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# OSCILLATORITY OF FRESNEL INTEGRALS AND CHIRP–LIKE FUNCTIONS

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*Dedicated to Luka Korkut, upon his retirement*

(*Communicated by Yuki Naito*)

*Abstract.* In this review article, we present results concerning fractal analysis of Fresnel and generalized Fresnel integrals. The study is related to computation of box dimension and Minkowski content of spirals defined parametrically by Fresnel integrals, as well as computation of box dimension of the graph of reflected component function which are chirp-like function. Also, we present some results about relationship between oscillatority of the graph of solution of differential equation, and oscillatority of a trajectory of the corresponding system in the phase space. We are concentrated on a class of differential equations with chirp-like solutions, and also spiral behavior in the phase space.

## 1. Introduction

As coauthors of Luka Korkut, we present here an overview of his scientific contribution in fractal analysis of Fresnel integrals and chirp-like functions. We cite the main results from joint articles of Korkut, Resman, Vlah, Žubrinić and Županović, [18, 19, 20, 21, 22, 23]. The articles are mostly based on qualitative theory of differential equations. A standard technique of this theory is *phase plane analysis*. We study trajectories of the corresponding system of differential equations in the phase plane, instead of studying the graph of the solution of the equation directly. Our main interest is fractal analysis of behavior of the graph of oscillatory solution, and of a trajectory of the associated system. This approach has been extended to the study of oscillatory integrals. From the point of view of fractal geometry, fractal properties of trajectories in the phase plane have been analyzed and compared with fractal properties of the graphs of solutions of differential equations. The particularly interesting case are curves with an accumulation point in whose neighborhood the curve itself is non-rectifiable, that is, of infinite length.

Fractal dimension theory in dynamics has over the years evolved into an independent field of mathematics. Fractal dimensions enable better insight into the dynamics appearing in various problems in physics, engineering, medicine and in many other

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branches of mathematics. Some of the fractal dimensions that are important for dynamics are Hausdorff dimension, box dimension, the Rényi spectrum for dimensions, correlation dimension, information dimension and packing dimension.

In our considerations, the *box dimension* is used, in order to give better insight to oscillatority of a class of integrals and to oscillatority of solutions of a class of differential equations. The box dimension is a tool for distinguishing between non-rectifiable curves, near an accumulation point. For planar curves, box dimension of a curve in the neighborhood of an accumulation point lies in the interval $[1,2]$. It measures the ”amount” of accumulation of the curve at the accumulation point. Note that another commonly used fractal dimension, the Hausdorff dimension, cannot distinguish between non-rectifiable smooth curves. Using the countable stability of the Hausdorff dimension, and a fact that every smooth non-rectifiable curve is a countable union of rectifiable curves, we get that the Hausdorff dimension of every non-rectifiable smooth curve is always trivial and equal to $1$.

The main observation in our work is that there exists an interesting relationship between oscillatority of the graph of the solution and oscillatority of a trajectory in the phase space. Box dimension of a trajectory is thus called the *phase dimension* of the solution of the equation. In this study, the two curves are significant. Their box dimensions have been calculated in the book of Claude Tricot [49]:

(i) The $\alpha$*-power-type spiral*, given by $r=\varphi^{-\alpha}$ in polar coordinates, where $\varphi \geqslant \varphi_0 > 0$, $\alpha \in (0,1]$. It is non-rectifiable near the origin, with box dimension

$$d=\frac{2}{1+\alpha}.$$

(ii) The $(\alpha,\beta)$*-chirp function*, given by the formula $y(x)=x^{\alpha} sin(x^{-\beta})$, where $x \in (0,x_0)$, $x_0>0$. For $0<\alpha\leqslant\beta$, its graph is non-rectifiable near the origin and accumulates in the neighborhood of the origin with box dimension

$$d=2-\frac{1+\alpha}{1+\beta}.$$

If we want to measure oscillatority at infinity instead at the origin, we can perform the change of coordinates which puts the infinity to the origin. Such graph is called the *reflected graph*.

In this overview article, we present some results that originate from the articles of Luka Korkut and his coauthors. Each subsection treats the results from a different article, and the contribution of Luka Korkut is explained at the beginning of each section.

This article is dedicated only to the results of Luka Korkut, upon his retirement. Other articles from collaborators of Luka Korkut dealing with the similar subjects in fractal analysis of differential equations and dynamical systems are: [28, 40, 41, 42, 43, 44, 45, 51] for fractal analysis of solutions of ordinary differential equations, [14, 32, 48, 56] for fractal analysis of trajectories of discrete dynamical systems, and [53, 54, 55] for fractal analysis of spiral trajectories of continuous dynamical systems in the phase plane and phase space. More specifically, Euler type equations are considered in [41, 40, 51], Hartman-Wintner type equations in [28], half linear equations in [44], the Bessel equation in [43] and the first results connecting fractal properties of chirps and spirals, with applications to Liénard and Bessel equations, can be found in [45].

In [14], bifurcations of 1-dimensional discrete systems were treated. It was noted that the box dimension around the bifurcation point reveals number of fixed points appearing in perturbations. As far as continuous systems are concerned, the main object of research was the connection between the cyclicity of simple limit periodic sets (elliptic singular points or periodic orbits) and the box dimension of a spiral trajectory in their neighborhood. It was noted in [54] that the density of accumulation is correlated with number of cycles born in perturbations. Continuous systems were related to discrete systems via Poincaré map in [55], using [8]. The result was then extended to some simple polycycles in [32].

It is worth knowing that chirp functions are considered in the time-frequency analysis, see references [3, 5, 15, 37, 47]. For some applications of the time-frequency analysis, see for instance [2, 16, 39, 46, 50]. Finally, for essential qualitative properties of chirp-like solutions, see classical references [6, 13, 38, 1, 17]. For oscillations of solutions of second order quasilinear differential equations, see for instance [27, 26], while for oscillations in biology, see for instance [12]. In this article we investigate the fractal approach to the theory of oscillations.

### 1.1. Notations

Let us now recall some basic definitions from fractal analysis. By $d(x,A)$ we denote the Euclidean distance from point $x$ to a given subset $A$ in $\mathbb{R}^N$. Let $A_\varepsilon$ be the open $\varepsilon$-neighborhood of $A$. The *upper $s$-dimensional Minkowski content* of a bounded subset $A$ in $\mathbb{R}^N$, $s \geqslant 0$, is defined by

$$\mathscr{M}^{*s}(A) := \limsup_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

where $|A_\varepsilon|$ denotes the $N$-dimensional Lebesgue measure of $A_\varepsilon$. The *lower $s$-dimensional Minkowski content* of $A$ is defined by

$$\mathscr{M}^{s}_{*}(A) := \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}.$$

If both of these quantities coincide, the common value is denoted by $\mathscr{M}^s(A)$. See Krantz and Parks [25, p. 74], Mattila [33, p. 79], and Žubrinić [52] for basic properties of the Minkowski contents. Value $s$ at which the function $s \mapsto \mathscr{M}^{*s}(A)$ jumps from infinity to zero is called the *upper box dimension* of $A$, denoted by $d = \overline{\dim}_B A$. More precisely,

$$\overline{\dim}_B A = \inf\{s \geqslant 0 : \mathscr{M}^{*s}(A) = 0\} = \sup\{s \geqslant 0 : \mathscr{M}^{*s}(A) = \infty\}.$$

The *lower box dimension* of $A$, denoted by $d = \underline{\dim}_B A$, is defined analogously. If both of these dimensions are equal, the common value is called the *box dimension* of $A$, and denoted by $d = \dim_B A$. See Falconer [11].

If $0 < \mathcal{M}_*^d(A) \leqslant \mathcal{M}^{*d}(A) < \infty$, we say that $A$ is *Minkowski nondegenerate*, see [52]. If $\mathcal{M}_*^d(A) = \mathcal{M}^{*d}(A)$, the common value is denoted by $\mathcal{M}^d(A)$, and called $d$-*dimensional Minkowski content* of $A$. If moreover $\mathcal{M}^d(A) \in (0,\infty)$, we say that $A$ is *Minkowski measurable*. Box dimension appears very often, in particular in dynamics. For more information, see the survey article [56]. The Minkowski content is a subject of extensive study undertaken by M. Lapidus and his collaborators in various directions, see [24], [29], [30] and the references therein.

Finally, we introduce some notations used in the articles. For two real functions $f$, $g$ of real variable, we write

$$f(t) \sim g(t), \text{ as } t \to 0 \ (t \to \infty),$$

if $\lim_{t\to 0\ (t\to\infty)} f(t)/g(t) = 1$. Let $k$ be a nonnegative integer, and $f$, $g$ of class $C^k$. We write

$$f(t) \sim_k g(t), \text{ as } t \to 0 \ (t \to \infty),$$

if $f^{(j)}(t) \sim g^{(j)}(t)$ as $t \to 0$ $(t \to \infty)$, for all $j = 0,1,\ldots,k$.

Similarly, we write

$$f(t) \simeq g(t), \text{ as } t \to 0 \ (t \to \infty),$$

if there exist two positive constants $C > 0$ and $D > 0$, such that $C f(t) \leqslant g(t) \leqslant D f(t)$, for all $t$ sufficiently close to $t = 0$ (for all $t$ sufficiently large). Let $k$ be a nonnegative integer and let $f$ and $g$ be of class $C^k$. We write

$$f(t) \simeq_k g(t), \text{ as } t \to 0 \ (t \to \infty),$$

if $f^{(j)}(t) \simeq g^{(j)}(t)$, as $t \to 0$ $(t \to \infty)$, for all $j = 0,1,\ldots,k$.

Furthermore, we write $f(t) = O(g(t))$, as $t \to 0$ $(t \to \infty)$, if there exists a positive constant $C > 0$ such that $|f(t)| \leqslant C|g(t)|$, for all $t$ sufficiently close to $t = 0$ (for all $t$ sufficiently large). Similary, we write $f(t) = o(g(t))$, as $t \to 0$ $(t \to \infty)$, if, for every positive constant $\varepsilon > 0$, it holds that $|f(t)| \leqslant \varepsilon|g(t)|$, for all $t$ sufficiently close to $t = 0$ (for all $t$ sufficiently large).

## 2. Fractal analysis of Fresnel integrals

In this section we study, from the point of view of fractal geometry, clothoids and generalized clothoids, defined by Fresnel and generalized Fresnel integrals. That is, we compute their box dimension and Minkowski content, as well as box dimension of graphs of their component functions.

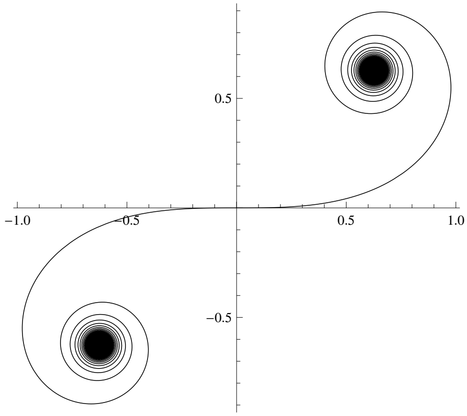



Figure 1: Graph of the clothoid.

## 2.1. Standard clothoid

Luka Korkut has been working on the problem of clothoid since Vladimir Kostov, see [7], proposed fractal study of clothoid as an interesting problem. The clothoid, also called the Cornu spiral or the Euler spiral, is widely used in robotics, civil engineering, number theory etc. It is used for finding the optimal path for a robot with prescribed initial and final angles and curvatures, in modeling road shapes and in computer aided geometric design applications. So called clothoid splines are used among others in computer typography and cartography, see for instance [36, 35, 34]. Also, the clothoid is associated with the concept of diffraction in optics, see [4, p. 428]. This subsection is devoted to results from article [23].

*Clothoid* is a planar curve defined parametrically by

$$x(t) = \int_0^t \cos(s^2)\,ds, \quad y(t) = \int_0^t \sin(s^2)\,ds, \tag{2.1}$$

where $t \in \mathbb{R}$. The graph consists of two spiral curves converging to two focus points, in the first and in the third quadrant, as $t \to \pm\infty$. The two spirals are symmetric with respect to the origin. At any point, the curvature is proportional to the arc length from the origin: the curvature at the point $(x(t), y(t))$ is equal to $2t$, and the arc length from the origin to the point $(x(t), y(t))$ is equal to $t$. The spirals are thus nonrectifiable, as $t \to \pm\infty$, see Figure 1 above.

In [23], several results about box dimension and Minkowski content of the clothoid are proved.

THEOREM 1. (Box dimension and Minkowski content of the clothoid, [23]) *Let* $\Gamma$ *be the clothoid defined by* (2.1). *Then*

$$\dim_B \Gamma = 4/3.$$

*Furthermore, $\Gamma$ is Minkowski measurable with Minkowski content*

$$\mathscr{M}^{4/3}(\Gamma) = 3 \cdot 2^{-2/3}\pi^{1/3}.$$

*The idea of the proof.* For finding box dimension of spiral trajectories of the clothoid, we use Theorem 5 from [54]. The theorem is a generalization of Tricot's formula [49, p. 121] for the box dimension of spiral trajectories $r(\varphi) = \varphi^{-\alpha}$, mentioned in the Introduction. It deals with spirals with the asymptotics $r(\varphi) \sim \varphi^{-\alpha}$, $\varphi \to \infty$. To prove that spirals of the clothoid satisfy the assumptions of Theorem 5 from [54], we exploit the known asymptotics of Fresnel integrals from Lebedev [31, p. 23],

$$C(x) = \int_0^x \cos\left(\frac{\pi s^2}{2}\right) ds, \quad S(x) = \int_0^x \sin\left(\frac{\pi s^2}{2}\right) ds.$$

For large values of $|x| \to \infty$, we have

$$\begin{cases} C(x) & = \frac{1}{2} - \frac{1}{\pi x}\left[B(x)\cos(\frac{\pi x^2}{2}) - A(x)\sin(\frac{\pi x^2}{2})\right] \\ S(x) & = \frac{1}{2} - \frac{1}{\pi x}\left[A(x)\cos(\frac{\pi x^2}{2}) + B(x)\sin(\frac{\pi x^2}{2})\right]. \end{cases}$$

Here,

$$A(x) = \sum_{k=0}^{N} \frac{(-1)^k \alpha_{2k}}{(\pi x^2)^{2k}} + O(|x|^{-4N-4}), \quad B(x) = \sum_{k=0}^{N} \frac{(-1)^k \alpha_{2k+1}}{(\pi x^2)^{2k+1}} + O(|x|^{-4N-6}),$$

for any $N \geqslant 0$, and

$$\alpha_k = 1 \cdot 3 \cdots (2k-1), \quad \text{for } k \geqslant 1, \quad \alpha_0 = 1.$$

We now apply the above expansions to (2.1), to get the expansions of component functions, as $t \to \infty$:

$$\begin{cases} x(t) = \int_0^t \cos(s^2)\, ds = \sqrt{\frac{\pi}{2}} C\left(\sqrt{\frac{2}{\pi}} t\right) \\ y(t) = \int_0^t \sin(s^2)\, ds = \sqrt{\frac{\pi}{2}} S\left(\sqrt{\frac{2}{\pi}} t\right). \end{cases}$$

Using the above expansions, after some computation, we get the asymptotics of the spiral trajectory of the clothoid in polar coordinates $r(\varphi)$, as $\varphi \to \infty$. □

Furthermore, in [23], fractal analysis of the graphs of the component functions $x(t)$ and $y(t)$ of the clothoid is provided. It turns out that the reflected component functions are almost chirp functions, and thus their box dimensions follow easily from Tricot's formula for chirps, see Introduction.

We first state two definitions from Pašić, Žubrinić and Županović [45] that we need in the sequel.

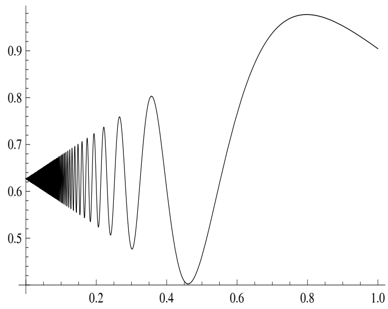



Figure 2: Graph of reflected component function $X(\tau) = x(1/\tau)$, of the clothoid.

DEFINITION 1. (Oscillatory function near $t = \infty$ or $t = 0$, [45]) Let $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that $x(t)$ is *oscillatory function near* $t = \infty$ if there exists a sequence $t_k \to \infty$ such that $x(t_k) = 0$, and the functions $x|_{(t_k, t_{k+1})}$ alternately change sign for $k \in \mathbb{N}$.

Analogously, let $u : (0, t_0] \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that $u$ is *oscillatory function near the origin* if there exists a sequence $s_k$ such that $s_k \searrow 0$ as $k \to \infty$, $u(s_k) = 0$ and restrictions $u|_{(s_{k+1}, s_k)}$ alternately change sign for $k \in \mathbb{N}$.

To measure the oscillatority of $x(t)$ at infinity, the following notion of *oscillatory dimension* has been introduced in [45]. It was mentioned in the context of solutions of nonlinear ODEs of the second order, defined on $(t_0, \infty)$, with oscillatory nature near $t = \infty$.

DEFINITION 2. (Oscillatory dimension, [45]) Let $x : [t_0, \infty) \to \mathbb{R}$ be an oscillatory function near $t = \infty$. Let $X : (0, 1/t_0] \to \mathbb{R}$ be its reflected function, that is, $X(\tau) = x(1/\tau)$. The *oscillatory dimension* $\dim_{osc}(x)$ of $x(t)$ near $t = \infty$ is defined as the box dimension of the graph $G(X)$ of $X(\tau)$ near $\tau = 0$,

$$\dim_{osc}(x) = \dim_B G(X),$$

provided that the box dimension exists.

The reflected function $X(\tau)$ is smooth, so the oscillatory dimension does not depend on $t_0$.

The second main theorem from [23], stated below, concerns oscillatory dimensions of component functions $x(t)$ and $y(t)$ of the clothoid, see Figure 2 above.

THEOREM 2. (Oscillatory dimension of the component functions, [23]) *Let $x(t)$ and $y(t)$ be component functions of the clothoid, defined by (2.1). The oscillatory dimension of both of them is equal to $4/3$. Furthermore, the graphs of the corresponding reflected functions $X(\tau)$ and $Y(\tau)$ are Minkowski nondegenerate.*

*Sketch of the proof.* The proof relies on the following result from [23, Lemma 2].

LEMMA 1. *Let* $g:(0,\tau_0)\to\mathbb{R}$ *be a smooth function, and* $G(g)$ *its graph in* $\mathbb{R}^2$. *If* $h:(0,\tau_0)\to\mathbb{R}$ *is a Lipschitz function then*

$$\underline{\dim}_B G(g+h)=\underline{\dim}_B G(g),\quad \overline{\dim}_B G(g+h)=\overline{\dim}_B G(g).$$

*In particular, if there exists* $\dim_B G(g)$, *then* $\dim_B G(g+h)=\dim_B G(g)$. *Furthermore, if the graph* $G(g)$ *is Minkowski nondegenerate, then the same holds for* $G(g+h)$.

We show that the reflected component functions $X(\tau)$ and $Y(\tau)$ can be written as sums of an appropriate chirp function and remainder function with the bounded first derivative, which is therefore Lipschitz. The result follows directly from Tricot's formula for graphs of chirp functions and Lemma 1 above. □

## 2.2. Generalized clothoids

This subsection is devoted to the results of Luka Korkut and his coauthors published in [20], where the generalization of the standard clothoid, so-called $p$-*clothoid*, has been considered. As in previous section, we cite here the results concerning fractal analysis of graphs of $p$-clothoids, as well as of their component functions.

By $p$-*clothoid*, $p>1$, we mean a planar curve defined parametrically by

$$x(t)=\int_0^t \cos(s^p)\,ds,\quad y(t)=\int_0^t \sin(s^p)\,ds, \tag{2.2}$$

where $t\geqslant 0$. We may replace $s^p$ by $|s|^p$ in (2.2), and allow $t\in\mathbb{R}$. Then the clothoid, as before, consists of two spirals with foci in the first and in the third quadrant that are symmetric with respect to the origin. Note that the standard clothoid from Subsection 2.1 corresponds to $p=2$. For $p$-clothoid, see Figure 3 below, the arc length from the origin to the point $(x(t),y(t))$ is equal to $t$, and the curvature at $(x(t),y(t))$ is equal to $pt^{p-1}$.

The focus point of the $p$-clothoid defined by (2.2) has the following coordinates:

$$\begin{cases} a=\int_0^\infty \cos(s^p)\,ds=\frac{1}{p}\Gamma(1/p)\cos(\pi/2p),\\ b=\int_0^\infty \sin(s^p)\,ds=\frac{1}{p}\Gamma(1/p)\sin(\pi/2p). \end{cases} \tag{2.3}$$

Here, $\Gamma(z)$ is the gamma function, see [9, p. 13, Vol. I]. It was proven in [20, Lemma 2] that the improper integrals converge due to $p>1$. For standard clothoid, $p=2$, the focus points in (2.3) have been computed by Euler, see [10].

The main result of [20] is the following Theorem 3, which is a generalization of Theorem 1 from Subsection 2.1. It was proven in a similar way as Theorem 1 in previous section, but using asymptotic expansions of generalized Fresnel integrals associated to generalized Euler spirals.

THEOREM 3. (Box dimension and Minkowski content of $p$-clothoids, [20]) *Let* $\Gamma_p$ *be the* $p$-*clothoid defined by* (2.2), $p>1$. *Then*

$$d=\dim_B\Gamma_p=2p/(2p-1).$$

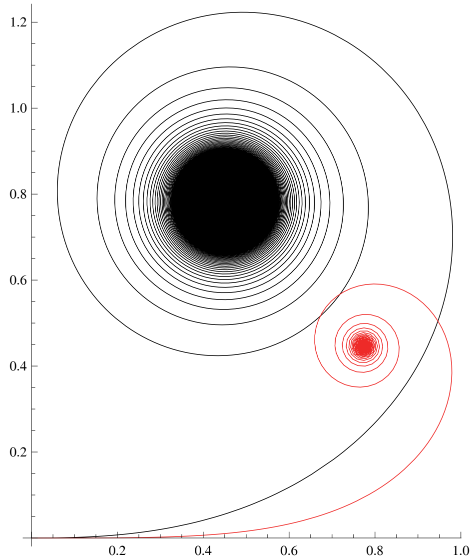



Figure 3: Graph of the $p$-clothoid, $t \geqslant 0$, for values $p = 3/2$ and $p = 3$, for bigger and smaller accumulation, respectively.

*Furthermore, $\Gamma_p$ is Minkowski measurable and its Minkowski content is equal to*

$$\mathscr{M}^d(\Gamma_p) = (2p-1)\left(p(p-1)^{p-1}\right)^{-2/(2p-1)}\pi^{1/(2p-1)}.$$

*Example.* For the standard 2-clothoid, we get from Theorem 3 that its box dimension is equal to $4/3$.

*Sketch of the proof.* The following asymptotic expansions and Theorem 5 from [54], are the keypoints of the proof. For these expansions, see [9, pp. 149-150, Vol. II]. Also, in [20], a short, elementary proof of these expansions is proposed. Let $x(t)$ and $y(t)$ be generalized Fresnel integrals defined by (2.2), $p > 1$, and $a = \lim_{t\to\infty} x(t)$, $b = \lim_{t\to\infty} y(t)$. Then, for any nonnegative integer $N$, we have

$$\begin{cases} x(t) = a + A_N(t)\sin(t^p) - B_N(t)\cos(t^p) + O(t^{-(2N+3)p+1}), \\ y(t) = b - B_N(t)\sin(t^p) - A_N(t)\cos(t^p) + O(t^{-(2N+3)p+1}), \end{cases}$$

when $t \to \infty$. Here,

$$A_N(t) = \sum_{k=0}^{N} (-1)^k a_{2k} t^{-(2k+1)p+1},$$

$$B_N(t) = \sum_{k=0}^{N} (-1)^k a_{2k+1} t^{-(2k+2)p+1},$$

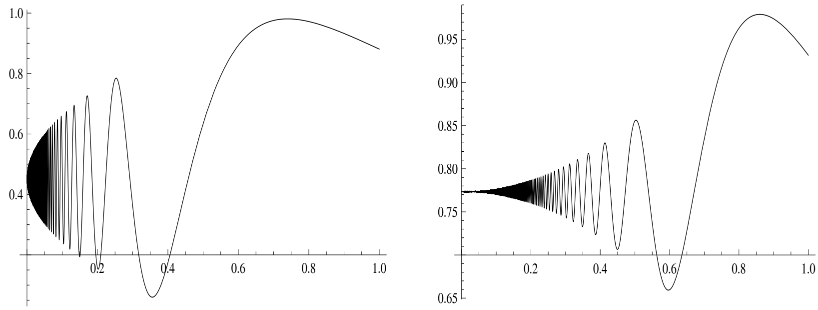



Figure 4: Graph of reflected component function $X(\tau) = x(1/\tau)$, of the $p$-clothoid for $p = 3/2$ and $p = 3$ respectively.

$$a_n = p^{-n-1}(p-1)(2p-1)\dots(np-1),\ n \geqslant 1 \text{ and } a_0 = p^{-1}.$$

□

With the same definitions of oscillatority as in Subsection 2.1, we have the following theorem about the component functions of $p$-clothoid.

THEOREM 4. (Box dimension of the component functions of $p$-clothoid, [20]) *Assume that $p \geqslant 2$ and let $x(t)$ and $y(t)$ be the component functions of the $p$-clothoid defined by* (2.2)*. The oscillatory dimension of both of them is equal to $(2+p)/(1+p)$. Furthermore, the graphs of the corresponding reflected functions $X(\tau)$ and $Y(\tau)$ are Minkowski nondegenerate.*

This theorem is proved in the same manner as Theorem 2. The reflected functions $X(\tau)$ and $Y(\tau)$, can be written as sum of chirp functions and remainder term whose first derivative is bounded, due to the assumption that $p \geqslant 2$. We can then apply Lemma 1. For graphs of reflected function $X(\tau)$ for different values of $p$, see Figure 4 above. Remember, for $p = 2$, see Figure 2.

REMARK 1. *There is a generalization of the previous theorem for $p > 1$, published in* [18]. *The same conclusion holds.* The difference in the proof is that the remainder term in the reflected functions in the case $1 < p < 2$ does not have bounded first derivative and we cannot directly apply Lemma 1. Instead, we have to consider the first two terms in the development of $X(\tau)$, as $\tau \to 0$, instead of only the first term. The sum of first two terms turns out to be the so-called *generalized chirp-like function*. Computing the box dimension of such functions was therefore needed. It is a subject of the following section.

## 3. Fractal analysis of chirp-like functions and spirals

The link between chirp-type oscillatority of graphs of solutions of ordinary differential equations and power-type oscillatority of spirals generated by these solutions in

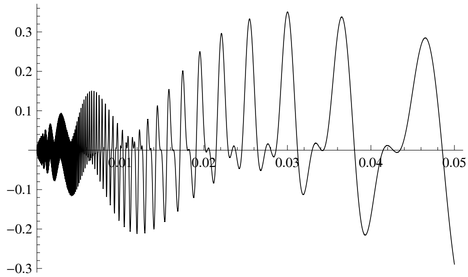



Figure 5: Graph of $(1/2,1)$-chirp-like function $y(x) = P(x)\sin(Q(x))$, near $x = 0$, where $P(x) = x^{1/2} + 2x^{2/3}\sin(x^{-1})$ and $Q(x) = x^{-1} + x^{-1/2}$.

the phase plane has been observed by Luka Korkut and his coauthors. In the paper [45], phase oscillatority and phase dimension of functions has been introduced. These results have been applied to the class of planar autonomous systems which have weak focus at the origin, that is which have strictly imaginary eigenvalues. For definitions and notations, see Subsection 3.2 below. It has been proven that $\alpha$-power-type oscillations in the phase plane imply oscillatority of component functions of $(\alpha,1)$-chirp type, where $\alpha \in (0,1)$.

Due to this observation and Remark 1, a need arose for fractal analyis of the so-called *chirp-like functions*. They behave asymptotically like chirps, but are not exactly chirps, see Definition 3 below. We expected the same Tricot's formula for box dimension to hold in this case. This was the subject of the article [18], described in Subsection 3.1 below.

### 3.1. Chirp-like functions

This subsection is devoted to results from article [18] and [22] about box dimension and Minkowski content of the so-called *chirp-like functions*.

DEFINITION 3. (Chirp-like functions) Functions of the form

$$y = P(x)\sin(Q(x)) \text{ or } y = P(x)\cos(Q(x)),$$

where $P(x) \simeq x^{\alpha}$, $Q(x) \simeq_1 x^{-\beta}$, as $x \to 0$, are called $(\alpha,\beta)$-*chirp-like functions* near $x = 0$.

For the example of $(\alpha,\beta)$-chirp-like function near $x = 0$, see Figure 5 above.

Let us remark here that we use the name *chirp-like* in a descriptive and imprecise manner. We will call all functions from Theorem 5 and Theorem 6 *chirp-like*, although their properties differ slightly from those in Definition 3.

In the sequel we need the following definition from Pašić [40].

DEFINITION 4. ($d$-dimensional fractal oscillatority) Suppose that $v : I \to \mathbb{R}$, $I = (0,1]$, is an oscillatory function near the origin. Let $d \in [1,2)$. We say that $v$ is *$d$-dimensional fractal oscillatory* near the origin if

$$\dim_B G(v) = d \text{ and } 0 < \mathscr{M}_*^d(G(v)) \leqslant \mathscr{M}^{*d}(G(v)) < \infty.$$

Here, $G(v)$ denotes the graph of $v$.

In [18], [22], Luka Korkut proved several results concerning box dimension of chirp-like functions. Sufficient chirp-like behavior conditions have been established for a function to have the box dimension of the standard chirp, $d = 2-(1+\alpha)/(1+\beta)$.

THEOREM 5. (Box dimension of chirp-like functions, [18]) *Let $0 < \alpha \leqslant \beta$, $\alpha \notin \{1,2,3,4\}$, $\delta > 0$, $I = (0,\delta]$. Suppose $y(x) = p(x)S(q(x))$, where $p$, $q \in C^5(I)$, $p(x) > 0$, $q(x) > 0$ on $I$ and $p(x)$, $q(x)$ satisfy the following estimates:*

$$p(x) \sim_5 x^{\alpha}, \qquad q(x) \sim_5 x^{-\beta},$$

*as $x \to 0$. Let $S(t) = C\cos(t) + D\sin(t)$, $C, D \in \mathbb{R}$, be an arbitrary linear combination of sine and cosine functions. Then $y(x)$ is d-dimensional fractal oscillatory near the origin, where $d = 2-(\alpha+1)/(\beta+1)$.*

REMARK 2. Theorem 5 is also true for $\alpha \in \{1,2,3,4\}$, but under slightly different assumptions on $p(x)$, as $x \to 0$:

$$\begin{aligned}
&\alpha = 1: \ p(x) \sim x,\ p'(x) \sim 1,\ p^{(j)}(x) = o(x^{-(j-1)}),\ j = 2,3,4,5,\\
&\alpha = 2: \ p(x) \sim x^2,\ p'(x) \sim 2x,\ p''(x) \sim 2,\ p^{(j)}(x) = o(x^{-(j-2)}),\ j = 3,4,5,\\
&\alpha = 3: \ p(x) \sim x^3,\ p'(x) \sim 3x^2,\ p''(x) \sim 6x,\ p'''(x) \sim 6,\\
&\qquad\quad p^{(j)}(x) = o(x^{-(j-3)}),\ j = 4,5,\\
&\alpha = 4: \ p(x) \sim x^4,\ p'(x) \sim 4x^3,\ p''(x) \sim 12x^2,\ p'''(x) \sim 24x,\\
&\qquad\quad p^{(iv)}(x) \sim 24,\ p^{(v)}(x) = o(x^{-1}).
\end{aligned}$$

From Theorem 5, the following corollary about box dimension of sum of chirp functions was deduced:

COROLLARY 1. ([18]) *Let $0 < \alpha_1$, $0 < \alpha_2$, $\beta \geqslant \alpha = \min\{\alpha_1, \alpha_2\}$. Let*

$$y(x) = C_1 x^{\alpha_1} \sin x^{-\beta} + C_2 x^{\alpha_2} \cos x^{-\beta},$$

*where $C_1$ and $C_2$ are nonzero real constants. Then $y(x)$ is $d$-dimensional fractal oscillatory near the origin, where $d = 2-(\alpha+1)/(\beta+1)$.*

*Proof.* The sum $y(x)$ can be written as $y(x) = p(x)\sin(q(x))$. It can be checked that $y(x)$ is chirp-like in the sense of Theorem 5.

Finally, the following theorem proven by Luka Korkut in [22] is an improved version of Theorem 5 from [18].

THEOREM 6. (Box dimension of chirp-like functions, [22]) *Let* $I = (0, c]$, $c > 0$. *Let* $y(x) = p(x)S(q(x))$, $x \in I$, *where* $p(x) \in C(I) \cap C^1(I)$, $q(x) \in C^1(I)$, $S(t) \in C^1(\mathbb{R})$. *Let* $S(t)$ *be a* $2T$*-periodic real function defined on* $\mathbb{R}$, $T > 0$, *such that*

$$\begin{cases} S(a) = S(a+T) = 0 \textit{ for some } a \in \mathbb{R}, \\ S(t) \neq 0 \textit{ for all } t \in (a, a+T) \cup (a+T, a+2T). \end{cases}$$

*Let moreover* $S(t)$ *alternately change sign on intervals* $(a + (k-1)T, a + kT)$, $k \in \mathbb{N}$. *Without loss of generality, we take* $a = 0$. *Let*

$$p(x) \simeq_1 x^{\alpha}, \quad as \quad x \to 0,$$
$$q(x) \simeq_1 x^{-\beta}, \quad as \quad x \to 0,$$

*where* $0 < \alpha \leqslant \beta$. *Then* $y(x)$ *is* $d$*-dimensional fractal oscillatory near the origin, with* $d = 2 - (\alpha + 1)/(\beta + 1)$.

*Idea of the proof of Theorems* 5 *and* 6. The proof relies on [45]. We check that the conditions of Theorem 1 and of modified Theorem 2 from [45] are fullfilled, and the conclusion follows. For this purpose, Luka Korkut made essential modifications of Theorem 2 in [18]. □

### 3.2. Connection between chirps and spirals

This subsection is devoted to the results from article [22]. Luka Korkut worked in this subject, as well as in subjects of the following two subsections, as PhD advisor of Domagoj Vlah.

Already mentioned several times, fractal connection between oscillatority in the phase plane and oscillatority of the graph of a function, showed as an interesting problem. In this subsection we explain this relation, while in two proceeding subsections we show some applications of the obtained results.

Similary to definitions of oscillatority and oscillatory dimension, here we first introduce definitions of a phase oscillatory function, phase dimension and precisely define a spiral.

DEFINITION 5. Assume now that $x$ is of class $C^1$. We say that $x$ is a *phase oscillatory* function if the following condition holds: the set $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ in the plane is a spiral converging to the origin.

DEFINITION 6. By a *spiral* here we mean the graph of a function $r = f(\varphi)$, $\varphi \geqslant \varphi_1 > 0$, in polar coordinates, where

$$\begin{cases} f : [\varphi_1, \infty) \to (0, \infty) \text{ is such that } f(\varphi) \to 0 \text{ as } \varphi \to \infty, \\ f \text{ is } \textit{radially decreasing} \text{ (i.e., for any fixed } \varphi \geqslant \varphi_1 \\ \text{the function } \mathbb{N} \ni k \mapsto f(\varphi + 2k\pi) \text{ is decreasing).} \end{cases}$$

This definition appears in [54]. Depending on the context, by a spiral we also mean the graph that is a mirror image of the spiral from Definition 6, with respect to the $x$-axis. As expected, by a *spiral near the origin* we mean the graph of function $r = f(\varphi)$, $\varphi \geqslant \varphi_1 > 0$, defined in polar coordinates, such that there exists $\varphi_2 \geqslant \varphi_1$ and the graph of the function $r = f(\varphi)$, $\varphi \geqslant \varphi_2$, viewed in polar coordinates, is a spiral.

DEFINITION 7. The *phase dimension* $\dim_{ph}(x)$ of a function $x : [t_0,\infty) \to \mathbb{R}$ of class $C^1$ is defined as the box dimension of the corresponding planar curve

$$\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}.$$

Phase dimension is, also as the oscillatory dimension, the fractal dimension, introduced in the study of chirp-like solutions of second order ODEs, see [45].

Next, in [22] we introduced the notion of a wavy spiral defined by a wavy function.

DEFINITION 8. Let $r : [t_0,\infty) \to (0,\infty)$ be a $C^1$ function. Assume that $r'(t_0) \leqslant 0$. We say that $r = r(t)$ is a *wavy function* if the sequence $(t_n)$ defined inductively by:

$$\begin{aligned} t_{2k+1} &:= \inf\{t : t > t_{2k}, r'(t) > 0\}, \quad k \in \mathbb{N}_0, \\ t_{2k+2} &:= \inf\{t : t > t_{2k+1}, r(t) = r(t_{2k+1})\}, \quad k \in \mathbb{N}_0, \end{aligned}$$

is well-defined, and satisfies the *waviness condition*:

$$\begin{cases} \text{(i) The sequence } (t_n) \text{ is increasing and } t_n \to \infty \text{ as } n \to \infty. \\ \text{(ii) There exists } \varepsilon > 0 \text{, such that for all } k \in \mathbb{N}_0 \text{ holds } t_{2k+1} - t_{2k} \geqslant \varepsilon. \\ \text{(iii) For all } k \text{ sufficiently large it holds } \underset{t\in[t_{2k+1},t_{2k+2}]}{\operatorname{osc}} r(t) = o\left(t_{2k+1}^{-\alpha-1}\right), \ \alpha \in (0,1), \end{cases}$$

where $\underset{t\in I}{\operatorname{osc}} r(t) = \max_{t\in I} r(t) - \min_{t\in I} r(t)$.

DEFINITION 9. Let a spiral $\Gamma'$, given in polar coordinates by $r = f(\varphi)$, where $f$ is a given function. If there exists increasing or decreasing function of class $C^1$, $\varphi = \varphi(t)$, such that $r(t) = f(\varphi(t))$ is a wavy function, then we say $\Gamma'$ is a *wavy spiral*.

For example, function $r(t) = \sqrt{x^2(t) + \dot{x}^2(t)}$, $t \geqslant t_0 > 0$, for carefully chosen $t_0$, where $x(t) = t^{-\alpha}\sin t$, $\alpha \in (0,1)$, is a wavy function, see Figure 6 below. By defining function $\varphi(t) = t$, in the sense of Definition 9, we get a wavy spiral, see Figure 7 below.

Now we can state the first result here, about the box dimension of a spiral generated by a chirp-like function, which is one of the main results from [22].

THEOREM 7. (Chirp–spiral comparison, [22]) *Let* $\alpha > 0$. *Assume that*

$$X : (0, 1/t_0] \to \mathbb{R}, \ t_0 > 0, \ X(\tau) = P(\tau)\sin 1/\tau,$$

*where* $P(\tau)$ *is a positive function such that* $P(\tau) \sim_3 \tau^\alpha$ *as* $\tau \to 0$. *Define* $x(t) = X(1/t)$ *and a continuous function* $\varphi(t)$ *by* $\tan\varphi(t) = \dot{x}(t)/x(t)$.

(i) *If* $\alpha \in (0,1)$ *then the planar curve* $\Gamma := \{(x(t),\dot{x}(t)) : t \in [t_0,\infty)\}$ *generated by* $X$ *is a wavy spiral* $r = f(\varphi)$, $\varphi \in (-\infty,-\varphi_0]$, $\varphi_0 > 0$, *near the origin. We have* $f(\varphi) \simeq |\varphi|^{-\alpha}$ *as* $\varphi \to -\infty$, *and*

$$\dim_{ph}(x) := \dim_B \Gamma = \frac{2}{1+\alpha}.$$

(ii) *If* $\alpha > 1$ *then the planar curve* $\Gamma := \{(x(t),\dot{x}(t)) : t \in [t_0,\infty)\}$ *is a rectifiable wavy spiral near the origin.*

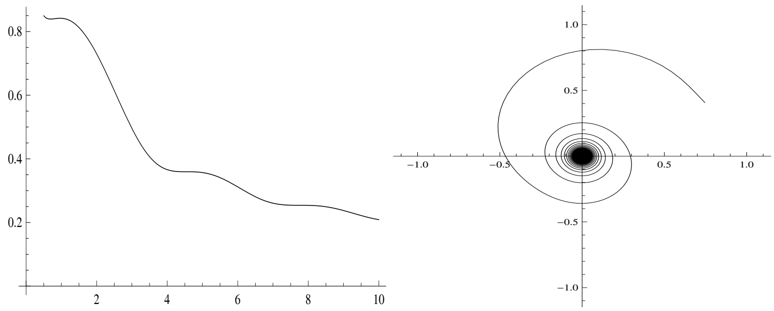



Figure 6: Left picture: graph of wavy function $r(t) = \sqrt{x^2(t)+\dot{x}^2(t)}$, where $x(t) = t^{-\alpha}\sin t$, $\alpha = 2/3$ and $t_0 = 0.5$.

Figure 7: Right picture: graph of wavy spiral given parametrically in polar coordinates $(r(t),\varphi(t))$, where $r(t) = \sqrt{x^2(t)+\dot{x}^2(t)}$, $t \geqslant 0.5$, $x(t) = t^{-\alpha}\sin t$, $\alpha = 2/3$ and $\varphi(t) = t$.

The other two results are, in some way, reversals of Theorem 7. They tell us about the box dimension and rectifiability of a chirp-like function generated by a spiral.

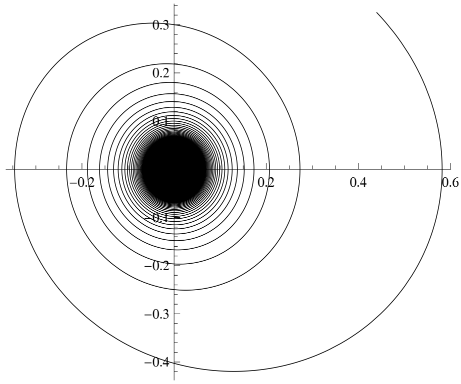
THEOREM 8. (Spiral-chirp comparison, [22]) *Let* $\alpha \in (0,1)$, *and assume that* $x : [t_0,\infty) \to \mathbb{R}$, $t_0 > 0$, *is a function of class* $C^2$, *such that the planar curve*

$$\Gamma = \{(x(t),\dot{x}(t)) : t \in [t_0,\infty)\}$$

*is a spiral* $r = f(\varphi)$, $\varphi \in (\varphi_0,\infty)$, $\varphi_0 > 0$, *in polar coordinates, near the origin, such that* $f(\varphi) \simeq_1 \varphi^{-\alpha}$, *as* $\varphi \to \infty$, *and* $\dot{\varphi}(t) \simeq 1$, *as* $t \to \infty$, *where* $\varphi(t)$ *is a function of class* $C^1$ *defined by* $\tan\varphi(t) = \dot{x}(t)/x(t)$. *Define* $X(\tau) = x(1/\tau)$. *Then* $X = X(\tau)$ *is* $(\alpha,1)$*-chirp-like function, and*

$$\dim_{osc}(x) := \dim_B G(X) = \frac{3-\alpha}{2},$$

*where* $G(X)$ *is graph of the function* $X$. *Furthermore,* $G(X)$ *is Minkowski nondegenerate.*



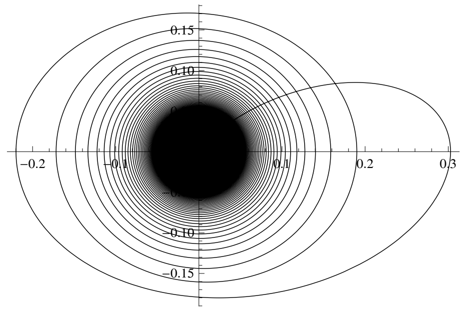




Figure 8: Graph of spiral $(x(t),\dot{x}(t))$, $t \geqslant 1$, generated by $x(t) = J_1(t)$ and $x(t) = J_{10}(t)$ respectively.

THEOREM 9. (Rectifiability of a chirp generated by a rectifiable spiral, [22]) *Let $\alpha > 1$, and assume that $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, is a function of class $C^2$ such that the planar curve $\Gamma = \{(x(t),\dot{x}(t)) : t \in [t_0,\infty)\}$ is a rectifiable spiral $r = f(\varphi)$, $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$ in polar coordinates, near the origin, such that $f(\varphi) \simeq_1 \varphi^{-\alpha}$, as $\varphi \to \infty$, $|f''(\varphi)| \leqslant C\varphi^{-\alpha-2}$ and $\dot{\varphi}(t) \simeq 1$ as $t \to \infty$, where $\varphi(t)$ is a function of class $C^1$ defined by $\tan\varphi(t) = \dot{x}(t)/x(t)$. Define $X(\tau) = x(1/\tau)$. Then $X = X(\tau)$ is $(\alpha,1)$-chirp-like rectifiable function near the origin.*

### 3.3. Bessel functions

These subsection is devoted to results from article [19].

Bessel system is a nonautonomous planar system with non-rectifiable spiral trajectories. It is proved in [19] that the phase dimension of the Bessel equation does not depend on the order of Bessel functions, which are the solutions. For any order, trajectories behave as $1/2$-power-type spirals. Given the fact that the planar Bessel system is nonautonomous, it can also be interpreted as a three-dimensional system with spatial spiral trajectories.

The *Bessel equation of order $\nu$*, widely known in literature, see e.g. [31, p. 98], is the linear second-order ordinary differential equation given by

$$t^2x''(t) + tx'(t) + (t^2 + \nu^2)x(t) = 0,$$

where $\nu \in \mathbb{R}$ is a parameter. Bessel equation has two linearly independent solutions, which are called *Bessel functions* of the first and second kind of order $\nu$, designated $J_\nu$ and $Y_\nu$, respectively. For graphs of spirals $(x(t),\dot{x}(t))$, generated by Bessel function $J_\nu$, for different values of parametar $\nu$, see Figure 8 above.

THEOREM 10. (Phase dimension of Bessel functions [19]) *Phase dimension of Bessel functions $J_\nu$ and $Y_\nu$ is equal to $4/3$, for every $\nu \in \mathbb{R}$.*

### 3.4. Autonomous spatial systems

These subsection is devoted to results from article [21], which also examines some of the other three-dimensional systems that are associated with the study of connection between oscillatority of chirp-type and oscillatority of power-type. It extends work from [53] about the box dimension of spatial spirals, lying on surfaces and accumulating to the origin. It turned out that it is important whether the surface is Lipschitz or Hölder type, that is, whether it has finite or infinite derivative at the origin, respectively.

Here we study, as a model, a class of second-order nonautonomous equations, exhibiting both chirp-like and spiral behavior,

$$\ddot{x} - \frac{2\,p'(t)}{p(t)}\dot{x} + \left[1 + \frac{2p'^2(t)}{p^2(t)} - \frac{p''(t)}{p(t)}\right]x = 0, \quad t \in [t_0,\infty),\ t_0 > 0, \tag{3.1}$$

where function $p$ is of class $C^2$. This equation has explicit solution

$$x(t) = C_1 p(t)\sin t + C_2 p(t)\cos t.$$

Introducing change in variables $z = 1/(t - C_3)$, we acquire cubic system

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \frac{2p'(\frac{1}{z})}{p(\frac{1}{z})}y - \left[1 - \frac{p''(\frac{1}{z})}{p(\frac{1}{z})} + \frac{2p'^2(\frac{1}{z})}{p^2(\frac{1}{z})}\right]x, \quad z \in (0, \frac{1}{t_0}] \\
\dot{z} &= -z^2.
\end{aligned} \tag{3.2}$$

In order to explain fractal behavior of the system (3.2) we need a lemma dealing with a bi-Lipschitz map, see [21]. It is a well known result from [11] that the box dimension of a set is invariant under bi-Lipschitz maps. Putting together these two results we obtain desired results about (3.2). For the sake of simplicity, but with no loss of generality, we work with trajectory $\Gamma$ of the solution of system (3.2) that is defined by

$$\begin{aligned}
x(t) &= p(t)\sin t \\
y(t) &= p'(t)\sin t + p(t)\cos t \\
z(t) &= \frac{1}{t}.
\end{aligned}$$

The following result relies on the fact that trajectory $\Gamma$ has projection $\Gamma_{xy}$ to $(x,y)$-plane which is a planar spiral. For graphs of several trajectories $\Gamma$ for different functions $p(t)$, see Figure 9 below.

THEOREM 11. (Trajectory in $\mathbb{R}^3$, [21]) *Let* $p(t) \sim_3 t^{-\alpha}$, $\alpha > 0$, *as* $t \to \infty$.

(i) *Phase dimension of any solution of the equation* (3.1) *near the origin is equal to* $\dim_{ph}(x) = 2/(1+\alpha)$ *for* $\alpha \in (0,1)$.

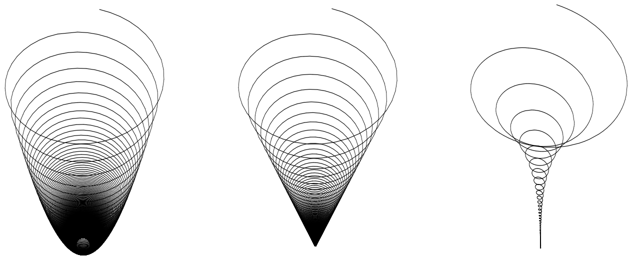


Figure 9: Trajectory $\Gamma$ of the solution of system (3.2) for the function $p(t)=t^{-1/2}$, $p(t)=t^{-1}$ and $p(t)=t^{-4}$ respectively.

(ii) *Trajectory* $\Gamma$ *of the system* (3.2) *near the origin has box dimension* $\dim_B\Gamma = 2/(1+\alpha)$ *for* $\alpha\in(0,1)$.

(iii) *Trajectory* $\Gamma$ *of the system* (3.2) *for* $\alpha>1$ *is a rectifiable spiral and* $\dim_B\Gamma=1$.

The following two theorems proved by Luka Korkut complete the study of system (3.2). A solution of system (3.2) projected to $(x,y)$-plane is a spiral. The following theorem gives us informations about projections to other two coordinate planes. Proof relies on Theorem 8.

THEOREM 12. (Projections, [21]) *Suppose* $p(t)\sim_3 t^{-\alpha}$, $\alpha>0$, *as* $t\to\infty$. *Then the projection* $G_{yz}$ *of a trajectory of the system* (3.2) *to* $(y,z)-$*plane is a* $(\alpha,1)$*-chirp-like function, and* $\dim_B G_{yz}=(3-\alpha)/2$ *if* $\alpha\in(0,1)$. *Analogously for projection* $G_{xz}$.

It is interesting to see that this nonrectifiable case corresponds to spiral contained in Lipschitzian surface, while rectifiable spiral trajectory of the system lies in Hölderian surface.

We would also like to concern rectifiability on the Hölderian surface. In the case of system (3.2), $\alpha>1$ the Hölderian surface does not affect the rectifiability. We have the following theorem that gives sufficient conditions in the case of a more general situation, that is, for a spiral lying in the Hölderian surface $z=g(r)$, $g(r)\simeq r^{\beta}$, $\beta\in(0,1)$. Notice that, if a spiral lies in the Hölderian surface, and tend to the origin, we call it a Hölder-focus spiral.

THEOREM 13. (Rectifiability in $\mathbb{R}^3$, [21]) *Let* $f\,[\varphi_1,\infty)\to(0,\infty)$, $\varphi_1>0$, $f(\varphi)\simeq\varphi^{-\alpha}$, $|f'(\varphi)|\leqslant C\varphi^{-\alpha-1}$, $\alpha>1$, $r=f(\varphi)$ *define a rectifiable spiral. Assume that* $g\,(0,f(\varphi_1))\to(0,\infty)$ *is a function of class* $C^1$ *such that*

$$g(r)\simeq r^{\beta},\quad |g'(r)|\leqslant Dr^{\beta-1},\quad \beta\in(0,1).$$

*Let $\Gamma$ be a Hölder-focus spiral defined by $r = f(\varphi)$, $\varphi \in [\varphi_1, \infty)$, $z = g(r)$, then $\Gamma$ is rectifiable spiral.*

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