{\alpha-1}u(t)),\ h(t,u(t),D_{0^+}^{\alpha-1}u(t)) \leqslant \phi_r(t), t \in (0,1).$$

Then Lemmas 3 and 4 imply that

$$\begin{aligned}
0 &\leqslant t^{2-\alpha}(Tu)(t) \\
&= \int_0^1 t^{2-\alpha}G(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\quad + \frac{at+(\alpha-1)b}{(\alpha-1)\delta}\int_0^1 h(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\quad + \frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{d}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta}\int_0^1 g(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\leqslant \Pi_1\int_0^1(1-s)^{\alpha-\mu-1}\phi_r(s)ds + \Pi_1\int_0^1(1-s)^{\alpha-1}\phi_r(s)ds \\
&\quad + 2\Pi_2\int_0^1\phi_r(s)ds.
\end{aligned}$$

On the other hand, from Remarks 1 and 2, we have

$$\begin{aligned}
&t^{\mu+2-\alpha}|D^\mu(Tu)(t)| \\
&= t^{\mu+2-\alpha}\int_0^1 H(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\quad + \frac{bd\Gamma(\alpha-\mu-1)\left(\frac{1}{\alpha-1}-t\right)+bc\Gamma(\alpha-1)(1-t)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}\int_0^1 g(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\quad + \left(\frac{b}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}t\right)\int_0^1 h(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\leqslant \Pi\int_0^1\Bigg([(1-s)^{\alpha-\mu-1}+(1-s)^{\alpha-1}]f(s,u(s),D_{0^+}^{\alpha-1}u(s))+g(s,u(s),D_{0^+}^{\alpha-1}u(s)) \\
&\qquad + h(s,u(s),D_{0^+}^{\alpha-1}u(s))\Bigg)ds \\
&\leqslant \Pi\int_0^1\left([(1-s)^{\alpha-\mu-1}+(1-s)^{\alpha-1}]\phi_r(s)+2\phi_r(s)\right)ds.
\end{aligned}$$

It follows that $(Tu)(t)$ is nonnegative, $t^{2-\alpha}(Tu) \in C(0,1]$, $t^{\mu+2-\alpha}D_{0^+}^{\mu}Tu \in C(0,1]$ and there exist the limits

$$\lim_{t\to 0} t^{2-\alpha}(Tu)(t),\ \lim_{t\to 0} t^{\mu+2-\alpha}D_{0^+}^{\mu}(Tu)(t).$$

Since

$$\min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}Tu(t) \geqslant \Lambda_1\int_0^1(1-s)^{\alpha-\mu-1}f(s,u(s))ds$$

$$+\Lambda_1\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds$$
$$+\Lambda_2\int_0^1 g(s,u(s))ds+\Lambda_2\int_0^1 h(s,u(s))ds,$$

$$\begin{aligned}\sup_{t\in(0,1]} t^{2-\alpha}u(t)\leqslant{}&\Pi_1\int_0^1(1-s)^{\alpha-\mu-1}f(s,u(s))ds\\&+\Pi_1\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds\\&+\Pi_2\int_0^1 g(s,u(s))ds+\Pi_2\int_0^1 h(s,u(s))ds.\end{aligned}$$

It is easy to see that

$$\min_{t\in[\mu_1,\mu_2]} t^{2-\alpha}Tu(t)\geqslant\frac{\min\{\Lambda_1,\ \Lambda_2\}}{\max\{\Pi_1,\ \Pi_2\}}\sup_{t\in(0,1]} t^{2-\alpha}Tu(t). \tag{2.9}$$

Hence $T:P\to P$ is well defined. We divide the remainder of the proof into three steps.

Step 1. T is continuous.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence such that $y_n\to y_0$ in X. Then

$$r=\sup_{n=0,1,2,\cdots}||y_n||<\infty.$$

So there exists $\phi_r\in L^1(0,1)$ such that

$$|f(t,y_n(t),D_{0+}^{\mu}y_n(t))|=|f(t,t^{\alpha-2}t^{2-\alpha}y_n(t),t^{\alpha-\mu-2}t^{\mu+2-\alpha}D_{0+}^{\mu}y_n(t))|\leqslant\phi_r(t),$$

and

$$|g(t,y_n(t),D_{0+}^{\mu}y_n(t))|,\ |h(t,y_n(t),D_{0+}^{\mu}y_n(t))|\leqslant\phi_r(t)$$

holds for $t\in(0,1)$, $n=0,1,2,\cdots$. Then for $t\in(0,1]$, we have

$$\begin{aligned}&t^{2-\alpha}|(Ty_n)(t)-(Ty_0)(t)|\\&\quad\leqslant\int_0^1 t^{2-\alpha}G(t,s)|f(s,y_n(s),D_{0+}^{\mu}y_n(s))-f(s,y_0(s),D_{0+}^{\mu}y_0(s))|ds\\&\qquad+\frac{at+(\alpha-1)b}{(\alpha-1)\delta}\int_0^1|h(s,y_n(s),D_{0+}^{\mu}y_n(s))-h(s,y_0(s),D_{0+}^{\mu}y_0(s))|ds\\&\qquad+\frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{d}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta}\times\\&\qquad\quad\int_0^1|g(s,y_n(s),D_{0+}^{\mu}y_n(s))-g(s,y_0(s),D_{0+}^{\mu}y_0(s))|ds\\&\quad\leqslant\Pi_1\int_0^1(1-s)^{\alpha-\mu-1}|f(s,y_n(s),D_{0+}^{\mu}y_n(s))-f(s,y_0(s),D_{0+}^{\mu}y_0(s))|ds\end{aligned}$$

$$
\begin{aligned}
&+\Pi_1\int_0^1(1-s)^{\alpha-1}|f(s,y_n(s),D_{0^+}^{\mu}y_n(s))-f(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&+\Pi_2\int_0^1|h(s,y_n(s),D_{0^+}^{\mu}y_n(s))-h(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&+\Pi_2\int_0^1|g(s,y_n(s),D_{0^+}^{\mu}y_n(s))-g(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
\leqslant\ & 2\Pi_1\int_0^1\phi_r(s)ds+2\Pi_1\int_0^1\phi_r(s)ds\\
&+2\Pi_2\int_0^1\phi_r(s)ds+2\Pi_2\int_0^1\phi_r(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&t^{\mu+2-\alpha}|D_{0^+}^{\mu}(Ty_n)(t)-D_{0^+}^{\mu}(Ty_0)(t)|\\
&\leqslant\int_0^1 t^{\mu+2-\alpha}H(t,s)|f(s,y_n(s),D_{0^+}^{\mu}y_n(s))-f(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&\quad+\frac{bd\Gamma(\alpha-\mu-1)\left(\frac{1}{\alpha-1}-t\right)+bc\Gamma(\alpha-1)(1-t)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}\times\\
&\qquad\int_0^1|h(s,y_n(s),D_{0^+}^{\mu}y_n(s))-h(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&\quad+\left(\frac{b}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}t\right)\times\\
&\qquad\int_0^1|g(s,y_n(s),D_{0^+}^{\mu}y_n(s))-g(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&\leqslant\Pi\left(\int_0^1(1-s)^{\alpha-\mu-1}|f(s,y_n(s),D_{0^+}^{\mu}y_n(s))-f(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\right.\\
&\quad+\int_0^1(1-s)^{\alpha-1}|f(s,y_n(s),D_{0^+}^{\mu}y_n(s))-f(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&\quad+\int_0^1|h(s,y_n(s),D_{0^+}^{\mu}y_n(s))-h(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\\
&\quad\left.+\int_0^1|g(s,y_n(s),D_{0^+}^{\mu}y_n(s))-g(s,y_0(s),D_{0^+}^{\mu}y_0(s))|ds\right)\\
&\leqslant 2\Pi\int_0^1\phi_r(s)ds+2\Pi\int_0^1\phi_r(s)ds\\
&\quad+2\Pi\int_0^1\phi_r(s)ds+2\Pi\int_0^1\phi_r(s)ds.
\end{aligned}
$$

Since $f(t,\cdot,\cdot),g(t,\cdot,\cdot),h(t,\cdot,\cdot):[0,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous, we have $||Ty_n-Ty_0||\to 0$ as $n\to\infty$. Then T is continuous.

Step 2. T maps bounded sets into bounded sets in X.

It suffices to show that for each $r>0$, there exists a positive number $L>0$ such that for each $x\in M=\{y\in X:||y||\leqslant r\}$, we have $||Ty||\leqslant L$. By the assumption, there exists $\phi_r \in L^1(0,1)$ such that

$$|f(t,y(t),D_{0+}^{\mu}y(t))|,\ |g(t,y(t),D_{0+}^{\mu}y(t))|,\ |h(t,y(t),D_{0+}^{\mu}y(t))| \leqslant \phi_r(t), t \in (0,1).$$

By the definition of T, similarly we get

$$t^{2-\alpha}|(Ty)(t)| \leqslant \Pi_1 \int_0^1 (1-s)^{\alpha-\mu-1}\phi_r(s)ds + \Pi_1 \int_0^1 (1-s)^{\alpha-1}\phi_r(s)ds + 2\Pi_2 \int_0^1 \phi_r(s)ds$$

and

$$t^{\mu+2-\alpha}|D^{\mu}(Ty)(t)| \leqslant \Pi \int_0^1 \Big([(1-s)^{\alpha-\mu-1} + (1-s)^{\alpha-1}]\phi_r(s) + 2\phi_r(s)\Big)\, ds.$$

It follows that there exists $L > 0$ such that $||Ty|| \leqslant L$ for each $y \in \{y \in X : ||y|| \leqslant r\}$. Then T maps bounded sets into bounded sets in X.

Step 3. Let $M = \{y \in X : ||y|| \leqslant r\}$. Prove that both $t^{2-\alpha}TM$ and $t^{\mu+2-\alpha}TM$ are equicontinuous on $[0,1]$.

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in M = \{y \in X : ||y|| \leqslant r\}$. By the assumption, there exists $\phi_r \in L^1(0,1)$ such that

$$|f(t,y(t),D_{0+}^{\mu}y(t))|,\ |g(t,y(t),D_{0+}^{\mu}y(t))|,\ |h(t,y(t),D_{0+}^{\mu}y(t))| \leqslant \phi_r(t), t \in (0,1).$$

One sees, for $s \in [0,t_1]$, from (7) and the definition of H, that

$$t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s) = \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1} - t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{ac[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{ad[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$t_1^{\mu+2-\alpha}H(t_1,s) - t_2^{\mu+2-\alpha}H(t_2,s) = \frac{t_2^{\mu+2-\alpha}(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} - \frac{t_1^{\mu+2-\alpha}(t_1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1} + ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}[t_1-t_2].$$

For $s \in [t_1,t_2]$, we have

$$t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s) = \frac{ac[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} + \frac{ad[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)$$
$$=\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}[t_1-t_2]-\frac{t_2^{\mu+2-\alpha}(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}.$$

For $s\in[t_2,1]$, we have

$$t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)=\frac{ac[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}+\frac{ad[t_1-t_2]}{(\alpha-1)\delta}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)$$
$$=\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}[t_1-t_2].$$

Then

$$|t_1^{2-\alpha}(Ty)(t_1)-t_1^{2-\alpha}(Ty)(t_2)|$$
$$\leqslant\int_0^{t_1}|t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0^+}^{\mu}y(s))|ds$$
$$+\int_{t_1}^{t_2}|t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0^+}^{\mu}y(s))|ds$$
$$+\int_{t_2}^{1}|t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0^+}^{\mu}y(s))|ds$$
$$+\frac{a|t_1-t_2|}{(\alpha-1)\delta}\int_0^1 h(s,u(s),D_{0^+}^{\mu}y(s))ds$$
$$+\frac{\frac{c}{\Gamma(\alpha-\mu-1)}|t_1-t_2|+\frac{d}{\Gamma(\alpha-1)}|t_1-t_2|}{(\alpha-1)\delta}\int_0^1 g(s,u(s),D_{0^+}^{\mu}y(s))ds$$
$$\leqslant\int_0^1\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_r(s)ds$$
$$+\left(\frac{1}{\Gamma(\alpha)}+\frac{ac}{(\alpha-1)\delta\Gamma(\alpha-\mu)}+\frac{ad}{(\alpha-1)\delta\Gamma(\alpha)}\right)\int_{t_1}^{t_2}\phi_r(s)ds$$
$$+2|t_1-t_2|\left(\frac{ac}{(\alpha-1)\delta\Gamma(\alpha-\mu)}+\frac{ad}{(\alpha-1)\delta\Gamma(\alpha)}\right)\times$$
$$\int_0^1\left(\frac{ac}{\delta}+\frac{ad(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\right)\phi_r(s)ds$$
$$+\frac{a|t_1-t_2|}{(\alpha-1)\delta}\int_0^1\phi_r(s)ds$$
$$+\frac{\frac{c}{\Gamma(\alpha-\mu-1)}|t_1-t_2|+\frac{d}{\Gamma(\alpha-1)}|t_1-t_2|}{(\alpha-1)\delta}\int_0^1\phi_r(s)ds.$$

On the other hand, we have

$$
\begin{aligned}
&|t_1^{\mu+2-\alpha}D_{0^+}^{\mu}(Ty)(t_1)-t_1^{\mu+2-\alpha}D_{0^+}^{\mu}(Ty)(t_2)|\\
&\quad\leqslant \int_0^1 |t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha-1)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}|t_1-t_2|\int_0^1 g(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}|t_1-t_2|\int_0^1 h(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\quad\leqslant \int_0^{t_1} |t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\int_{t_1}^{t_2} |t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\int_{t_2}^{1} |t_1^{\mu+2-\alpha}H(t_1,s)-t_2^{\mu+2-\alpha}H(t_2,s)|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha-1)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}|t_1-t_2|\int_0^1 \phi_r(s)ds\\
&\qquad+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}|t_1-t_2|\int_0^1 \phi_r(s)ds\\
&\quad\leqslant \int_0^{t_1} \Bigg|\frac{t_2^{\mu+2-\alpha}(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}-\frac{t_1^{\mu+2-\alpha}(t_1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}\\
&\qquad+\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}[t_1-t_2]\Bigg|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\int_{t_1}^{t_2} \Bigg|\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}[t_1-t_2]\\
&\qquad-\frac{t_2^{\mu+2-\alpha}(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}\Bigg|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\int_{t_2}^{1} \frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}|t_1-t_2|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\\
&\qquad+\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha-1)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}|t_1-t_2|\int_0^1 \phi_r(s)ds\\
&\qquad+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}|t_1-t_2|\int_0^1 \phi_r(s)ds\\
&\quad\leqslant \int_0^{1} \left|\frac{t_2^{\mu+2-\alpha}(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}-\frac{t_1^{\mu+2-\alpha}(t_1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}\right|\phi_r(s)ds\\
&\qquad+\int_0^{1} \frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}\phi_r(s)ds|t_1-t_2|
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\left|\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}+\frac{(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}\right|\phi_r(s)ds\\
&+\int_0^1\frac{ac\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}(1-s)^{\alpha-\mu-1}+ad(1-s)^{\alpha-1}}{\delta(\alpha-1)\Gamma(\alpha-\mu)}\phi_r(s)ds|t_1-t_2|\\
&+\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha-1)}{b\delta\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)}\int_0^1\phi_r(s)ds|t_1-t_2|\\
&+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}\int_0^1\phi_r(s)ds|t_1-t_2|.
\end{aligned}
$$

As $t_1\to t_2$, the right-hand sides of the above inequalities tends to zero uniformly. Therefore, both $t^{2-\alpha}TM$ and $t^{\mu+2-\alpha}TM$ are equicontinuous.

The Arzela-Ascoli theorem implies that $\overline{T(M)}$ is compact. Thus, the operator $T:P\to P$ is completely continuous.

3. Main theorem

Now, we prove the main result. It is supposed that f,g,h are defined on $(0,1)\times[0,\infty)\times\mathbb{R}$ and are nonnegative Caratheodory functions, $\alpha\in(1,2),\mu\in(0,\alpha-1)$. Choose $0<\mu_1<\mu_2<1$, let Λ_1,Λ_2, and Π,Π_1,Π_2 be defined in Section 2. Denote

$$
\begin{aligned}
F(t,x,y)=[(1-t)^{\alpha-\mu-1}+(1-t)^{\alpha-1}]f(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y)\\
+g(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y)+h(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y).
\end{aligned}
$$

THEOREM 1. *If there exist positive numbers e_1, e_2, c and nonnegative function $\phi\in L^1(0,1)$ such that*

$$c>\frac{e_2}{\sigma}>e_2>e_1>0,\ Q\geqslant W,$$

(A1) *$F(t,u,v)\leqslant Q\phi(t)$ for all $t\in(0,1),u\in[0,c],v\in[-c,c]$;*
(A2) *$F(t,u,v)\geqslant W\phi(t)$ for all $t\in[\mu_1,\mu_2],u\in[e_2,e_2/\sigma],v\in[-c,c]$;*
(A3) *$F(t,u,v)\leqslant E\phi(t)$ for all $t\in(0,1),u\in[0,e_1],v\in[-c,c]$;*
where Q,W and E given by

$$
\begin{aligned}
Q&=\min\left\{\frac{c}{\Pi\int_0^1\phi(s)ds},\ \frac{c}{\max\{\Pi_1,\Pi_2\}\int_0^1\phi(s)ds}\right\};\\
W&=\frac{e_2}{\min\{\Lambda_1,\Lambda_2\}\int_{\mu_1}^{\mu_2}\phi(s)ds};\\
E&=\frac{e_1}{\max\{\Pi_1,\Pi_2\}\int_0^1\phi(s)ds},
\end{aligned}
$$

then BVP(1.4) has at least three positive solutions y_1,y_2,y_3 such that

$$\sup_{t\in(0,1]}t^{2-\alpha}y_1(t)<e_1,\ \inf_{t\in[\mu_1,\mu_2]}t^{2-\alpha}y_2(t)>e_2, \tag{3.1}$$

and

$$\sup_{t\in(0,1]} t^{2-\alpha}y_1(t) > e_1, \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}y_3(t) < e_2. \tag{3.2}$$

Proof. Define the functionals on $P \to R$ by

$$\begin{aligned} \beta_1(y) &= \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)|,\ y\in P,\\ \beta_2(y) &= \sup_{t\in(0,1]} t^{2-\alpha}|y(t)|,\ y\in P,\\ \beta_3(y) &= \sup_{t\in(0,1]} t^{2-\alpha}|y(t)|,\ y\in P,\\ \alpha_1(y) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}|y(t)|,\ y\in P,\\ \alpha_2(y) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}|y(t)|,\ y\in P. \end{aligned}$$

By the definitions, it is easy to see that α_1,α_2 are nonnegative continuous concave functional on the cone P, β_1,β_2,β_3 nonnegative continuous convex functional on the cone P.

One gets that $\alpha_1(x) \leqslant \beta_2(x)$ easily for all $x\in P$ and Remark 2.4 implies that $||x|| \leqslant \max\left\{1, \frac{\max\{\Pi_1,\Pi_2\}}{\Pi}\right\}\beta_1(x)$ for all $x\in P$.

Lemmas 5 and 6 imply that $TP\subset P$, $x = x(t)$ is a positive solution of BVP(1.4) if and only if $x(t)$ is a solution of the operator equation $x = Tx$ in P, T is completely continuous.

Hence $(B1)$ and $(B2)$ of Lemma 1 are satisfied.

Corresponding to Lemma 1, choose

$$c_1 = e_1,\ c_2 = e_2,\ c_3 = \sigma e_1,\ c_4 = \frac{e_2}{\sigma},\ c_5 = c.$$

Now, we prove that $(B3)$ of Lemma 1 are satisfied. One sees that $c_1 < c_2$ since $e_1 < e_2$. The remainder is divided into five steps.

Step 1. Prove that $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;

For $y\in\overline{P_{c_5}}$, we have $||y|| \leqslant c$. Then

$$0 \leqslant t^{2-\alpha}y(t) \leqslant c,\ -c \leqslant t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c \text{ for all } t\in(0,1).$$

So (A1) implies that

$$F(t,y(t),D_{0+}^{\mu}y(t)) < Q\phi(t),\ t\in(0,1).$$

Hence

$$\begin{aligned} \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}(Ty)(t)| &\leqslant \Pi\int_0^1 F(s,y(s),D_{0+}^{\mu}y(s))ds\\ &= \Pi Q\int_0^1 \phi(s)ds \end{aligned}$$

$$\leqslant c.$$

Since $Ty \in P$, we have

$$\begin{aligned}
t^{2-\alpha}|(Ty)(t)| &\leqslant \max\{\Pi_1,\Pi_2\}\int_0^1 F(s,y(s),D_{0+}^{\mu}y(s))ds \\
&\leqslant \max\{\Pi_1,\Pi_2\}Q\int_0^1 \phi(s)ds \\
&\leqslant c.
\end{aligned}$$

It follows that

$$||Ty|| = \max\left\{\sup_{t\in(0,1]} t^{2-\alpha}(Ty)(t)|, \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0^+}^{\mu}(Ty)(t)|\right\} \leqslant c.$$

Then $T(\overline{P_{c_5}}) \subseteq \overline{P_{c_5}}$. This completes the proof of (B3)(i) of Lemma 1.

Step 2. Prove that $\{y \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)|\alpha_1(x) > c_2\} \neq \emptyset$ and

$$\alpha_1(Tx) > c_2 \text{ for every } x \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5);$$

From the definitions of β_1,β_3 and α_1, one has that

$$\begin{aligned}
\alpha_1(y) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}y(t) \geqslant e_2, \\
\beta_3(y) &= \sup_{t\in(0,1]} t^{2-\alpha}y(t) \leqslant \frac{e_2}{\sigma}, \\
\beta_1(y) &= \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c.
\end{aligned}$$

It is easy to see that $\{y \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)|\alpha_1(y) > c_2\} \neq \emptyset$.

For $y \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$, then

$$e_2 \leqslant t^{2-\alpha}y(t) \leqslant \frac{e_2}{\sigma},\ t \in [\mu_1,\mu_2],\ \ t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c.$$

Thus $(A2)$ implies that

$$F(t,y(t),D_{0+}^{\mu}y(t)) \geqslant W\phi(t),\ \ t \in [\mu_1,\mu_2].$$

Since $Ty \in P$, we get from the definition of T that

$$\begin{aligned}
\alpha_1(Ty) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}(Ty)(t) \\
&\geqslant \inf_{t\in[\mu_1,\mu_2]}\Bigg(t^{2-\alpha}\int_0^1 G(t,s)f(s,u(s),D_{0+}^{\mu}u(s))ds \\
&\quad + \frac{at^{\alpha-1}+(\alpha-1)b}{(\alpha-1)\delta}\int_0^1 h(s,u(s),D_{0+}^{\mu}u(s))ds
\end{aligned}$$

$$+\frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-t\right)+\frac{d}{\Gamma(\alpha-1)}(1-t)}{(\alpha-1)\delta}\int_0^1 g(s,u(s),D_{0+}^{\mu}u(s))ds\Bigg)$$

$$\begin{aligned}
&\geqslant \min\{\Lambda_1,\Lambda_2\}\int_0^1 F(s,y(s),D_{0+}^{\mu}y(s))ds\\
&> \min\{\Lambda_1,\Lambda_2\}\int_{\mu_1}^{\mu_2} F(s,y(s),D_{0+}^{\mu}y(s))ds\\
&\geqslant \min\{\Lambda_1,\Lambda_2\}\int_{\mu_1}^{\mu_2} W\phi(s)ds\\
&\geqslant e_2.
\end{aligned}$$

It follows that $\alpha_1(Tx) > c_2$ for every $x \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$. This completes the proof of (B3)(ii) of Lemma 1.

Step 3. Prove that $\{y \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)|\beta_2(x) < c_1\} \neq \emptyset$ and

$$\beta_2(Tx) < c_1 \text{ for every } x \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5);$$

From the definitions of α_1,β_2 and β_1, one has that

$$\begin{aligned}
\alpha_2(y) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}y(t) \geqslant c_3,\\
\beta_2(y) &= \sup_{t\in(0,1]} t^{2-\alpha}y(t) \leqslant c_1,\\
\beta_1(y) &= \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c_5.
\end{aligned}$$

It is easy to see that $\{x \in P(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5) : \beta_2(x) < c_1\} \neq \emptyset$.

For $y \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)$, Then

$$0 \leqslant t^{2-\alpha}y(t) \leqslant c_1, \quad t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c, \ \ t \in (0,1].$$

Thus (A3) implies that

$$F(t,y(t),D_{0+}^{\mu}y(t)) \leqslant E\phi(t), \ \ t \in (0,1).$$

Then

$$\begin{aligned}
t^{2-\alpha}|(Ty)(t)| &\leqslant \max\{\Pi_1,\Pi_2\}\int_0^1 F(s,y(s),D_{0+}^{\mu}y(s))ds\\
&\leqslant \max\{\Pi_1,\Pi_2\}E\int_0^1 \phi(s)ds\\
&\leqslant e_1.
\end{aligned}$$

It follows that $\beta_2(Ty) < c_1$. This completes the proof of (B3)(iii) of Lemma 1.

Step 4. Prove that $\alpha_1(Ty) > c_2$ for $y \in P(\beta_1,\alpha_1;c_2,c_5)$ with $\beta_3(Ty) > c_4$;

For $y \in P(\beta_1,\alpha_1;c_2,c_5)$ with $\beta_3(Ty) > c_4$, we have that

$$\alpha_1(y) = \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}y(t) \geqslant c_2 = e_2$$

$$\beta_1(y) = \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c_5$$

and

$$\beta_3(Ty) = \sup_{t\in(0,1]} t^{2-\alpha}(Ty)(t) > \frac{e_2}{\sigma} = c_4.$$

Then

$$\begin{aligned}\alpha_1(Ty) &= \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}(Ty)(t)\\ &\geqslant \sigma \sup_{t\in(0,1]} t^{2-\alpha}(Ty)(t)\\ &> \sigma\frac{e_2}{\sigma}\\ &= e_2 = c_2.\end{aligned}$$

This completes the proof of (B3)(iv) of Lemma 1.

Step 5. Prove that $\beta_2(Tx) < c_1$ for each $x \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(Tx) < c_3$;

For $y \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(Ty) < c_3$, we have that

$$\beta_2(y) = \sup_{t\in(0,1]} t^{2-\alpha}y(t) \leqslant e_1$$

and

$$\beta_1(y) = \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y(t)| \leqslant c_5$$

and

$$\alpha_2(Ty) = \inf_{t\in[\mu_1,\mu_2]} t^{2-\alpha}(Ty)(t) < c_3 = \mu e_1.$$

Then

$$0 \leqslant t^{2-\alpha}y(t) \leqslant e_1 = c_1,\ -c \leqslant t^{\mu+2-\alpha}D_{0+}^{\mu}y(t) \leqslant c,\ t \in (0,1].$$

Using $(A3)$ and the methods in Step 3, we get

$$\beta_2(Ty) = \sup_{t\in(0,1]} t^{2-\alpha}(Ty)(t) < c_1.$$

This completes the proof of (B3)(v) of Lemma 1.

Then Lemma 1 implies that T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta_2(y_1) < e_1,\ \alpha_1(y_2) > e_2,\ \beta_2(y_3) > e_1,\ \alpha_1(y_3) < e_2.$$

Hence BVP(1.4) has at least three positive solutions y_1, y_2 and y_3 such that (3,1) and (3.2) hold.

4. An example

In this section, we give an example to illustrate the main theorem (Theorem 1).

EXAMPLE 1. Let $\lambda > 0$. Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t) + \frac{t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}u(t)\right)+f_1\left(t^{\frac{3}{4}}D_{0^+}^{\frac{1}{4}}u(t)\right)\right]}{(1-t)^{\frac{1}{2}}+(1-t)^{\frac{1}{4}}} = 0, \quad t\in(0,1), \\ \lim_{t\to 0} I_{0^+}^{\frac{1}{2}}u(t) - \lim_{t\to 0}\left[I_{0^+}^{\frac{1}{2}}u(t)\right]' = \int_0^1 t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}u(t)\right)+f_1\left(t^{\frac{3}{4}}D_{0^+}^{\frac{1}{4}}u(t)\right)\right]dt, \\ D_{0^+}^{\frac{1}{4}}u(1)+u(1) = \int_0^1 t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}u(t)\right)+f_1\left(t^{\frac{3}{4}}D_{0^+}^{\frac{1}{4}}u(t)\right)\right]dt, \end{cases} \tag{4.1}$$

where $f_0, f_1 : R \to [0,\infty)$ are continuous functions such that $f_0(x) > 0$ for all $x \neq 0$ and $0 \leqslant f_1(x) \leqslant 1$ with

$$f_0(x) = \begin{cases} 2.4076, \quad x\in[0,10], \\ 2.4676 + \frac{28682-2.4676}{100-10}(x-10), \quad x\in[10,100], \\ 28682, \quad x\in[100,10^5], \\ 28682\times e^{x-10^5}, \quad x\in[10^5,\infty). \end{cases}$$

Corresponding to BVP(1.4), we find that $\alpha = \frac{3}{2}$, $a=b=c=d=1$, $\mu = \frac{1}{4}$ and

$$f(t,x,y) = \frac{t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}x\right)+f_1\left(t^{\frac{3}{4}}y\right)\right]}{(1-t)^{\frac{1}{2}}+(1-t)^{\frac{1}{4}}},$$
$$g(t,x,y) = t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}x\right)+f_1\left(t^{\frac{3}{4}}y\right)\right],$$
$$h(t,x,y) = t^{-\frac{1}{8}}\left[f_0\left(t^{\frac{1}{2}}x\right)+f_1\left(t^{\frac{3}{4}}y\right)\right].$$

It is easy to see that f, g and h are Caratheodory functions, and

$$\begin{aligned} F(t,x,y) &= [(1-t)^{\alpha-\mu-1}+(1-t)^{\alpha-1}]f(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y)+g(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y) \\ &\quad + h(t,t^{\alpha-2}x,t^{\alpha-\mu-2}y) \\ &= 3t^{-\frac{1}{8}}[f_0(x)+f_1(y)]. \end{aligned}$$

Choose $\mu_1 = \frac{1}{4}, \mu_2 = \frac{3}{4}$, $\phi(t) = 3t^{-\frac{1}{8}}$. $\Gamma(1/2) \approx 1.7725$ and $\Gamma(1/4) \approx 3.6526$ imply that

$$\begin{aligned} \delta &= bc\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)} + bd + ac\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)} + ad\frac{1}{\alpha-1} \\ &= 3 + 5\frac{\Gamma(1/2)}{\Gamma(1/4)} \approx 5.4260, \end{aligned}$$

$$\Lambda_1 = \min\left\{\frac{bc\,(\alpha-1-(\alpha-\mu-1)\mu_2)}{(\alpha-1)\delta}\frac{1}{\Gamma(\alpha-\mu)},\ \frac{bd(\alpha-1)(1-\mu_2)}{(\alpha-1)\delta}\frac{1}{\Gamma(\alpha)}\right\}$$
$$= \min\left\{\frac{7}{6\Gamma(1/4)+10\Gamma(1/2)}, \frac{3\Gamma(1/4)}{2\Gamma(1/2)[3\Gamma(1/4)+5\Gamma(1/2)]}\right\} \approx 0.1559,$$
$$\Pi_1 = \max\left\{\left(\frac{ac}{(\alpha-1)\delta}+\frac{bc}{\delta}\right)\frac{1}{\Gamma(\alpha-\mu)},\ \left(\frac{ad}{(\alpha-1)\delta}+\frac{bd}{\delta}\right)\frac{1}{\Gamma(\alpha)}\right\}$$
$$= \max\left\{\frac{12}{3\Gamma(1/4)+5\Gamma(1/2)}, \frac{6\Gamma(1/4)}{\Gamma(1/2)[3\Gamma(1/4)+5\Gamma(1/2)]}\right\} \approx 0.6057,$$
$$\Lambda_2 = \min\left\{\frac{a\mu_1+(\alpha-1)b}{(\alpha-1)\delta},\ \frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-\mu_2\right)+\frac{d}{\Gamma(\alpha-1)}(1-\mu_2)}{(\alpha-1)\delta}\right\}$$
$$= \min\left\{\frac{3\Gamma(1/4)}{2[3\Gamma(1/4)+5\Gamma(1/2)]}, \frac{\Gamma(1/4)+5\Gamma(1/2)}{2\Gamma(1/2)[3\Gamma(1/4)+5\Gamma(1/2)]}\right\} \approx 0.2765,$$
$$\Pi_2 = \max\left\{\frac{a+(\alpha-1)b}{(\alpha-1)\delta},\ \frac{\frac{c}{\Gamma(\alpha-\mu-1)}\left(\frac{\alpha-1}{\alpha-\mu-1}-\mu_1\right)+\frac{d}{\Gamma(\alpha-1)}(1-\mu_1)}{(\alpha-1)\delta}\right\}$$
$$= \max\left\{\frac{3\Gamma(1/4)}{3\Gamma(1/4)+5\Gamma(1/2)}, \frac{7\Gamma(1/2)+3\Gamma(1/4)}{2\Gamma(1/2)[3\Gamma(1/4)+5\Gamma(1/2)]}\right\} \approx 0.5531,$$
$$\Pi = \max\left\{\frac{[ac+bc(\alpha-\mu-1)]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}+\delta(\alpha-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)},\right.$$
$$\frac{ad+bd(\alpha-\mu-1)}{\delta(\alpha-1)(\alpha-\mu-1)\Gamma(\alpha-\mu-1)}\times$$
$$\frac{bd\Gamma(\alpha-\mu-1)+bc\Gamma(\alpha)}{b\delta(\alpha-1)\Gamma(\alpha-\mu)\Gamma(\alpha-\mu-1)},$$
$$\left.\frac{b}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu-1)}+\frac{a}{\delta}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\mu)}\right\}$$
$$= \max\left\{\frac{6\Gamma(1/4)+20\Gamma(1/2)}{\Gamma(1/4)[3\Gamma(1/4)+5\Gamma(1/2)]}, \frac{10}{3\Gamma(1/4)+5\Gamma(1/2)},\right.$$
$$\left.\frac{4\Gamma(1/2)+8\Gamma(1/4)}{\Gamma(1/4)[3\Gamma(1/4)+5\Gamma(1/2)]}, \frac{5\Gamma(1/2)}{3\Gamma(1/4)+5\Gamma(1/2)}\right\}$$
$$\approx 0.7927,$$
$$\sigma = \frac{\min\{\Lambda_1,\ \Lambda_2\}}{\max\{\Pi_1,\ \Pi_2\}} \approx 0.2573.$$

Choose $e_1 = 10, e_2 = 100, c = 10^5$. Then Q, W and E are defined by

$$Q = \min\left\{\frac{c}{\Pi\int_0^1\phi(s)ds},\ \frac{c}{\max\{\Pi_1,\Pi_2\}\int_0^1\phi(s)ds}\right\} \approx 36794.0792;$$
$$W = \frac{e_2}{\min\{\Lambda_1,\Lambda_2\}\int_{\mu_1}^{\mu_2}\phi(s)ds} \approx 37.4171;$$

$$E = \frac{e_1}{\max\{\Pi_1, \Pi_2\} \int_0^1 \phi(s)ds} \approx 4.8153.$$

It is easy to see that

$$c > \frac{e_2}{\sigma} > e_2 > e_1 > 0,\ Q \geqslant W$$

and

(A1) $F(t,u,v) \leqslant Q\phi(t)$ for all $t \in (0,1), u \in [0,10^5], v \in [-10^5,10^5]$;

(A2) $F(t,u,v) \geqslant W\phi(t)$ for all $t \in [0.25,0.75], u \in [100,388.6513], v \in [-10^5,10^5]$;

(A3) $F(t,u,v) \leqslant E\phi(t)$ for all $t \in (0,1), u \in [0,10], v \in [-10^5,10^5]$;

Hence Theorem 1 implies that BVP(4.1) has at least three positive solutions y_1, y_2, y_3 such that

$$\sup_{t\in(0,1]} t^{\frac{1}{2}} y_1(t) < 10, \quad \inf_{t\in[1/4,3/4]} t^{\frac{1}{2}} y_2(t) > 100, \tag{4.2}$$

and

$$\sup_{t\in(0,1]} t^{\frac{1}{2}} y_1(t) > 10, \quad \inf_{t\in[1/4,3/4]} t^{\frac{1}{2}} y_3(t) < 100. \tag{4.3}$$

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