Differential
Equations
& Applications
Volume 5, Number 4 (2013), 519–525
doi:10.7153/dea-05-30

A LEAST SQUARES SOLUTION TO LINEAR BOUNDARY VALUE PROBLEMS WITH IMPULSES

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(*Communicated by Johnny Henderson*)

Abstract. In this paper we are concerned with obtaining least squares solutions for a linear nonhomogeneous boundary value problem with impulses. In particular, we obtain a complete description for the least squares solution of minimal norm, in the sense of $L^2[0,1]$.

1. Introduction

In the following we will be concerned with finding least squares solutions to

$$x'(t) = A(t)x(t) + h(t), \quad \text{a.e. } [0,1] \tag{1}$$

$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1,\ldots,k \tag{2}$$

subject to

$$Bx(0) + Dx(1) = 0. \tag{3}$$

The points $t_i, i = 1,\cdots,k$, are fixed with $0 < t_1 < t_2 < \cdots < t_k < 1$. For each $t \in [0,1]$, $A(t)$ is an $n \times n$ matrix. The components of $A(\cdot)$ are assumed to be in $L^2([0,1],\mathbb{R})$ and the function h is assumed to be in $L^2([0,1],\mathbb{R}^n)$. The v_i, $i = 1,\cdots,k$, are elements of $\mathbb{R}^n$, and B and D are $n \times n$ matrices.

In our analysis we obtain a complete description for the least squares solution of minimal $L^2([0,1],\mathbb{R}^n)$ norm. Our analysis is intimately related to the idea of generalized inverses. For those readers interested in the method of least squares as well as ideas regarding generalized inverses and generalized Green's functions as they apply to differential equations, we suggest [1, 2, 3, 4, 5, 6, 7, 8, 9].

Mathematics subject classification (2010): 34B15.
Keywords and phrases: boundary value problems, least squares solutions, impulsive differential equations.

2. Preliminaries

The linear boundary value problem will be viewed as an operator equation. To formulate the problem, we introduce the following. $PAC_{\{t_i\}}[0,1]$ will represent the subset of $L^2([0,1],\mathbb{R}^n)$ consisting of functions which are absolutely continuous on every compact subinterval of $[0,1]\setminus\{t_1,\cdots,t_k\}$. We define

$$\mathrm{dom}(\mathscr{L})=\{\phi\in PAC_{\{t_i\}}[0,1]\mid \phi'\in L^2([0,1],\mathbb{R}^n) \text{ and } B\phi(0)+D\phi(1)=0\}.$$

We define an inner-product on $L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^{nk}$ by

$$\left\langle \begin{bmatrix} h_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} h_2 \\ v_2 \end{bmatrix} \right\rangle = \int_0^1 {h_1}^T(s)h_2(s)ds + \sum_{i=1}^k {v_{1,i}}^T v_{2,i},$$

where for $j=1,2$,

$$v_j=\begin{bmatrix} v_{j,1} \\ \vdots \\ v_{j,k} \end{bmatrix}.$$

It is clear that $L^2([0,1],\mathbb{R}^n)\times\mathbb{R}^{nk}$ becomes a Hilbert space under the above inner-product.

We define an operator $\mathscr{L}:\mathrm{dom}(\mathscr{L})\to L^2\Big([0,1],\mathbb{R}^n\Big)\times\mathbb{R}^{nk}$ by

$$\mathscr{L}x=\begin{bmatrix} x'(\cdot)-A(\cdot)x(\cdot) \\ x(t_1^+)-x(t_1^-) \\ \vdots \\ x(t_k^+)-x(t_k^-) \end{bmatrix}.$$

REMARK 2.1. It is clear, from the previous defintions, that finding a least squares solution to (1)-(3) is equivalent to finding a least squares solution to the operator equation $\mathscr{L}x=\begin{bmatrix} h \\ v \end{bmatrix}$.

To obtain a description of our least squares solution, we will construct projections onto the $\mathrm{Ker}(\mathscr{L})$ and $\mathrm{Im}(\mathscr{L})$. To aid in the construction of these projections, we now completely characterize both the kernel and image of $\mathscr{L}$.

PROPOSITION 2.2. *A function $x\in\mathrm{Ker}(\mathscr{L})$ if and only if $x(t)=\Phi(t)c$ for some $c\in\mathrm{Ker}(B+D\Phi(1))$. Here $\Phi(\cdot)$ is the principal fundamental matrix solution to $x'=A(t)x$.*

Proof. $\mathscr{L}x=0$ if and only if $x'=A(t)x$ a.e. $[0,1]$ and $Bx(0)+Dx(1)=0$, which happens if and only if

$$x=\Phi(\cdot)x(0) \text{ and the boundary conditions hold,}$$

which is equivalent to

$$\exists c \in \mathbb{R}^n \text{ such that } x = \Phi(\cdot)c \text{ and } Bc + D\Phi(1)c = 0.$$

We now turn to a characterization of the $\mathrm{Im}(\mathscr{L})$. To do so, we introduce the following notation. We let $\{c_1, \cdots, c_p\}$ be a basis for $\mathrm{Ker}\big((B + D\Phi(1))^T\big)$. We define

$W = [c_1, ..., c_p]$

and

$\Psi(t)^T = W^T D\Phi(1)\Phi^{-1}(t)$.

Lastly, we define $S = \mathrm{span}\left\{ \begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi}_j \end{bmatrix} = 1, ..., p \right\}$,

where

$$\begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi}_j \end{bmatrix} = \begin{bmatrix} \Psi_j(\cdot) \\ \Psi_j(t_1), \\ \vdots \\ \Psi_j(t_k) \end{bmatrix}.$$

Here $\Psi_j(\cdot)$ denotes the jth column of $\Psi(\cdot)$.

PROPOSITION 2.3. $\mathscr{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ *if and only if* $\int_0^1 \Psi^T(s)h(s)ds + \sum_{i=1}^{k} \Psi^T(t_i)v_i = 0$*; that is, if and only if* $\left\langle \begin{bmatrix} \Psi_j(\cdot) \\ \vec{\Psi}_j \end{bmatrix}, \begin{bmatrix} h \\ v \end{bmatrix} \right\rangle = 0$ *for each* $j = 1, \cdots, p$.

Proof. It is well documented that $\mathscr{L}x = \begin{bmatrix} h \\ v \end{bmatrix}$ if and only if

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)h(s)ds + \sum_{t_i < t} \Phi^{-1}(t_i)v_i \right).$$

Imposing the boundary conditions, we have
$\begin{bmatrix} h \\ v \end{bmatrix} \in \mathrm{Im}(\mathscr{L})$ if and only if there exists $w \in \mathbb{R}^n$ such that

$$Bw + D\Phi(1)\left(w + \int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \right).$$

This is clearly equivalent to there existing a $w \in \mathbb{R}^n$ such that

$$[B + D\Phi(1)]w = -D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \right),$$

which is equivalent to

$$D\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k} \Phi^{-1}(t_i)v_i \right) \in \mathrm{Im}\big(B + D\Phi(1)\big).$$

Since $\mathrm{Im}(B + D\Phi(1)) = \mathrm{Ker}\Big((B + D\Phi(1))^T\Big)^{\perp}$, the result follows.

COROLLARY 2.4. *The image of $\mathscr{L}$ is equal to $S^{\perp}$.*

3. Least squares solution with minimal norm

In this section we characterize the least squares solution with minimal norm for the linear boundary value problem

$$x'(t) = A(t)x(t) + h(t), \quad a.e. [0,1]$$
$$x(t_i^+) - x(t_i^-) = v_i, \quad i = 1, ..., k$$

subject to

$$Bx(0) + Dx(1) = 0.$$

From Proposition (2.2), it follows that there exist a basis, $\alpha_1, \cdots, \alpha_p$, for $\operatorname{Ker}(B + D\Phi(1))$ such that

$$\{\Phi(\cdot)\alpha_1, ..., \Phi(\cdot)\alpha_p\}$$

is an orthonormal basis for the $\operatorname{Ker}(\mathscr{L})$.
We define

$$P : L^2([0,1], \mathbb{R}^n) \to L^2([0,1], \mathbb{R}^n)$$

by

$$Px = \sum_{j=1}^{p} \langle \Phi(\cdot)\alpha_j, x \rangle \Phi(\cdot)\alpha_j$$

and

$$Q : L^2([0,1], \mathbb{R}^n) \times \mathbb{R}^{nk} \leftarrow L^2([0,1], \mathbb{R}^n) \times \mathbb{R}^{nk}$$

by

$$Q \begin{bmatrix} h \\ v \end{bmatrix} = \sum_{j=1}^{p} \left\langle \begin{bmatrix} \psi_j(\cdot) \\ \vec{\psi}_j \end{bmatrix}, \begin{bmatrix} h \\ v \end{bmatrix} \right\rangle \begin{bmatrix} \psi_j(\cdot) \\ \vec{\psi}_j \end{bmatrix}.$$

It is clear that P and $I - Q$ are the orthogonal projections onto $\operatorname{Ker}(\mathscr{L})$ and $\operatorname{Im}(\mathscr{L})$, respectively.

PROPOSITION 3.1. *The least squares solution to* (1)-(3) *with minimal* $L^2([0,1], \mathbb{R}^n)$ *norm is given by* $M_p(I - Q)\begin{bmatrix} h \\ v \end{bmatrix}$, *where* $M_p = \mathscr{L}^{-1}_{|\operatorname{Ker}(P) \cap \operatorname{dom}(\mathscr{L})}$.

Proof.

It is clear that any least squares solution, x, satisfies $\mathscr{L}x = (I - Q)\begin{bmatrix} h \\ v \end{bmatrix}$.

Since

$$\begin{aligned} \|x\|^2 &= \|Px + (I - P)x\|^2 \\ &= \left\| Px + M_p(I - Q)\begin{bmatrix} h \\ v \end{bmatrix} \right\|^2 \\ &= \|Px\|^2 + \left\| M_p(I - Q)\begin{bmatrix} h \\ v \end{bmatrix} \right\|^2, \end{aligned}$$

we see that $\|x\|$ is a minimum precisely when $Px = 0$. The result now follows.

THEOREM 3.2. *The least squares solution to* (1)-(3) *with minimal* $L^2([0,1],\mathbb{R}^n)$ *norm is given by*

$$x(t) = \Phi(t)\Bigg((Ec+\beta) + \int_0^t \Phi^{-1}(s)\Bigg[h(s) - \sum_{j=1}^{p}\Bigg[\int_0^1 \psi_j^T(u)h(u)du + \sum_{i=1}^{k}\psi_j^T(t_i)v_i\Bigg]\Psi_j(s)\Bigg]ds + \sum_{t_i<t}\Phi^{-1}(t_i)\Bigg(v_i - \sum_{j=1}^{p}\Bigg[\int_0^1 \psi_j^T(u)h(u)du + \sum_{i=1}^{k}\psi_j^T(t_i)v_i\Bigg]\Psi_j(t_i)\Bigg)\Bigg).$$

Here $E = [\alpha_1,\ldots,\alpha_p]$, *and* $c \in \mathbb{R}^p$ *and* $\beta \in \mathrm{Ker}(B+D\Phi(1))^{\perp}$ *are the unique elements satisfying*

$$c_i = -\int_0^1 \alpha_i^T\Phi^T(s)\Phi(s)\beta - \int_0^1 \alpha_i^T\Phi^T(s)\Phi(s)\Bigg(\int_0^s \Phi^{-1}(u)\Bigg[h(u) - \sum_{j=1}^{p}\Bigg[\int_0^1 \psi_j^T(y)h(y)dy + \sum_{i=1}^{k}\psi_j^T(t_i)v_i\Bigg]\Psi_j(u)\Bigg]du + \sum_{t_i<s}\Phi^{-1}(t_i)\Bigg(v_i - \sum_{j=1}^{p}\Bigg[\int_0^1 \psi_j^T(y)h(y)dy + \sum_{i=1}^{k}\psi_j^T(t_i)v_i\Bigg]\Psi_j(t_i)\Bigg)ds\Bigg).$$

and

$$\beta = -TD\Phi(1)\left(\int_0^1 \Phi^{-1}(s)h(s)ds + \sum_{i=1}^{k}\Phi^{-1}(t_i)v_i\right),$$

where

$$T = [B + D\Phi(1)]^{-1}_{|\mathrm{Ker}(B+D\Phi(1))^{\perp}}.$$

REMARK 3.3. We would like to point out, as will be evident from the proof below, that when $A(\cdot)$ and h are continuous the the least squares solution will actually satisfy

$$x'(t) = A(t)x(t) + h(t) \quad \text{for all } t \in [0,1] \setminus \{t_1,\cdots,t_k\}.$$

Proof. With Proposition (3.1) in mind, we search for a description of M_p. Now, for $\begin{bmatrix} g \\ u \end{bmatrix} \in \mathrm{Im}(\mathscr{L})$, $M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right)$ is the unique element in $\mathrm{dom}(\mathscr{L})$ satisfying the following:

(i) $\mathscr{L}M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right) = \begin{bmatrix} g \\ u \end{bmatrix}$.

(ii) $PM_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right) = 0$.

We now show that

$$M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right)(t) = \Phi(t)\big(Ec^* + \beta\big) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right),$$

for all $\begin{bmatrix} g \\ u \end{bmatrix} \in \mathrm{Im}(\mathscr{L})$, where

$$c_i^* = -\int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(\beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right)ds.$$

From Proposition (2.3), it is clear that

$$\mathscr{L}\left(\Phi(t)\big(Ec^* + \beta\big) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right)\right) = \begin{bmatrix} g \\ u \end{bmatrix}.$$

Now,

$$\begin{aligned}
&\int_0^1 \alpha_i^T \Phi(s)^T \left[\Phi(s)\left(Ec^* + \beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right)\right]ds \\
&= \int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(c_i^*\alpha_i + \beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right)ds \\
&= c_i^* + \int_0^1 \alpha_i^T \Phi^T(s)\Phi(s)\left(\beta + \int_0^s \Phi^{-1}(u)g(u)du + \sum_{t_i<s} \Phi^{-1}(t_i)u_i\right)ds \\
&= 0.
\end{aligned}$$

Since $Px = 0$ if and only if for each i, $i = 1, \cdots, p$, we have $\langle \Phi(\cdot)\alpha_i, x\rangle = 0$, it follows that

$$P\left(\Phi(t)\big(Ec^* + \beta\big) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right)\right) = 0.$$

Thus,

$$M_p\left(\begin{bmatrix} g \\ u \end{bmatrix}\right)(t) = \Phi(t)\big(Ec^* + \beta\big) + \Phi(t)\left(\int_0^t \Phi^{-1}(s)g(s)ds + \sum_{t_i<t} \Phi^{-1}(t_i)u_i\right).$$

The result now follows for an arbitrary $\begin{bmatrix} h \\ v \end{bmatrix} \in L^2([0,1], \mathbb{R}^n) \times \mathbb{R}^{nk}$ by replacing $\begin{bmatrix} g \\ u \end{bmatrix}$ in the description of M_p with $(I - Q)\begin{bmatrix} h \\ v \end{bmatrix}$.

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(Received April 24, 2013)
(Revised July 24, 2013)

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