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EXISTENCE AND PRECISE ASYMPTOTIC BEHAVIOR OF STRONGLY MONOTONE SOLUTIONS OF SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. We analyze positive solutions of the two-dimensional systems of nonlinear differential equations

$$x' + p(t)y^{\alpha} = 0, \qquad y' + q(t)x^{\beta} = 0, \tag{A}$$

$$x' = p(t)y^{\alpha}, \qquad y' = q(t)x^{\beta}, \tag{B}$$

in the framework of regular variation and indicate the situation in which system (A) (resp. (B)) possesses strongly decreasing solutions (resp. strongly increasing solutions) with accurate asymptotic behavior as $t \to \infty$.

1. Introduction

Since the publication of the book of Marić [9] theory of regular variation (in the sense of Karamata) has gradually been recognized as a powerful tool for the asymptotic analysis of positive solutions of linear and nonlinear ordinary differential equations. Particularly noteworthy is the marked role played by Karamata's integration theorem in establishing the accurate asymptotic behavior at infinity of possible positive solutions for nonlinear differential equations of Emden-Fowler and Thomas-Fermi types; see e.g. the papers [4 - 8]. It is expected that similar analysis in the framework of regular variation could be effectively applied to a much larger class of differential equations. Motivated by this expectation we experiment in this paper with deriving precise information about the asymptotic behavior of positive solutions for the two simplest classes of nonlinear systems of differential equations

$$x' + p(t)y^{\alpha} = 0, \qquad y' + q(t)x^{\beta} = 0, \tag{A}$$

$$x' = p(t)y^{\alpha}, \qquad y' = q(t)x^{\beta}, \tag{B}$$

where the following assumptions are always assumed to hold:

(a) α and β are positive constants such that $\alpha\beta < 1$;

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(b) $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a>0$, both of which are regularly varying or nearly regularly varying in the sense specified in the next section.

Our primary concern is with solutions $(x(t),y(t))$ of (A) such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, which are referred to as *strongly decreasing solutions* of (A), and with solutions $(x(t),y(t))$ of (B) such that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty$, which are referred to as *strongly increasing solution* of (B). Such solutions are constructed as solutions of the integral equations

$$x(t) = \int_t^\infty p(s)y(s)^\alpha ds, \quad y(t) = \int_t^\infty q(s)x(s)^\beta ds, \quad t \geqslant T, \tag{1.1}$$

and

$$x(t) = x_0 + \int_T^t p(s)y(s)^\alpha ds, \quad y(t) = y_0 + \int_T^t q(s)x(s)^\beta ds, \quad t \geqslant T, \tag{1.2}$$

$x_0 > 0$, $y_0 > 0$ and $T > a$ being constants, in the class of regularly varying or nearly regularly varying functions with specific asymptotic behavior at infinity. The Schauder-Tychonoff fixed point theorem is employed for this purpose (see [2]). It will be shown that complete knowledge can be acquired of strongly monotone solutions with nonzero indices of (A) and (B) in the particular case where $p(t)$ and $q(t)$ are regularly varying.

After stating the definition and some basic properties of regularly varying functions in Section 2 we establish our main results on the existence and asymptotic behavior of nearly regularly varying solutions with explicit nonzero indices which provides strongly monotone solutions for systems (A) and (B) in Sections 3 and 4, respectively.

For the in-depth analysis of oscillation and asymptotic behavior for systems of nonlinear differential equations the reader is referred to the book of Mirzov [10].

2. Regularly varying functions

For the reader's convenience we recall the definition of regularly varying functions.

DEFINITION 2.1. A measurable function $f:[0,\infty)\to(0,\infty)$ is called *regularly varying of index* $\rho\in\mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by RV(ρ). We often use the symbol SV to denote RV(0), and call members of SV *slowly varying functions*. Any function $f(t)\in$ RV(ρ) is written as $f(t) = t^\rho g(t)$ with $g(t)\in$ SV, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. One of the most important properties of regularly varying functions is the following *representation theorem*.

PROPOSITION 2.1. *$f(t) \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t)\exp\left\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\right\}, \quad t \geqslant t_0,$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho. \tag{2.1}$$

If $c(t) \equiv c_0$ in (2.1), then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ.

Typical examples of slowly varying functions are: all functions tending to some positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \ \alpha_n \in \mathbb{R}, \quad \text{and} \quad \exp\left\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\right\}, \ \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\left\{ (\log t)^{\frac{1}{3}} \cos\,(\log t)^{\frac{1}{3}} \right\}$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0.$$

The following result concerns operations which preserve slow variation.

PROPOSITION 2.2. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^{\alpha}$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

PROPOSITION 2.3. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for deciding the regularity of differentiable positive functions follows.

PROPOSITION 2.4. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t \frac{f'(t)}{f(t)} = \rho.$$

The following result which is called Karamata's integration theorem is useful in handling slowly and regularly varying functions analytically.

PROPOSITION 2.5. *Let* $L(t) \in \mathrm{SV}$. *Then,*
(i) *If* $\alpha > -1$,

$$\int_a^t s^{\alpha} L(s) ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty.$$

(ii) *If* $\alpha < -1$,

$$\int_t^{\infty} s^{\alpha} L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty.$$

(iii) *If* $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$

and

$$m(t) = \int_t^{\infty} \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

Here and throughout the symbol $\sim$ is used to mean the asymptotic equivalence, i.e.

$$f(t) \sim g(t), \quad t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^{\rho} L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t\to\infty} L(t) = \mathrm{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-RV(ρ) (or ntr-RV(ρ)) denotes the set of all trivial RV(ρ)-functions (or the set of all nontrivial RV(ρ)-functions).

A measurable function $f : (0,\infty) \to (0,\infty)$ is called *regularly bounded* if for any $\lambda_0 > 1$ there exist positive constants m and M such that

$$1 < \lambda < \lambda_0 \quad \Longrightarrow \quad m \leqslant \frac{f(\lambda t)}{f(t)} \leqslant M \quad \text{for all large } t.$$

The totality of regularly bounded functions is denoted by RO.

It is clear that $\mathrm{RV}(\rho) \subset \mathrm{RO}$ for any $\rho \in \mathbb{R}$. Any function which is bounded both from above and from below by positive constants is regularly bounded. For example, $2+\sin t$ and $2+\sin(\log t)$ are regularly bounded. Note that $2+\sin t$ and $2+\sin(\log t)$ are not slowly varying, whereas $2+\sin(\log_n t)$, $n \geqslant 2$, are slowly varying.

We now define the class of nearly regularly varying functions which is a useful subclass of RO including all regularly varying functions. To this end it is convenient to introduce the following notation.

Let $f(t)$ and $g(t)$ be two positive continuous functions in a neighborhood of infinity, say for $t \geqslant T$. We use the notation $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants m and M such that

$$m g(t) \leqslant f(t) \leqslant M g(t) \quad \text{for} \ \ t \geqslant T.$$

Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. It is easy to see that if $f(t) \asymp g(t)$, $t \to \infty$, and if $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

DEFINITION 2.2. If $f(t)$ satisfies $f(t) \asymp g(t)$, $t \to \infty$, for some $g(t)$ which is regularly varying of index ρ, then $f(t)$ is called a *nearly regularly varying function of index* ρ.

Since $2+\sin t \asymp 2+\sin(\log_n t)$, $t \to \infty$, for all $n \geqslant 2$, the function $2+\sin t$ is nearly slowly varying, and the same is true of $2+\sin(\log t)$. If $g(t) \in \mathrm{RV}(\rho)$, then the functions $(2+\sin t)g(t)$ and $(2+\sin(\log t))g(t)$ are nearly regularly varying of index ρ, but not regularly varying of index ρ.

A vector function $(f(t), g(t))$ is called regularly varying (or nearly regularly varying) of index (ρ, σ) if $f(t)$ and $g(t)$ are regularly varying (or nearly regularly varying) of indices ρ and σ, respectively.

The reader is referred to Bingham et al [1] for the most complete exposition of theory of regular variation and its applications and to Marić [9] for the comprehensive survey of results up to 2000th on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

3. Strongly decreasing solutions of (A)

One of the main results of this section is the following theorem ensuring the existence of strongly decreasing solutions for (A) in the class of nearly regularly varying vector functions of negative indices.

THEOREM 3.1. *Let λ and μ be constants satisfying the linear system of inequalities*

$$\lambda + 1 + \alpha(\mu+1) < 0, \quad \beta(\lambda+1) + \mu + 1 < 0, \tag{3.1}$$

and define ρ and σ by

$$\rho = \frac{\lambda+1+\alpha(\mu+1)}{1-\alpha\beta}, \quad \sigma = \frac{\beta(\lambda+1)+\mu+1}{1-\alpha\beta}. \tag{3.2}$$

Suppose that $p(t)$ and $q(t)$ are nearly regularly varying functions of indices λ and μ, respectively, such that

$$p(t) \asymp t^{\lambda} l(t), \quad q(t) \asymp t^{\mu} m(t), \quad l(t),\, m(t) \in \mathrm{SV}. \tag{3.3}$$

Then, system (A) *possesses a nearly regularly varying solution $(x(t), y(t))$ of negative index (ρ, σ) such that*

$$x(t) \asymp \left[\frac{t^{1+\alpha} p(t) q(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \asymp \left[\frac{t^{1+\beta} p(t)^{\beta} q(t)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty. \tag{3.4}$$

The proof of this theorem is based on the fact (see Lemma 3.1 below) that complete analysis of strongly decreasing regularly varying solutions of the system of integral asymptotic relations

$$\text{(AR)} \qquad x(t) \sim \int_t^\infty p(s)y(s)^\alpha ds, \quad y(t) \sim \int_t^\infty q(s)x(s)^\beta ds, \quad t \to \infty,$$

can be made provided $p(t)$ and $q(t)$ are regularly varying.

LEMMA 3.1. *Suppose that* $p(t) \in \mathrm{RV}(\lambda)$ *and* $q(t) \in \mathrm{RV}(\mu)$ *are expressed in the form*

$$p(t) = t^\lambda l(t), \quad q(t) = t^\mu m(t), \quad l(t),\, m(t) \in \mathrm{SV}. \tag{3.5}$$

Relation (AR) *possesses regularly varying solutions of index* (ρ,σ) *with* $\rho < 0$ *and* $\sigma < 0$ *if and only if* (λ,μ) *satisfies the system of inequalities* (3.1)*, in which case* ρ *and* σ *are given by* (3.2) *and the asymptotic behavior of any such solution* $(x(t),y(t))$ *of* (AR) *is governed by the unique formula*

$$x(t) \sim \left[\frac{t^{1+\alpha}p(t)q(t)^\alpha}{(-\rho)(-\sigma)^\alpha}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \left[\frac{t^{1+\beta}p(t)^\beta q(t)}{(-\rho)^\beta(-\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty. \tag{3.6}$$

Proof. (The “only if” part) Let $(x(t),y(t))$ be a regularly varying solution of (AR) such that

$$x(t) = t^\rho \xi(t), \quad y(t) = t^\sigma \eta(t), \quad \xi(t),\, \eta(t) \in \mathrm{SV}, \quad \rho < 0,\ \sigma < 0. \tag{3.7}$$

Then, we have

$$x(t) \sim \int_t^\infty s^{\lambda+\alpha\sigma} l(s)\eta(s)^\alpha ds, \quad y(t) \sim \int_t^\infty s^{\mu+\beta\rho} m(s)\xi(s)^\beta ds, \quad t \to \infty. \tag{3.8}$$

The convergence of the above integrals implies that $\lambda + \alpha\sigma \leqslant -1$ and $\mu + \beta\rho \leqslant -1$. If $\lambda + \alpha\sigma = -1$ and $\mu + \beta\rho = -1$, then by Karamata's integration theorem ((iii) of Proposition 2.5)

$$\int_t^\infty s^{-1} l(s)\eta(s)^\alpha ds \in \mathrm{SV}, \quad \int_t^\infty s^{-1} m(s)\xi(s)^\beta ds \in \mathrm{SV},$$

so that neither $x(t)$ nor $y(t)$ can be regularly varying functions of negative index. Therefore, we must have $\lambda + \alpha\sigma < -1$ and $\mu + \beta\rho < -1$, in which case applying Karamata's integration theorem ((ii) of Proposition 2.5) to the integrals in (3.8), we obtain

$$x(t) \sim \frac{t^{\lambda+\alpha\sigma+1} l(t)\eta(t)^\alpha}{-(\lambda+\alpha\sigma+1)}, \quad y(t) \sim \frac{t^{\mu+\beta\rho+1} m(t)\xi(t)^\beta}{-(\mu+\beta\rho+1)}, \quad t \to \infty, \tag{3.9}$$

which shows that $x(t)$ and $y(t)$ must be regularly varying of negatives indices $\lambda + \alpha\sigma + 1$ and $\mu + \beta\rho + 1$, respectively. It follows that

$$\rho = \lambda + \alpha\sigma + 1, \quad \sigma = \mu + \beta\rho + 1,$$

from which we see that ρ and σ are uniquely determined by (3.2). We now rewrite (3.9) as

$$x(t) \sim \frac{t^{\lambda+1}l(t)y(t)^{\alpha}}{-\rho} = \frac{tp(t)y(t)^{\alpha}}{-\rho}, \quad y(t) \sim \frac{t^{\mu+1}m(t)x(t)^{\beta}}{-\sigma} = \frac{tq(t)x(t)^{\beta}}{-\sigma}, \quad t \to \infty,$$

from which it follows that

$$x(t) \sim \frac{t^{1+\alpha}p(t)q(t)^{\alpha}x(t)^{\alpha\beta}}{(-\rho)(-\sigma)^{\alpha}}, \quad y(t) \sim \frac{t^{1+\beta}p(t)^{\beta}q(t)y(t)^{\alpha\beta}}{(-\rho)^{\beta}(-\sigma)}, \quad t \to \infty.$$

This immediately yields the asymptotic formulas (3.6) for $x(t)$ and $y(t)$.

(The "if" part) Suppose that (λ,μ) satisfies (3.1), define (ρ,σ) by (3.2) and consider the function $(X(t),Y(t))$ given by

$$X(t) = \left[\frac{t^{1+\alpha}p(t)q(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = \left[\frac{t^{1+\beta}p(t)^{\beta}q(t)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{1}{1-\alpha\beta}}, \tag{3.10}$$

It is convenient to notice that $X(t)$ and $Y(t)$ are rewritten as

$$X(t) = t^{\rho}\left[\frac{l(t)m(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = t^{\sigma}\left[\frac{l(t)^{\beta}m(t)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{1}{1-\alpha\beta}}.$$

Using these expressions and applying Karamata's integration theorem, we obtain

$$\begin{aligned}\int_t^{\infty} p(s)Y(s)^{\alpha}ds &= \int_t^{\infty} s^{\lambda+\alpha\sigma}l(s)\left[\frac{l(s)^{\beta}m(s)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{\alpha}{1-\alpha\beta}}ds \\ &= \int_t^{\infty} s^{\rho-1}l(s)\left[\frac{l(s)^{\beta}m(s)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{\alpha}{1-\alpha\beta}}ds \\ &\sim \frac{t^{\rho}l(t)}{(-\rho)}\left[\frac{l(t)^{\beta}m(t)}{(-\rho)^{\beta}(-\sigma)}\right]^{\frac{\alpha}{1-\alpha\beta}} = X(t),\end{aligned}$$

as $t \to \infty$, and similarly,

$$\int_t^{\infty} q(s)X(s)^{\beta}ds \sim \frac{t^{\sigma}m(t)}{(-\sigma)}\left[\frac{l(t)m(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\right]^{\frac{\beta}{1-\alpha\beta}} = Y(t), \quad t \to \infty.$$

Thus, we conclude that $(X(t),Y(t))$ satisfies

$$\int_t^{\infty} p(s)Y(s)^{\alpha}ds \sim X(t), \quad \int_t^{\infty} q(s)X(s)^{\beta}ds \sim Y(t), \quad t \to \infty,$$

that is, provides a regularly varying solution of index (ρ,σ) of the asymptotic relation (AR). This completes the proof of Lemma 3.1.

Proof of Theorem 3.1. Let (λ,μ) satisfy (3.1) and define (ρ,σ) by (3.2). Let $p_{\lambda}(t)$ and $q_{\mu}(t)$ denote the functions

$$p_{\lambda}(t) = t^{\lambda}l(t) \in \mathrm{RV}(\lambda), \quad q_{\mu}(t) = t^{\mu}m(t) \in \mathrm{RV}(\mu), \tag{3.11}$$

where $l(t)$ and $m(t)$ are SV-functions appearing in (3.3). By (3.3) there exist positive constants k, l, K and L such that

$$kp_\lambda(t) \leqslant p(t) \leqslant Kp_\lambda(t), \quad lq_\mu(t) \leqslant q(t) \leqslant Lq_\mu(t), \quad t \geqslant a. \tag{3.12}$$

Define the function $(X_\lambda(t), Y_\mu(t))$ by the formulas (3.10) with $p(t)$ and $q(t)$ replaced by $p_\lambda(t)$ and $q_\mu(t)$, respectively. Since $(X_\lambda(t), Y_\mu(t))$ satisfies (AR) by Lemma 3.1, there exists $T > a$ such that

$$\begin{cases} \frac{1}{2}X_\lambda(t) \leqslant \int_t^\infty p_\lambda(s) Y_\mu(s)^\alpha ds \leqslant 2X_\lambda(t), \\ \frac{1}{2}Y_\mu(t) \leqslant \int_t^\infty q_\mu(s) X_\lambda(s)^\beta ds \leqslant 2Y_\mu(t), \end{cases} \tag{3.13}$$

for $t \geqslant T$. We may assume that $X_\lambda(t)$ and $Y_\mu(t)$ are decreasing for $t \geqslant T$. Let us now choose $(a,b), (A,B) \in \mathbb{R}^2$ so that $a < A$, $b < B$ and

$$a \leqslant \frac{1}{2}kb^\alpha, \quad b \leqslant \frac{1}{2}la^\beta, \quad 2KB^\alpha \leqslant A, \quad 2LA^\beta \leqslant B. \tag{3.14}$$

It is elementary to see that such a choice of $(a,b), (A,B)$ is really possible. For example, one can choose as follows:

$$a = (2^{-(1+\alpha)} l^\alpha k)^{\frac{1}{1-\alpha\beta}}, \quad b = (2^{-(1+\beta)} lk^\beta)^{\frac{1}{1-\alpha\beta}},$$

$$A = (2^{1+\alpha} L^\alpha K)^{\frac{1}{1-\alpha\beta}}, \quad B = (2^{1+\beta} LK^\beta)^{\frac{1}{1-\alpha\beta}}.$$

We define $\mathscr{X}$ to be the subset of $C[T,\infty) \times C[T,\infty)$ consisting of vector functions $(x(t), y(t))$ satisfying

$$aX_\lambda(t) \leqslant x(t) \leqslant AX_\lambda(t), \quad bY_\mu(t) \leqslant y(t) \leqslant BY_\mu(t), \quad t \geqslant T. \tag{3.15}$$

Clearly, $\mathscr{X}$ is a closed convex subset of $C[T,\infty) \times C[T,\infty)$. Furthermore, define the mapping $\Phi : \mathscr{X} \to C[T,\infty) \times C[T,\infty)$ by

$$\Phi(x(t), y(t)) = (\mathscr{F}y(t), \mathscr{G}x(t)), \quad t \geqslant T, \tag{3.16}$$

where $\mathscr{F}$ and $\mathscr{G}$ denote the integral operators

$$\mathscr{F}y(t) = \int_t^\infty p(s) y(s)^\alpha ds, \quad \mathscr{G}x(t) = \int_t^\infty q(s) x(s)^\beta ds, \quad t \geqslant T. \tag{3.17}$$

It can be shown that Φ is a continuous self-map of $\mathscr{X}$ which sends $\mathscr{X}$ into a relatively compact subset of $C[T,\infty) \times C[T,\infty)$.

(i) $\Phi(\mathscr{X}) \subset \mathscr{X}$. If $(x(t), y(t)) \in \mathscr{X}$, then using (3.12) - (3.15), we see that

$$\mathscr{F}y(t) \geqslant \int_t^\infty kp_\lambda(s)(bY_\mu(s))^\alpha ds \geqslant \frac{1}{2}kb^\alpha X_\lambda(t) \geqslant aX_\lambda(t),$$

$$\mathscr{F}y(t) \leqslant \int_t^\infty Kp_\lambda(s)(BY_\mu(s))^\alpha ds \leqslant 2KB^\alpha X_\lambda(t) \leqslant AX_\lambda(t),$$

$$\mathscr{G}x(t) \geqslant \int_t^\infty lq_\mu(s)(aX_\lambda(s))^\beta ds \geqslant \frac{1}{2}la^\beta Y_\mu(t) \geqslant bY_\mu(t),$$

$$\mathscr{G}x(t) \leqslant \int_t^\infty Lq_\mu(s)(AX_\lambda(s))^\beta ds \leqslant 2LA^\beta Y_\mu(t) \leqslant BY_\mu(t),$$

for $t \geqslant T$, which implies that $(\mathscr{F}y(t), \mathscr{G}x(t)) \in \mathscr{X}$.

(ii) $\Phi(\mathscr{X})$ is relatively compact. The inclusion $\Phi(\mathscr{X}) \subset \mathscr{X}$ shows that $\Phi(\mathscr{X})$ is uniformly bounded on $[T,\infty)$. The inequalities

$$0 \geqslant (\mathscr{F}y)'(t) \geqslant -B^\alpha p(t) Y_\mu(t)^\alpha, \quad 0 \geqslant (\mathscr{G}x)'(t) \geqslant -A^\beta q(t) X_\lambda(t)^\beta, \quad t \geqslant T,$$

holding for all $(x(t),y(t)) \in \mathscr{X}$ ensure that $\Phi(\mathscr{X})$ is equicontinuous on $[T,\infty)$. The relative compactness of $\Phi(\mathscr{X})$ then follows from the Arzela-Ascoli lemma (see [2], pp. 7-8).

(iii) Φ is continuous. Let $\{(x_n(t), y_n(t))\}$ be a sequence in $\mathscr{X}$ converging to $(x(t),y(t)) \in \mathscr{X}$ as $n \to \infty$ uniformly on any compact subinterval of $[T,\infty)$. We have to prove that $\Phi(x_n(t),y_n(t)) \to \Phi(x(t),y(t))$, that is,

$$\mathscr{F}y_n(t) \to \mathscr{F}y(t), \quad \mathscr{G}x_n(t) \to \mathscr{G}x(t) \quad \text{as } n \to \infty, \tag{3.18}$$

uniformly on compact subintervals of $[T,\infty)$. But this follows immediately from the Lebesgue dominated convergence theorem applied to the integrals in the inequalities

$$|\mathscr{F}y_n(t) - \mathscr{F}y(t)| \leqslant \int_t^\infty p(s)|y_n(s)^\alpha - y(s)^\alpha| ds,$$

$$|\mathscr{G}x_n(t) - \mathscr{G}x(t)| \leqslant \int_t^\infty q(s)|x_n(s)^\beta - x(s)^\beta| ds.$$

This establishes the continuity of Φ.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled for Φ, and so there exists $(x(t),y(t)) \in \mathscr{X}$ such that $\Phi(x(t),y(t)) = (x(t),y(t))$, $t \geqslant T$, that is,

$$x(t) = \mathscr{F}y(t) = \int_t^\infty p(s)y(s)^\alpha ds, \quad y(t) = \mathscr{G}x(t) = \int_t^\infty q(s)x(s)^\beta ds, \quad t \geqslant T,$$

which implies that $(x(t),y(t))$ gives a strongly decreasing solution of system (A). Since $(x(t),y(t))$ is a member of $\mathscr{X}$, it becomes nearly regularly varying of negative index (ρ,σ). This completes the proof of Theorem 3.1. □

As for the solutions constructed in Theorem 3.1, their regularity can be characterized completely under the stronger assumption that $p(t)$ and $q(t)$ are regularly varying functions.

The generalized L'Hospital's rule given in the following lemma (see [3]) plays a crucial role in the proof of Theorem 3.2 below.

LEMMA 3.2. *Let $f(t), g(t) \in \mathrm{C}^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \textit{and } g'(t) > 0 \textit{ for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and } g'(t) < 0 \textit{ for all large } t.$$

Then,

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leqslant \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \quad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leqslant \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

THEOREM 3.2. *Suppose that $p(t)$ and $q(t)$ are regularly varying of indices λ and μ, respectively. System* (A) *possesses regularly varying solutions $(x(t),y(t))$ such that*

$$x(t) \in \mathrm{RV}(\rho), \quad y(t) \in \mathrm{RV}(\sigma), \quad \rho < 0, \quad \sigma < 0$$

if and only if (3.1) *holds, in which case ρ and σ are given by* (3.2) *and the asymptotic behavior of any such solution $(x(t),y(t))$ is governed by the formulas*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^{\beta}q(t)}{(-\rho)^{\beta}(-\sigma)}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t\to\infty. \tag{3.19}$$

Proof of Theorem 3.2.

(The "only if" part) This follows from Lemma 3.1.

(The "if" part) Suppose that (3.1) holds and define the negative constants ρ and σ by (3.2). By Theorem 3.1 system (A) has a nearly regularly varying solution $(x(t),y(t))$ on $[T,\infty)$ such that

$$aX(t) \leqslant x(t) \leqslant AX(t), \quad bY(t) \leqslant y(t) \leqslant BY(t), \quad t \geqslant T, \tag{3.20}$$

for some positive constants T,a,A,b and B, where

$$X(t) = \Big[\frac{t^{\alpha+1}p(t)q(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}} \in \mathrm{RV}(\rho), \quad Y(t) = \Big[\frac{t^{\beta+1}p(t)^{\beta}q(t)}{(-\rho)^{\beta}(-\sigma)}\Big]^{\frac{1}{1-\alpha\beta}} \in \mathrm{RV}(\sigma). \tag{3.21}$$

It is clear that $(x(t),y(t))$ satisfies

$$x(t) = \int_t^{\infty} p(s)y(s)^{\alpha}ds, \quad y(t) = \int_t^{\infty} q(s)x(s)^{\beta}ds, \quad t \geqslant T. \tag{3.22}$$

Let $U(t)$ and $V(t)$ denote the functions defined by

$$U(t) = \int_t^{\infty} p(s)Y(s)^{\alpha}ds, \quad V(t) = \int_t^{\infty} q(s)X(s)^{\beta}ds, \quad t \geqslant a. \tag{3.23}$$

Note that $U(t)$ and $V(t)$ satisfy the asymptotic relations

$$U(t) \sim X(t), \quad V(t) \sim Y(t), \quad t\to\infty. \tag{3.24}$$

Put

$$k = \liminf_{t\to\infty} \frac{x(t)}{U(t)}, \quad K = \limsup_{t\to\infty} \frac{x(t)}{U(t)}, \quad l = \liminf_{t\to\infty} \frac{y(t)}{V(t)}, \quad L = \limsup_{t\to\infty} \frac{y(t)}{V(t)}. \tag{3.25}$$

From (3.20) and (3.22) we see that $0 < k \leqslant K < \infty$ and $0 < l \leqslant L < \infty$. Applying the generalized L'Hospital rule repeatedly, we obtain

$$\begin{aligned} k &= \liminf_{t\to\infty} \frac{x(t)}{U(t)} \geqslant \liminf_{t\to\infty} \frac{x'(t)}{U'(t)} = \liminf_{t\to\infty} \frac{p(t)y(t)^{\alpha}}{p(t)Y(t)^{\alpha}} \\ &= \liminf_{t\to\infty} \left(\frac{y(t)}{Y(t)}\right)^{\alpha} = \liminf_{t\to\infty} \left(\frac{y(t)}{V(t)}\right)^{\alpha} = \left(\liminf_{t\to\infty} \frac{y(t)}{V(t)}\right)^{\alpha} = l^{\alpha}, \end{aligned}$$

and

$$\begin{aligned} l &= \liminf_{t\to\infty} \frac{y(t)}{V(t)} \geqslant \liminf_{t\to\infty} \frac{y'(t)}{V'(t)} = \liminf_{t\to\infty} \frac{q(t)x(t)^{\beta}}{q(t)X(t)^{\beta}} \\ &= \liminf_{t\to\infty} \left(\frac{x(t)}{X(t)}\right)^{\beta} = \liminf_{t\to\infty} \left(\frac{x(t)}{U(t)}\right)^{\beta} = \left(\liminf_{t\to\infty} \frac{x(t)}{U(t)}\right)^{\beta} = k^{\beta}, \end{aligned}$$

where (3.24) has been used in the final step of each of the above computations. Since $\alpha\beta < 1$, the inequalities $k \geqslant l^{\alpha}$ and $l \geqslant k^{\beta}$ thus obtained imply

$$1 \leqslant k < \infty, \quad 1 \leqslant l < \infty. \tag{3.26}$$

Similarly, we obtain $K \leqslant L^{\alpha}$ and $L \leqslant K^{\beta}$, from which it follows that

$$0 < K \leqslant 1, \quad 0 < L \leqslant 1. \tag{3.27}$$

From (3.26) and (3.27) we conclude that $k = K = 1$ and $l = L = 1$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{U(t)} = 1, \quad \lim_{t\to\infty} \frac{y(t)}{V(t)} = 1,$$

which combined with (3.24) shows that

$$x(t) \sim U(t) \sim X(t), \quad y(t) \sim V(t) \sim Y(t), \quad t \to \infty.$$

This completes the proof. □

EXAMPLE 3.1. Consider system (A) with

$$p(t) \asymp 2t^{\alpha-3}(\log t)^{\alpha+1}, \quad q(t) \asymp t^{2(\beta-1)}(\log t)^{-(\beta+1)}, \quad t \to \infty.$$

This means that (3.3) holds with

$$\lambda = \alpha - 3, \quad \mu = 2(\beta - 1), \quad l(t) = 2(\log t)^{\alpha+1}, \quad m(t) = (\log t)^{-(\beta+1)}.$$

Since $\lambda + 1 + \alpha(\mu + 1) = -2(1 - \alpha\beta) < 0$ and $\beta(\lambda + 1) + \mu + 1 = 1 - \alpha\beta < 0$, (3.2) determines the constants ρ and σ to be $\rho = -2$ and $\sigma = -1$, and one finds that

$$\frac{l(t)m(t)^{\alpha}}{(-\rho)(-\sigma)^{\alpha}} = (\log t)^{1-\alpha\beta}, \quad \frac{l(t)^{\beta}m(t)}{(-\rho)^{\beta}(-\sigma)} = (\log t)^{\alpha\beta-1}.$$

Therefore, by Theorem 3.1 the system (A) possesses a strongly decreasing solution $(x(t), y(t))$ such that

$$x(t) \asymp t^{-2}\log t, \quad y(t) \asymp (t\log t)^{-1}, \quad t \to \infty.$$

Assume more strongly that $p(t)$ and $q(t)$ are regularly varying functions such that

$$p(t) \sim 2t^{\alpha-3}(\log t)^{\alpha+1}, \quad q(t) \sim t^{2(\beta-1)}(\log t)^{-(\beta+1)}, \quad t \to \infty.$$

Then, applying Theorem 3.2 we conclude that system (A) possesses strongly decreasing solutions of index (-2,-1) all of which enjoy the unique asymptotic formulas

$$x(t) \sim t^{-2}\log t, \quad y(t) \sim (t\log t)^{-1}, \quad t \to \infty.$$

If in particular

$$p(t) = 2t^{\alpha-3}(\log t)^{\alpha+1}\left(1 - \frac{1}{2\log t}\right), \quad q(t) = t^{2(\beta-1)}(\log t)^{-(\beta+1)}\left(1 + \frac{1}{\log t}\right),$$

then the system (A) has an exact strongly decreasing solution $(t^{-2}\log t, (t\log t)^{-1})$.

4. Strongly increasing solutions of (B)

We turn our attention to the study of strongly increasing solutions of system (B) satisfying conditions (a) and (b). One of our main results here is the following theorem which enables us to find the desired solutions in the class of nearly regularly varying solutions of positive indices.

THEOREM 4.1. *Let λ and μ be constants satisfying the linear system of inequalities*

$$\lambda + 1 + \alpha(\mu + 1) > 0, \quad \beta(\lambda + 1) + \mu + 1 > 0, \tag{4.1}$$

and define $\rho > 0$ and $\sigma > 0$ by (3.2). *Suppose that $p(t)$ and $q(t)$ are nearly regularly varying functions of indices λ and μ, respectively, given by* (3.3). *Then, system* (B) *possesses a nearly regularly varying solution $(x(t), y(t))$ of positive index (ρ, σ) such that*

$$x(t) \asymp \left[\frac{t^{1+\alpha}p(t)q(t)^{\alpha}}{\rho\sigma^{\alpha}}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \asymp \left[\frac{t^{1+\beta}p(t)^{\beta}q(t)}{\rho^{\beta}\sigma}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty. \tag{4.2}$$

We notice that a strongly increasing solution $(x(t), y(t))$ of (B), if exists on $[T, \infty)$, satisfies the following system of integral asymptotic relations

$$x(t) \sim \int_T^t p(s)y(s)^{\alpha}ds, \quad y(t) \sim \int_T^t q(s)x(s)^{\beta}ds, \quad t \to \infty. \tag{BR}$$

The proof of Theorem 4.1 heavily depends on the fact that accurate information can be acquired about regularly varying solutions of (BR) provided $p(t)$ and $q(t)$ are regularly varying.

LEMMA 4.1. *Suppose that* $p(t) \in \mathrm{RV}(\lambda)$ *and* $q(t) \in \mathrm{RV}(\mu)$ *are expressed in the form* (3.5). *Relation* (BR) *possesses regularly varying solutions of index* (ρ,σ) *with* $\rho > 0$ *and* $\sigma > 0$ *if and only if* (λ,μ) *satisfies the system of inequalities* (4.1), *in which case* ρ *and* σ *are given by* (3.2) *and the asymptotic behavior of any such solution* $(x(t),y(t))$ *of* (BR) *is governed by the unique formula*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^{\alpha}}{\rho\sigma^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^{\beta}q(t)}{\rho^{\beta}\sigma}\Big]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty. \tag{4.3}$$

Proof of lemma 4.1. (The "only if" part) Suppose that (BR) has a regularly varying solution $(x(t),y(t))$ of positive index (ρ,σ) which exists on $[T,\infty)$ and is expressed in the form (3.7). Then,

$$x(t) \sim \int_T^t s^{\lambda+\alpha\sigma}l(s)\eta(s)^{\alpha}ds, \quad y(t) \sim \int_T^t s^{\mu+\beta\rho}m(s)\xi(s)^{\beta}ds, \quad t \to \infty. \tag{4.4}$$

The divergence of the above integrals as $t \to \infty$ implies that $\lambda + \alpha\sigma \geqslant -1$ and $\mu + \beta\rho \geqslant -1$. The possibilities $\lambda + \alpha\sigma = -1$ and $\mu + \beta\rho = -1$ should be excluded because

$$\int_T^t s^{-1}l(s)\eta(s)^{\alpha}ds \in \mathrm{SV}, \quad \int_T^t s^{-1}m(s)\xi(s)^{\beta}ds \in \mathrm{SV},$$

by (iii) of Proposition 2.5. Therefore, we must have $\lambda + \alpha\sigma > -1$ and $\mu + \beta\rho > -1$, in which case from Karamata's integration theorem ((i) of Proposition 2.5) applied to the integrals in (4.4) we obtain

$$x(t) \sim \frac{t^{\lambda+\alpha\sigma+1}l(t)\eta(t)^{\alpha}}{\lambda+\alpha\sigma+1}, \quad y(t) \sim \frac{t^{\mu+\beta\rho+1}l(t)\xi(t)^{\beta}}{\mu+\beta\rho+1}, \quad t \to \infty. \tag{4.5}$$

which shows that $x(t)$ and $y(t)$ are regularly varying of positive indices $\lambda + \alpha\sigma + 1$ and $\mu + \beta\rho + 1$, respectively. Consequently, we must have $\rho = \lambda + \alpha\sigma + 1$ and $\sigma = \mu + \beta\rho + 1$, from which it follows that the positive constants ρ and σ are uniquely determined by (3.2). We note that (4.5) can be rewritten as

$$x(t) \sim \frac{t^{\lambda+1}l(t)y(t)^{\alpha}}{\rho} = \frac{tp(t)y(t)^{\alpha}}{\rho},$$
$$y(t) \sim \frac{t^{\mu+1}m(t)x(t)^{\beta}}{\sigma} = \frac{tq(t)x(t)^{\beta}}{\rho}, \quad t \to \infty,$$

which implies

$$x(t) \sim \frac{t^{1+\alpha}p(t)q(t)^{\alpha}x(t)^{\alpha\beta}}{\rho\sigma^{\alpha}}, \quad y(t) \sim \frac{t^{1+\beta}p(t)^{\beta}q(t)y(t)^{\alpha\beta}}{\rho^{\beta}\sigma}, \quad t \to \infty.$$

Clearly, this leads to the asymptotic formula (4.3) for $(x(t),y(t))$.

(The "if" part) Let (λ,μ) satisfy (4.1) and define (ρ,σ) by (3.2). Consider the regularly varying function $(X(t),Y(t))$ defined by

$$X(t) = \Big[\frac{t^{1+\alpha}p(t)q(t)^{\alpha}}{\rho\sigma^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = \Big[\frac{t^{1+\beta}p(t)^{\beta}q(t)}{\rho^{\beta}\sigma}\Big]^{\frac{1}{1-\alpha\beta}}. \tag{4.6}$$

It can be verified that $(X(t),Y(t))$ satisfies the system of asymptotic relations (BR) for any $T \geqslant a$. In fact, rewriting (4.6) as

$$X(t) = t^{\rho}\Big[\frac{l(t)m(t)^{\alpha}}{\rho\sigma^{\alpha}}\Big]^{\frac{1}{1-\alpha\beta}}, \quad Y(t) = t^{\sigma}\Big[\frac{l(t)^{\beta}m(t)}{\rho^{\beta}\sigma}\Big]^{\frac{1}{1-\alpha\beta}},$$

we compute via Karamata's integration theorem as follows:

$$\begin{aligned}\int_T^t p(s)Y(s)^{\alpha}ds &= \int_T^t s^{\lambda+\alpha\sigma}l(s)\Big[\frac{l(s)^{\beta}m(s)}{\rho^{\beta}\sigma}\Big]^{\frac{\alpha}{1-\alpha\beta}}ds \\ &= \int_T^t s^{\rho-1}l(s)\Big[\frac{l(s)^{\beta}m(s)}{\rho^{\beta}\sigma}\Big]^{\frac{\alpha}{1-\alpha\beta}}ds \\ &\sim \frac{t^{\rho}l(t)}{\rho}\Big[\frac{l(t)^{\beta}m(t)}{\rho^{\beta}\sigma}\Big]^{\frac{\alpha}{1-\alpha\beta}} \\ &= X(t),\end{aligned}$$

as $t \to \infty$, and similarly

$$\int_T^t q(s)X(s)^{\beta}ds \sim \frac{t^{\sigma}m(t)}{\sigma}\Big[\frac{l(t)m(t)^{\alpha}}{\rho\sigma^{\alpha}}\Big]^{\frac{\beta}{1-\alpha\beta}} = Y(t), \quad t \to \infty.$$

This completes the proof of Lemma 4.1. □

Proof of Theorem 4.1. Let (λ,μ) satisfy (4.1) and define (ρ,σ) by (3.2). Define $p_{\lambda}(t)$ and $q_{\mu}(t)$ by (3.11), which satisfy (3.12) for some positive constants k,K,l,L, and let $X_{\lambda}(t)$ and $Y_{\mu}(t)$ denote the functions defined by (4.6) with $p(t)$ and $q(t)$ replaced by $p_{\lambda}(t)$ and $q_{\mu}(t)$, respectively. Since $(X_{\lambda}(t),Y_{\mu}(t))$ satisfies relation (BR), there exists $T_0 \geqslant a$ such that

$$\int_{T_0}^t p_{\lambda}(s)Y_{\mu}(s)^{\alpha}ds \leqslant 2X_{\lambda}(t), \quad \int_{T_0}^t q_{\mu}(s)X_{\lambda}(s)^{\beta}ds \leqslant 2Y_{\mu}(t), \quad t \geqslant T_0. \tag{4.7}$$

We may assume that $X_{\lambda}(t)$ and $Y_{\mu}(t)$ are increasing for $t \geqslant T_0$. Using (BR) again, we see that there exists $T_1 > T_0$ such that

$$\int_{T_0}^t p_{\lambda}(s)Y_{\mu}(s)^{\alpha}ds \geqslant \frac{1}{2}X_{\lambda}(t), \quad \int_{T_0}^t q_{\mu}(s)X_{\lambda}(s)^{\beta}ds \geqslant \frac{1}{2}Y_{\mu}(t), \quad t \geqslant T_1. \tag{4.8}$$

We now choose $(a,b),(A,B) \in \mathbb{R}^2$ so that $a < A$, $b < B$,

$$a \leqslant \frac{1}{2}kb^{\alpha}, \quad b \leqslant \frac{1}{2}la^{\beta}, \quad 4KB^{\alpha} \leqslant A, \quad 4LA^{\beta} \leqslant B, \tag{4.9}$$

and

$$aX_{\lambda}(T_1) \leqslant \frac{1}{2}AX_{\lambda}(T_0), \quad bY_{\mu}(T_1) \leqslant \frac{1}{2}BY_{\mu}(T_0). \tag{4.10}$$

It is easy to check that (4.9) and (4.10) are consistent. Let $\mathscr{X}$ be defined to be the set of continuous vector functions $(x(t),y(t))$ on $[T_0,\infty)$ such that

$$aX_{\lambda}(t) \leqslant x(t) \leqslant AX_{\lambda}(t), \quad bY_{\mu}(t) \leqslant y(t) \leqslant BY_{\mu}(t), \quad t \geqslant T_0, \tag{4.11}$$

and consider the mapping $\Phi : \mathscr{X} \to C[T_0,\infty)\times C[T_0,\infty)$ given by

$$\Phi(x(t),y(t)) = (\mathscr{F}y(t),\mathscr{G}x(t)), \quad t \geqslant T_0, \tag{4.12}$$

where $\mathscr{F}$ and $\mathscr{G}$ stand for the integral operators

$$\mathscr{F}y(t) = x_0 + \int_{T_0}^{t} p(s)y(s)^{\alpha}ds, \quad \mathscr{G}x(t) = y_0 + \int_{T_0}^{t} q(s)x(s)^{\beta}ds, \quad t \geqslant T_0, \tag{4.13}$$

where x_0 and y_0 are positive constants satisfying

$$aX_\lambda(T_1) \leqslant x_0 \leqslant \frac{1}{2}AX_\lambda(T_0), \quad bY_\mu(T_1) \leqslant y_0 \leqslant \frac{1}{2}BY_\mu(T_0). \tag{4.14}$$

It is proved without difficulty that Φ is a continuous self-map of $\mathscr{X}$ with the property that $\Phi(\mathscr{X})$ is relatively compact in $C[T_0,\infty)\times C[T_0,\infty)$. Let $(x(t),y(t)) \in \mathscr{X}$. Using (4.7) - (4.11), we see that

$$\mathscr{F}y(t) \geqslant x_0 \geqslant aX_\lambda(T_1) \geqslant aX_\lambda(t), \quad T_0 \leqslant t \leqslant T_1,$$

and

$$\begin{aligned}\mathscr{F}y(t) \geqslant \int_{T_0}^{t} p(s)y(s)^{\alpha}ds &\geqslant \int_{T_0}^{t} kp_\lambda(s)(bY_\mu(s))^{\alpha}ds \\ &\geqslant \frac{1}{2}kb^{\alpha}X_\lambda(t) \geqslant aX_\lambda(t), \quad t \geqslant T_1.\end{aligned}$$

On the other hand, we have for $t \geqslant T_0$

$$\begin{aligned}\mathscr{F}y(t) &\leqslant \frac{1}{2}AX_\lambda(T_0) + \int_{T_0}^{t} Kp_\lambda(s)(BY_\mu(s))^{\alpha}ds \\ &\leqslant \frac{1}{2}AX_\lambda(t) + 2KB^{\alpha}X_\lambda(t) \\ &\leqslant \frac{1}{2}AX_\lambda(t) + \frac{1}{2}AX_\lambda(t) = AX_\lambda(t).\end{aligned}$$

This implies that $aX_\lambda(t) \leqslant \mathscr{F}y(t) \leqslant AX_\lambda(t)$ for $t \geqslant T_0$. And entirely analogous computations apply to $\mathscr{G}$, showing that $bY_\mu(t) \leqslant \mathscr{G}x(t) \leqslant BY_\mu(t)$ for $t \geqslant T_0$. It follows that $\Phi(x(t),y(t)) \in \mathscr{X}$. The relative compactness of $\Phi(\mathscr{X})$ follows from the inclusion $\Phi(\mathscr{X}) \subset \mathscr{X}$ and the inequalities

$$0 \leqslant (\mathscr{F}y)'(t) \leqslant B^{\alpha}p(t)Y_\mu(t)^{\alpha}, \quad 0 \leqslant (\mathscr{G}x)'(t) \leqslant A^{\beta}q(t)X_\lambda(t)^{\beta}, \quad t \geqslant T_0,$$

holding for all $(x(t),y(t)) \in \mathscr{X}$. To confirm the continuity of Φ it suffices to consider any sequence $\{(x_n(t),y_n(t))\}$ in $\mathscr{X}$ converging to $(x(t),y(t)) \in \mathscr{X}$ uniformly on compact subintervals of $[T_0,\infty)$ and verify that $\mathscr{F}y_n(t) \to \mathscr{F}y(t)$ and $\mathscr{G}x_n(t) \to \mathscr{G}x(t)$ uniformly on any compact subinterval of $[T_0,\infty)$ by applying the Lebesgue dominated convergence theorem to the following integrals

$$|\mathscr{F}y_n(t) - \mathscr{F}y(t)| \leqslant \int_{T_0}^{t} p(s)|y_n(s)^{\alpha} - y(s)^{\alpha}|ds,$$

$$|\mathscr{G}x_n(t) - \mathscr{G}x(t)| \leqslant \int_{T_0}^{t} q(s)|x_n(s)^\beta - x(s)^\beta| ds, \quad t \geqslant T_0.$$

Consequently by the Schauder-Tychonoff fixed point theorem Φ has a fixed point $(x(t), y(t)) \in \mathscr{X}$, which satisfies the system of integral equations

$$x(t) = x_0 + \int_{T_0}^{t} p(s)y(s)^\alpha ds, \quad y(t) = y_0 + \int_{T_0}^{t} q(s)x(s)^\beta ds, \quad t \geqslant T_0.$$

It follows therefore that $(x(t), y(t))$ provides a strongly increasing solution of system (B) which is nearly regularly varying of index (ρ, σ). This completes the proof of Theorem 4.1. $\square$

As the next theorem shows, under the stronger assumption that $p(t)$ and $q(t)$ are regularly varying functions, the full regularity of the solutions obtained in Theorem 4.1 can be proved via the generalized L'Hospital rule (Lemma 3.2), so that the existence of regularly varying solutions with positive indices is characterized completely in this particular case. The proof is similar to that of Theorem 3.2 and we omit it.

THEOREM 4.2. *Suppose that $p(t)$ and $q(t)$ are regularly varying of indices λ and μ, respectively. System* (B) *possesses regularly varying solutions $(x(t), y(t))$ such that*

$$x(t) \in \mathrm{RV}(\rho), \quad y(t) \in \mathrm{RV}(\sigma), \quad \rho > 0,\ \sigma > 0,$$

if and only if (4.1) *holds, in which case ρ and σ are given by* (3.2) *and the asymptotic behavior of any such solution is governed by the formulas*

$$x(t) \sim \left[\frac{t^{\alpha+1} p(t) q(t)^\alpha}{\rho \sigma^\alpha}\right]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \left[\frac{t^{\beta+1} p(t)^\beta q(t)}{\rho^\beta \sigma}\right]^{\frac{1}{1-\alpha\beta}}, \quad t \to \infty. \tag{4.15}$$

EXAMPLE 4.1. Consider system (B) with

$$p(t) \asymp 2t^{1-3\alpha} \exp\big((1+\alpha)\sqrt{\log t}\big),$$
$$q(t) \asymp 3t^{2(1-\beta)} \exp\big(-(1+\beta)\sqrt{\log t}\big), \quad t \to \infty.$$

This means that (3.3) holds with $\lambda = 1 - 3\alpha$, $\mu = 2(1-\beta)$,

$$l(t) = 2\exp\big((1+\alpha)\sqrt{\log t}\big) \quad \text{and} \quad m(t) = 3\exp\big(-(1+\beta)\sqrt{\log t}\big).$$

Since

$$\lambda + 1 + \alpha(\mu + 1) = 2(1 - \alpha\beta) > 0 \quad \text{and} \quad \beta(\lambda + 1) + \mu + 1 = 3(1 - \alpha\beta) > 0,$$

(3.2) determines the constants $\rho = 2$ and $\sigma = 3$, and we have

$$\frac{l(t)m(t)^\alpha}{\rho\sigma^\alpha} = \exp\big((1-\alpha\beta)\sqrt{\log t}\big), \quad \frac{l(t)^\beta m(t)}{\rho^\beta \sigma} = \exp\big((\alpha\beta - 1)\sqrt{\log t}\big).$$

Therefore, by Theorem 4.1 this system (B) possesses a strongly increasing solution $(x(t),y(t))$ such that

$$x(t) \asymp t^2 \exp\big(\sqrt{\log t}\big), \quad y(t) \asymp t^3 \exp\big(-\sqrt{\log t}\big), \quad t \to \infty.$$

If $p(t)$ and $q(t)$ are regularly varying functions such that

$$p(t) \sim 2t^{1-3\alpha} \exp\big((1+\alpha)\sqrt{\log t}\big),$$
$$q(t) \sim 3t^{2(1-\beta)} \exp\big(-(1+\beta)\sqrt{\log t}\big), \quad t \to \infty,$$

then by Theorem 4.2 system (B) possesses strongly increasing solutions which are regularly varying of index (2,3) all of which enjoy the unique asymptotic formulas

$$x(t) \sim t^2 \exp\big(\sqrt{\log t}\big), \quad y(t) \sim t^3 \exp\big(-\sqrt{\log t}\big), \quad t \to \infty.$$

If in particular

$$p(t) = 2t^{1-3\alpha} \exp\big((1+\alpha)\sqrt{\log t}\big)\left(1 + \frac{1}{4\sqrt{\log t}}\right),$$
$$q(t) = 3t^{2(1-\beta)} \exp\big(-(1+\beta)\sqrt{\log t}\big)\left(1 - \frac{1}{6\sqrt{\log t}}\right),$$

then the system (B) has the following exact strongly increasing solution

$$\big(t^2 \exp\big(\sqrt{\log t}\big), t^3 \exp\big(-\sqrt{\log t}\big)\big).$$

5. Application to generalized Thomas-Fermi equations

We conclude this paper with a remark that our main results for systems (A) and (B) can be used to produce new results on the existence and precise asymptotic behavior of strongly monotone regularly varying solutions for the generalized Thomas-Fermi differential equation

$$(p(t)|x'|^{\alpha-1}x')' = q(t)|x|^{\beta-1}x, \tag{5.1}$$

where α, β are positive constants such that $\alpha > \beta$ and $p(t)$, $q(t)$ are positive continuous functions on $[a,\infty)$. An important feature of our results is that we do not need to distinguish the two cases

$$\text{(I)} \ \int_a^\infty p(t)^{-1/\alpha} dt = \infty, \qquad \text{(II)} \ \int_a^\infty p(t)^{-1/\alpha} dt < \infty,$$

as was done by some authors.

A positive solution $x(t)$ of (5.1) is said to be *strongly decreasing* if it satisfies

$$\lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} p(t)^{\frac{1}{\alpha}} x'(t) = 0 \tag{5.2}$$

and *strongly increasing* if it satisfies

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} p(t)^{\frac{1}{\alpha}} x'(t) = \infty. \tag{5.3}$$

(It is easy to see that if we define the functions

$$P(t) = \int_a^t p(s)^{-1/\alpha} ds \quad \text{in the case (I)}, \quad \pi(t) = \int_t^\infty p(s)^{-1/\alpha} ds \quad \text{in the case (II)},$$

then (5.2) is equivalent to a single condition

$$\lim_{t\to\infty} x(t) = 0 \quad \text{if (I) holds} \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{\pi(t)} = 0 \quad \text{if (II) holds.}$$

Similarly, (5.3) reduces to:

$$\lim_{t\to\infty} \frac{x(t)}{P(t)} = \infty \ \text{ in the case (I),} \quad \text{and} \quad \lim_{t\to\infty} x(t) = \infty \ \text{ in the case (II).)}$$

Let $x(t)$ be a strongly decreasing (resp. strongly increasing) solution of equation (5.1) and put $y(t) = p(t)(-x'(t))^\alpha$ (resp. $y(t) = p(t)x'(t)^\alpha$). Then, $(x(t), y(t))$ is a strongly decreasing (resp. strongly increasing) solution of the following system of first order differential equations

$$x' + p(t)^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}} = 0, \qquad y' + q(t)x^\beta = 0, \tag{5.4}$$

$$\left(\text{resp. } \ x' = p(t)^{-\frac{1}{\alpha}} y^{\frac{1}{\alpha}}, \qquad y' = q(t)x^\beta\right). \tag{5.5}$$

In order to study these systems in the framework of regular variation we need to require that $p \in \mathrm{RV}(\lambda)$ and $q \in \mathrm{RV}(\mu)$ and that they are expressed in the form $p(t) = t^\lambda l(t)$ and $q(t) = t^\mu m(t)$ with $l, m \in \mathrm{SV}$.

We are now in a position to apply Theorems 3.2 and 4.2 to the systems (5.4) and (5.5), respectively. From Theorem 3.2 applied to (5.4) we see that (5.4) has strongly decreasing solutions $(x(t), y(t))$ such that $x \in \mathrm{RV}(\rho)$ and $y \in \mathrm{RV}(\sigma)$ for some negative ρ and σ if and only if

$$\alpha - \lambda + \mu + 1 < 0 \quad \text{and} \quad \beta(\alpha - \lambda) + \alpha(\mu + 1) < 0,$$

in which case ρ and σ are determined uniquely by

$$\rho = \frac{\alpha - \lambda + \mu + 1}{\alpha - \beta} \quad \text{and} \quad \sigma = \frac{\beta(\alpha - \lambda) + \alpha(\mu + 1)}{\alpha - \beta}. \tag{5.6}$$

On the other hand, from Theorem 4.2 it follows that (5.5) has strongly increasing solutions $(x(t), y(t))$ such that $x \in \mathrm{RV}(\rho)$ and $y \in \mathrm{RV}(\sigma)$ for some positive ρ and σ if and only if

$$\alpha - \lambda + \mu + 1 > 0 \quad \text{and} \quad \beta(\alpha - \lambda) + \alpha(\mu + 1) > 0,$$

in which case ρ and σ are given by (5.6).

As easily seen, if $\alpha \geqslant \lambda$, then

$$\alpha - \lambda + \mu + 1 < 0 \quad \text{implies} \quad \beta(\alpha - \lambda) + \alpha(\mu + 1) < 0,$$

and

$$\beta(\alpha - \lambda) + \alpha(\mu + 1) > 0 \quad \text{implies} \quad \alpha - \lambda + \mu + 1 > 0.$$

Similarly, if $\alpha \leqslant \lambda$, then

$$\beta(\alpha-\lambda)+\alpha(\mu+1)<0 \quad \text{implies} \quad \alpha-\lambda+\mu+1<0,$$

and

$$\alpha-\lambda+\mu+1>0 \quad \text{implies} \quad \beta(\alpha-\lambda)+\alpha(\mu+1)>0.$$

The above-mentioned statements concerning the strongly monotone solutions of systems (5.4) and (5.5) can be transformed into the following theorems on the strongly monotone regularly varying solutions of the generalized Thomas-Fermi equation (5.1).

THEOREM 5.1. *Suppose that* $p \in \mathrm{RV}(\lambda)$ *and* $q \in \mathrm{RV}(\mu)$. (i) *Let* $\lambda \leqslant \alpha$. *Then, equation* (5.1) *possesses strongly decreasing regularly varying solutions in* $\mathrm{RV}(\rho)$ *with* $\rho < 0$ *if and only if* $\alpha-\lambda+\mu+1<0$, *in which case the regularity index* ρ *is uniquely determined by*

$$\rho = \frac{\alpha-\lambda+\mu+1}{\alpha-\beta}, \tag{5.7}$$

and the asymptotic behavior of any such solution $x(t)$ *is governed by the unique decay law*

$$x(t) \sim \left[\frac{t^{\alpha+1}p(t)^{-1}q(t)}{(-\rho)^{\alpha}(\alpha-\lambda-\alpha\rho)}\right]^{\frac{1}{\alpha-\beta}}, \quad t\to\infty. \tag{5.8}$$

(ii) *Let* $\alpha < \lambda$. *Then, equation* (5.1) *possesses strongly decreasing regularly varying solutions in* $\mathrm{RV}(\rho)$ *with* $\rho < \frac{\alpha-\lambda}{\alpha}$ *if and only if* $\beta(\alpha-\lambda)+\alpha(\mu+1)<0$, *in which case the regularity index is uniquely determined by* (5.7) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique decay law* (5.8).

THEOREM 5.2. *Suppose that* $p \in \mathrm{RV}(\lambda)$ *and* $q \in \mathrm{RV}(\mu)$. (i) *Let* $\lambda \geqslant \alpha$. *Then, equation* (5.1) *possesses strongly increasing regularly varying solutions in* $\mathrm{RV}(\rho)$ *with* $\rho > 0$ *if and only if* $\alpha-\lambda+\mu+1>0$, *in which case the regularity index* ρ *is uniquely determined by* (5.7) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique growth law*

$$x(t) \sim \left[\frac{t^{\alpha+1}p(t)^{-1}q(t)}{\rho^{\alpha}(\alpha\rho+\lambda-\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \quad t\to\infty. \tag{5.9}$$

(ii) *Let* $\alpha > \lambda$. *Then, equation* (5.1) *possesses strongly increasing regularly varying solutions in* $\mathrm{RV}(\rho)$ *with* $\rho > \frac{\alpha-\lambda}{\alpha}$ *if and only if* $\beta(\alpha-\lambda)+\alpha(\mu+1)>0$, *in which case the regularity index is uniquely determined by* (5.7) *and the asymptotic behavior of any such solution* $x(t)$ *is governed by the unique growth law* (5.9).

We note that the above theorems generalize some of the recent results of Kusano et al. [8] concerning the special case of (5.1) with $p(t) \equiv 1$.

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