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STABILITY OF POSITIVE SOLUTIONS TO $p\&2$–LAPLACE TYPE EQUATIONS

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Abstract. In this article, we first show the existence of a positive solution to

$$\begin{cases} -\Delta_p u - \alpha\Delta u = \lambda(u - f(u)) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

by the method of lower and upper solutions and then under certain conditions on f, we show the stability of positive solution.

1. Introduction

Let us consider the following boundary value problem

$$\begin{cases} -\Delta_p u - \alpha\Delta u = g(x,u) \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for some $\alpha \in \mathbb{R}^+$, where $\Omega \subset \mathbb{R}^N$ is an open, smooth and bounded subset, $N \geqslant 2$ and $2 \leqslant p < \infty$. Solutions of (1.1) are the steady state solutions of the reaction diffusion equation

$$u_t = div(A(u)\nabla u) + g(x,u), \tag{1.2}$$

where $A(u) = \alpha + |\nabla u|^{p-2}$. This equation has applications in science and engineering, see [1] for chemical reactions, [22] for plasma physics, [6] for biophysics and solid states.

For the existence and uniqueness of a positive solution to (1.1), in case $\alpha = 0$, we refer the reader to [12]. In case $\alpha = 1$, (1.1) appears in the investigation of soliton like solutions of

$$i\psi_t = -\Delta_p\psi - \Delta\psi + g(x,\psi), \tag{1.3}$$

which was dealt by G. H. Derrick [10] as a model for elementary particles.

Problems involving the operator $-\Delta_p - \Delta$ have not been studied much so far. For instance, using the fibering method or the mountain pass theorem, N. E. Sidiropoulos [21] obtain the existence of a nonnegative solution of (1.1) for $g(x,u) = a(x)u^{q-1} - b(x)u^{s-1}$,

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where $p < 2, 1 < q$ and $s < 2^*$. For the existence of three non-negative solutions of (1.1) with indefinite nonlinearities $g(x,u)$, we refer the reader to [20]. We refer to [6, 15, 16] for a class of equations involving $-\Delta_p - \Delta_r$, where the existence and regularity of the weak solutions are discussed.

Recently, there are some investigations on the stability of solutions to equations of type (1.1) (when $\alpha = 0$, see [26] and the references therein). We refer the reader to the work of V.Benci and D.Fortunato [2], where they establish the existence, nonexistence and stability results for solitary-wave or kink solution to partial differential equations with variational structure. The authors considered the nonlinear wave equation

$$\psi_{tt} - c^2\Delta\phi + G'(\psi) = 0 \text{ for } \psi : \mathbb{R}^4 \to \mathbb{R}^k$$

and a system in which the nonlinear wave equation is coupled to Maxwell's equation. In [26], we obtain a stability theorem for a class of quasilinear elliptic equations of the form

$$\begin{cases} -\Delta_p u = a(x)u - f(x,u) \ \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $a \in L^\infty(\Omega)$, $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and $f \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ in the y variable such that

$$f_y(x,y) \geqslant \frac{f(x,y)}{y}, \ \forall\, 0 \neq y \in \mathbb{R}.$$

The main aim of this paper is to see whether the stability theorem of [26] can be extended to the following problem:

$$\begin{cases} -\Delta_p u - \alpha\Delta u = \lambda(u - f(u)) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \ \lambda > 0, \end{cases} \tag{1.4}$$

which has a number of applications, see [1, 6, 22].

In fact, using the method of upper and lower solutions as in [17], we show the existence of a positive solution to (1.4) and by extending the results of [26], we obtain the stability theorem to (1.4). We remark that when $\alpha = 1 = \lambda$ in (1.4), our existence results are extension of earlier research work.

We make the following hypotheses on the nonlinearity:

(H1) Let $f \in C(\mathbb{R}^+, \mathbb{R})$ and for any $t_0 > 0$, there exists $A > 0$ such that

$$|t - f(t)| \leqslant A, \ \forall t \in [0, t_0].$$

(H2) $f(0) < 0$ and there exists $\beta > 0$ such that $\beta = f(\beta)$.

(H3) $f'(y) \geqslant \frac{f(y)}{y}, \ \forall\, 0 < y \in \mathbb{R}$.

The organization of this paper is as follows: Section 2 deals with the existence of a positive solution to (1.1) and qualitative results to an eigenvalue problem associated with (1.1). In Section 3, we show the stability of positive solution.

2. Auxiliary results

In this section, we show the existence of a solution to (1.1) by the method of lower-upper solution and also discuss the qualitative results to an eigenvalue problem associated with (1.1). Let $g(x,t)$ be a Carathédory function on $\Omega\times\mathbb{R}$ with the property that for any $t_0>0$, there exists a constant A such that $|g(x,t)|\leqslant A$ for a.e. $x\in\Omega$ and all $t\in[-t_0,t_0]$.

A function $u\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ is called a (weak) lower solution of the problem (1.1) if $u\leqslant 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx+\alpha\int_\Omega \nabla u\nabla\phi dx\leqslant\int_\Omega g(x,u)\phi dx$$

for all $\phi\in C_c^\infty(\Omega)$, $\phi\geqslant 0$. Similarly, we can define upper solution by reversing the inequality signs.

PROPOSITION 2.1. *Assume that $\underline{u}$ and $\overline{u}$ are respectively lower and upper solutions for* (1.1)*, with $\underline{u}\leqslant\overline{u}$ a.e. in Ω. Let us consider the associated functional*

$$E(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx+\frac{\alpha}{2}\int_\Omega|\nabla u|^2dx-\int_\Omega G(x,u)dx,$$

where

$$G(x,s)=\int_0^s g(x,t)dt.$$

Let

$$M=\{u\in W_0^{1,p}(\Omega)|\underline{u}\leqslant u\leqslant\overline{u}\; a.e.\; in\;\Omega\}.$$

Then E attains the infimum at some point $u\in M$ and u is a solution of (1.1).

Proof. The proof is adapted from [9] or p. 17[22] which deal with the quasilinear and semilinear cases, respectively. Since the proof is short and interesting, so we repeat it here. By coercivity and weak lower semicontinuity, one can easily see that the infimum of E is achieved at some $u\in M$. Let $\phi\in C_c^\infty(\Omega)$, $\varepsilon>0$, and define

$$v_\varepsilon:=\min\{\overline{u},\max\{\underline{u},u+\varepsilon\phi\}\}=u+\varepsilon\phi-\phi^\varepsilon+\phi_\varepsilon,$$

where

$$\phi^\varepsilon:=\max\{0,u+\varepsilon\phi-\overline{u}\}\quad\text{and}\quad\phi_\varepsilon:=-\min\{0,u+\varepsilon\phi-\underline{u}\}.$$

Since u minimizes E on M and E is a C^1 functional on $W_0^{1,p}(\Omega)$, so we have $<E'(u),v_\varepsilon-u>\geqslant 0$, which gives

$$<E'(u),\phi>\geqslant\frac{(<E'(u),\phi^\varepsilon>-<E'(u),\phi_\varepsilon>)}{\varepsilon}. \tag{2.1}$$

Since $\overline{u}$ is an upper solution and $-\Delta_p$ is monotone, we have

$$<E'(u),\phi^\varepsilon>\geqslant<E'(u)-E'(\overline{u}),\phi^\varepsilon>$$

$$\geqslant \varepsilon\left[\int_{\Omega_\varepsilon}(|\nabla u|^{p-2}\nabla u-|\nabla\overline{u}|^{p-2}\nabla\overline{u})\nabla\phi\right.$$
$$\left.+(\nabla u-\nabla\overline{u})\nabla\phi\right]-\varepsilon\int_{\Omega_\varepsilon}|g(x,u)-g(x,\overline{u})||\phi|, \tag{2.2}$$

where $\Omega_\varepsilon=\{x\in\Omega : u(x)+\varepsilon\phi x\geqslant\overline{u}(x)>u(x)\}$. Now $|\Omega_\varepsilon|\to 0$ as $\varepsilon\to 0$, the last inequality implies that

$$<E'(u),\,\phi^\varepsilon>\geqslant 0(\varepsilon)\;\text{ as }\varepsilon\to 0.$$

Similarly,

$$<E'(u),\,\phi_\varepsilon>\leqslant 0(\varepsilon)\;\text{ as}\,\varepsilon\to 0$$

and by (2.1), we get

$$<E'(u),\,\phi>\geqslant 0.$$

Replacing ϕ by $-\phi$, one concludes that u solves (1.1).

Using the ideas as in [17], we show the existence of a positive solution to (1.4).

THEOREM 2.2. *Let* (H1) and (H2) hold. Then (1.4) has a positive solution.

Proof. By (H2), it is easy to see that $\underline{u}=0$ is a subsolution of (1.4). In fact, using the fact that $f(0)<0$, it is a strict subsolution of (1.4). Again by (H2), one can see that $\overline{u}=\beta>0$ is an upper solution of (1.4). Now since (H1) holds so an application of Proposition 2.1 yields the existence of a positive solution to (1.4).

REMARK 2.3. We remark that in the above theorem, $f(0)<0$ is used to construct a strict subsolution to (1.4). In case $f(0)=0$, using the similar ideas of [17], it seems possible to construct positive subsolution and therefore one can establish the existence of a positive solution to (1.4), by lower-upper solution method. We leave this as an exercise for interesting reader.

Next, we discuss the existence of first eigenvalue and qualitative questions to the following weighted eigenvalue problem

$$\begin{cases}-\Delta_p\psi-\alpha\Delta\psi=\lambda c(x)|\psi|^{p-2}\psi & \text{in }\Omega,\\ \psi=0 \quad \text{on }\partial\Omega,\end{cases} \tag{2.3}$$

where $c\in L^\infty(\Omega)$ and it may be allowed to be of sign changing nature.

LEMMA 2.4. *Let*

$$M=\{u\in W_0^{1,p}(\Omega)|\;\frac{1}{p}\int_\Omega c(x)|u|^p dx=1\}.$$

Then M is a weakly closed subset of $W_0^{1,p}(\Omega)$.

Proof. Let $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. We claim that $u \in M$. Since $u_n \in M$, we have

$$\frac{1}{p}\int_\Omega c(x)|u_n|^p dx = 1.$$

Since $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$, $u_n \to u$ strongly in $L^p(\Omega)$. Up to a subsequence (still denoted by $\{u_n\}$), $u_n(x) \to u(x)$ a.e. $x \in \Omega$ and there exists $h \in L^p(\Omega)$ such that $|u_n(x)| \leqslant h(x)$ a.e. $x \in \Omega$. Now using the fact that $c \in L^\infty(\Omega)$, by Lebesgue dominated convergence theorem, one can see that

$$1 = \frac{1}{p}\int_\Omega c(x)|u_n|^p dx \to \frac{1}{p}\int_\Omega c(x)|u|^p dx.$$

This implies that $\frac{1}{p}\int_\Omega c(x)|u|^p dx = 1$ and therefore $u \in M$.

Using similar arguments as in [25], the next lemma deals with the first eigenvalue of (2.3).

LEMMA 2.5. *Let*

$$\lambda_1(c) = \inf\Big\{ \int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{\alpha}{2}|\nabla u|^2)dx \\ : u \in W_0^{1,p}(\Omega) \text{ and } \frac{1}{p}\int_\Omega c(x)|u|^p dx = 1 \Big\}. \quad (2.4)$$

Then $\lambda_1(c)$ is achieved and $\lambda_1(c)$ is the least positive eigenvalue of (2.3)*. Moreover, $\lambda_1(c) = \Phi(u)$ for some $u \in M$ if and only if u is an eigenfunction associated with $\lambda_1(c)$, where*

$$\Phi(u) = \int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{1}{2}|\nabla u|^2)dx$$

and

$$M = \{u \in W_0^{1,p}(\Omega) | \ \frac{1}{p}\int_\Omega c(x)|u|^p dx = 1\}.$$

Proof. Let us define the functional associated with (2.3):

$$E : M \longrightarrow \mathbb{R} \text{ by}$$

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx, \ u \in M.$$

By Lemma 2.4, M is a weakly closed subset of $W_0^{1,p}(\Omega)$. It is easy to see that E is coercive and weakly lower semicontinuous functional on M. Then by Theorem 1.2 [22], E is bounded from below on M and attains its infimum, denoted by m, i.e.,

$$E(m) = \min_M E.$$

One can also see that E is continuous, Gâteaux differentiable and the derivative of E is given by

$$< E'(u), v >= \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx + \alpha \int_\Omega \nabla u.\nabla v dx,$$

$\forall u, v \in W_0^{1,p}(\Omega)$. It is easy to see that E' is continuous on $W_0^{1,p}(\Omega)$. Then by standard Lagrange multiplier rule, the minimizer m solves (2.3) in the weak sense.

LEMMA 2.6. *Suppose that u is a weak solution of*

$$\begin{cases} -\Delta_p u - \alpha \Delta u = b(x) & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2.5}$$

where $b : \Omega \to [0, \infty)$ is a $L^\infty(\Omega)$ function. Then $u \geqslant 0$ in $\overline{\Omega}$.

Proof. Let

$$\Omega^+ = \{x \in \Omega | u(x) \geqslant 0\} \qquad \text{and} \qquad \Omega^- = \{x \in \Omega | u(x) < 0\}.$$

Let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The weak formulation of (2.5) with test function u^- yields

$$\int_{\Omega^-} |\nabla u^-|^p dx + \alpha \int_{\Omega^-} |\nabla u^-|^2 dx = -\int_{\Omega^-} b(x) u^- dx.$$

This implies that $\nabla u^- = 0$ in $\overline{\Omega}$ and so u^- is constant in $\overline{\Omega}$. By standard regularity theory, u is continuous in $\overline{\Omega}$ and therefore is u^-. Since $u^- = 0$ on $\partial\Omega$. This implies that $u^- = 0$ in Ω and hence $u = u^+ \geqslant 0$. This completes the proof.

From [13, 14], the following "Strong maximum principle" holds.

LEMMA 2.7. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative weak solution of*

$$-\Delta_p u - \alpha \Delta u = \lambda a(x) u \ in\ \Omega; \ \ u = 0 \ on\ \partial\Omega, \tag{2.6}$$

where $0 < a \in L^\infty(\Omega)$. Then either $u \equiv 0$ or $u > 0$ in Ω.

PROPOSITION 2.8. *The eigenfunctions associated with $\lambda_1(c)$ are either positive or negative in Ω.*

Proof. Let $u \in M$ be an eigenfunction associated with $\lambda_1(c)$. Then u achieves the infimum in (2.4). Since $||\nabla|u|||_p + ||\nabla|u|||_2 = ||\nabla u||_p + ||\nabla u||_2$ and $|u| \in M$, it follows that $|u|$ achieves the infimum in (2.4) also and therefore, from Lemma 2.5, $|u|$ is an eigenfunction for $\lambda_1(c)$. By Lemma 2.7, we conclude that $|u(x)| > 0$, $\forall x \in \Omega$ and therefore u is either positive or negative in Ω.

3. Stability

In this section, we consider the stability of nontrivial weak solutions of (1.4). The weak formulation of (1.4) is the following:

$$\int_\Omega (|\nabla u|^{p-2}\nabla u.\nabla\phi + \alpha\nabla u.\nabla\phi)dx = \lambda(\int_\Omega u\phi\, dx - \int_\Omega f(u)\phi\, dx),\ \forall\ \phi \in C_c^1(\Omega), \quad (3.1)$$

where $C_c^1(\Omega)$ is the space of C^1 functions in Ω having a compact support in Ω. A solution u of (1.4) satisfies (3.1) and by the well-known elliptic regularity theory, thanks to [7, 8] for $C^{1,\alpha}(\Omega)$ regularity $(\alpha \in (0,1))$ of solutions to (1.4) with positive nonlinearities, which are independent of x and to [11, 18, 23] for the same regularity of solutions to (1.4) where the solution u is assumed to be in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In this paper, we assume u to be in $C^{1,\alpha}(\Omega)$.

The functional associated with (1.4) is

$$E : W_0^{1,p}(\Omega) \to \mathbb{R} \quad \text{defined by}$$

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx + \lambda\int_\Omega F(u)dx,$$

where

$$F(s) = \int_0^s f(t)dt.$$

In order to define the stability of solutions to (1.4), let us consider the weighted Sobolev space with weight

$$w(x) = |\nabla u(x)|^{p-2}.$$

As in [7, 19, 24], let us denote the space by $H_w^{1,2}(\Omega)$, which is defined as the closure of $C^1(\Omega)$ or $(C^\infty(\Omega))$ with respect to the $||.||_{H_w^{1,2}(\Omega)}$ norm defined as follows:

$$\begin{aligned} ||v||_{H_w^{1,2}(\Omega)} := & \quad ||v||_{L^2(\Omega)} + ||\nabla v||_{L_w^2(\Omega)} \\ = & \quad \left(\int_\Omega |v(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega w(x)|\nabla v(x)|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

We define $H_{w,0}^{1,2}(\Omega)$ to be the closure of $C_c^1(\Omega)$ with respect to the $H_w^{1,2}(\Omega)$-norm.

The linearized operator L_u associated with (1.4) at a given solution u is defined by the following duality:

$$L_u : v \in H_{w,0}^{1,2}(\Omega) \to L_u(v) \in (H_{w,0}^{1,2}(\Omega))', \text{ where}$$

$$L_u(v) : \psi \in H_{w,0}^{1,2}(\Omega) \to L_u(v,\psi) \ \text{ and}$$

$$\begin{aligned} L_u(v,\psi) = \int_\Omega (&|\nabla u|^{p-2}(\nabla v.\nabla\psi) + (p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)(\nabla u.\nabla\psi) \\ &+ \alpha\nabla v.\nabla\psi - \lambda v\psi + \lambda f'(u)v\psi)dx. \end{aligned}$$

It is easy to see that L_u is well-defined and the first eigenvalue of L_u is given by

$$\lambda_1 = \inf_{v\in H^{1,2}_{w,0}(\Omega),\, v\neq 0} \frac{L_u(v,v)}{\int_\Omega v^2 dx}. \tag{3.2}$$

For the details of spectral theory of linearized p-Laplace equations, we refer to [3]. We notice that the stability of the solution to (1.4) can be defined without its associated heat equation. We recall that a solution u of (1.4) is said to be stable if

$$\int_\Omega \Big((p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)^2 + |\nabla u|^{p-2}|\nabla v|^2 + \alpha|\nabla v|^2 \\ - \lambda v^2 + \lambda f'(u)v^2 \Big) dx \geqslant 0, \tag{3.3}$$

for every $v \in C^1_c(\Omega)$, (see [3, 4]). We remark that the left hand side of (3.3) is nothing but the second variation of the energy functional $E(u)$ and we point out that for $p \neq 2$, it is well-defined only in the weighted Sobolev space, see [5].

Actually, (3.3) implies that the principal eigenvalue of the linearized equation associated with (1.4) is nonnegative and hence the solution u of (1.4) is stable.

In the ensuing theorem, we show the stability of a positive solution to (1.4).

THEOREM 3.1. *Let $(H1)$-$(H3)$ be satisfied. Then the positive solution u of* (1.4) *is stable.*

Proof. Since (H1)-(H2) hold so by an application of Theorem 2.2, (1.4) has a positive solution $u \in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$ and by regularity theory, u is in $C^{1,\alpha}(\Omega)$. Let for any $v \in C^1_c(\Omega)$, we take

$$\phi = \frac{v^2}{u}$$

as a test function in (3.1). Since

$$\nabla\phi = \frac{2uv\nabla v - v^2\nabla u}{u^2},$$

so from (3.1), we get

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\left[\frac{2uv\nabla v - v^2\nabla u}{u^2}\right]dx + \alpha\int_\Omega \nabla u.\left[\frac{2uv\nabla v - v^2\nabla u}{u^2}\right]dx \\ = \int_\Omega \lambda v^2 dx - \lambda\int_\Omega \frac{f(u)v^2}{u}dx.$$

This implies that

$$0 = \int_\Omega |\nabla u|^{p-2}\nabla u.\left[\frac{2uv\nabla v - v^2\nabla u}{u^2}\right]dx + \alpha\int_\Omega \nabla u.\left[\frac{2uv\nabla v - v^2\nabla u}{u^2}\right]dx$$

$$
\begin{aligned}
&\quad -\lambda\int_\Omega v^2dx+\lambda\int_\Omega \frac{f(u)v^2}{u}dx\\
&=\int_\Omega\left[\frac{2v|\nabla u|^{p-2}\nabla u.\nabla v}{u}-\frac{v^2|\nabla u|^{p-2}\nabla u.\nabla u}{u^2}-\lambda v^2+\lambda\frac{f(u)v^2}{u}\right]dx\\
&\quad +\alpha\int_\Omega\left(\frac{2v\nabla u.\nabla v}{u}-\frac{v^2\nabla u.\nabla u}{u^2}\right)dx\\
&=\int_\Omega\Bigg[-\lambda v^2+\lambda\frac{f(u)v^2}{u}-\left(\left(\frac{v(\nabla u.\nabla u)^{\frac{p}{4}}}{u}\right)^2-\frac{2v|\nabla u|^{p-2}\nabla u.\nabla v}{u}\right)\\
&\qquad -(\nabla u.\nabla u)^{\frac{p-4}{2}}(\nabla u.\nabla v)^2\Bigg]dx\\
&\quad +\int_\Omega(\nabla u.\nabla u)^{\frac{p-4}{2}}(\nabla u.\nabla v)^2dx+\int_\Omega\left[\alpha|\nabla v|^2-\alpha\left(\frac{v\nabla u}{u}-\nabla v\right)^2\right]dx\\
&=\int_\Omega\Bigg[-\lambda v^2+\lambda\frac{f(u)v^2}{u}-\left(\frac{v(\nabla u.\nabla u)^{\frac{p}{4}}}{u}-(\nabla u.\nabla u)^{\frac{p-4}{4}}\nabla u.\nabla v\right)^2\\
&\quad +|\nabla u|^{p-4}(\nabla u.\nabla v)^2\Bigg]dx+\int_\Omega\left[\alpha|\nabla v|^2-\alpha\left(\frac{v\nabla u}{u}-\nabla v\right)^2\right]dx. \qquad (3.4)
\end{aligned}
$$

From (3.4), we see that

$$
\begin{aligned}
&\int_\Omega\left[|\nabla u|^{p-4}(\nabla u.\nabla v)^2+\alpha|\nabla v|^2-\lambda v^2+\lambda\frac{f(u)v^2}{u}\right]dx\\
&\quad=\int_\Omega\left[\left(\frac{v(\nabla u.\nabla u)^{\frac{p}{4}}}{u}-(\nabla u.\nabla u)^{\frac{p-4}{4}}\nabla u.\nabla v\right)^2+\alpha\left(\frac{v\nabla u}{u}-\nabla v\right)^2\right]dx\\
&\quad\geqslant 0. \qquad (3.5)
\end{aligned}
$$

Since

$$|\nabla u|^2|\nabla v|^2\geqslant(\nabla u.\nabla v)^2,$$

so we get

$$
\begin{aligned}
&\int_\Omega[(p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)^2+|\nabla u|^{p-2}|\nabla v|^2]dx\\
&\geqslant\int_\Omega[(p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)^2+|\nabla u|^{p-4}(\nabla u.\nabla v)^2]dx\\
&=\int_\Omega[(p-1)|\nabla u|^{p-4}(\nabla u.\nabla v)^2]dx\\
&\geqslant\int_\Omega[|\nabla u|^{p-4}(\nabla u.\nabla v)^2]dx.
\end{aligned} \qquad (3.6)
$$

From (3.5) and (3.6), we get

$$\int_\Omega \Big[(p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)^2 + |\nabla u|^{p-2}|\nabla v|^2 + \alpha|\nabla v|^2 - \lambda v^2 + \lambda \frac{f(u)v^2}{u}\Big] dx \geqslant 0. \quad (3.7)$$

Now by hypothesis (H3), we obtain

$$\int_\Omega \Big[(p-2)|\nabla u|^{p-4}(\nabla u.\nabla v)^2 + |\nabla u|^{p-2}|\nabla v|^2 + \alpha|\nabla v|^2 - \lambda v^2 + \lambda f'(u)v^2\Big] dx \geqslant 0, \quad (3.8)$$

which completes the proof.

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