Differential
Equations
& Applications
Volume 5, Number 4 (2013), 561–576
doi:10.7153/dea-05-33

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS INVOLVING HARDY–SOBOLEV CRITICAL NONLINEARITY

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(*Communicated by Dongsheng Kang*)

Abstract. This paper is concerned with a singular elliptic system, which involves the Hardy-Sobolev critical nonlinearity. The existence and multiplicity of solutions for this system are obtained by the variational methods.

1. Introduction

The aim of this paper is to establish the existence and multiplicity of solutions to the following semilinear elliptic system

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) - \mu\frac{u}{|x|^{2(1+a)}} = \frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^{bp}} + \lambda\frac{\partial}{\partial u}F(x,u,v), & x\in\Omega, \\ -\mathrm{div}(|x|^{-2a}\nabla v) - \mu\frac{v}{|x|^{2(1+a)}} = \frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{bp}} + \lambda\frac{\partial}{\partial v}F(x,u,v), & x\in\Omega, \\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $0\in\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N\geqslant 3)$ with smooth boundary $\partial\Omega$,

$$0\leqslant a<\sqrt{\overline{\mu}},\ \ \overline{\mu}\triangleq\left(\frac{N-2}{2}\right)^2, \quad \text{and} \quad 0\leqslant\mu<(\sqrt{\overline{\mu}}-a)^2,$$

$a\leqslant b<a+1$, $\lambda>0$, $\alpha,\beta>1$ satisfy

$$\alpha+\beta=p=p(a,b)\triangleq\frac{2N}{N-2(1+a-b)}$$

is the Hardy- Sobolev critical exponent. Note that

$$p=p(a,a)=\frac{2N}{N-2}=2^*$$

is the Sobolev critical exponent. F is a real function satisfying some assumptions.

Mathematics subject classification (2010): 35A15, 35B33, 35J70.
Keywords and phrases: solution, critical Hardy-Sobolev exponent, elliptic system.

Paper DEA-05-33

In the case $\mu = 0$, problem (1.1) is related to the well known Caffarelli-Kohn-Nirenberg inequalities in [4],

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p dx\right)^{\frac{2}{p}} \leqslant C_{a,b}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N), \tag{1.2}$$

where

$$-\infty \leqslant a < \sqrt{\overline{\mu}}, \ \ a \leqslant b \leqslant a+1, \quad \text{and } p = \frac{2N}{N-2(1+a-b)}.$$

For particular constants and extremal functions, see [6]. As $b = 1+a$ and $p = 2$ in (1.2), we have the following weighted Hardy inequality [6, 7],

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2(1+a)}}dx \leqslant \frac{1}{(\sqrt{\overline{\mu}}-a)^2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N). \tag{1.3}$$

When $a = 0$, (1.3) becomes the well known Hardy inequality,

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}dx \leqslant \frac{1}{\overline{\mu}}\int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N).$$

By using the inequality (1.2) and the boundedness of Ω, it was proved in [15] that there exists $C > 0$ such that

$$\left(\int_{\Omega} |x|^{-\delta}|u|^r dx\right)^{\frac{2}{r}} \leqslant C\int_{\Omega} |x|^{-2a}|\nabla u|^2 dx, \quad \text{for all} \quad u \in H_0^1(\Omega,|x|^{-2a}), \tag{1.4}$$

where $1 \leqslant r \leqslant \frac{2N}{N-2}$, $\delta \leqslant (a+1)r + N[1-(r/2)]$, which is known Caffarelli-Kohn-Nirenberg's inequality. In other words, the embedding $H_0^1(\Omega,|x|^{-2a}) \hookrightarrow L^r(\Omega,|x|^{-\delta})$ is continuous if

$$1 \leqslant r \leqslant \frac{2N}{N-2} \quad \text{and} \quad \delta \leqslant (a+1)r + N[1-(r/2)].$$

Moreover, this embedding is compact if

$$1 \leqslant r < \frac{2N}{N-2} \quad \text{and} \quad \delta < (a+1)r + N[1-(r/2)],$$

(see [15] Theorem 2.1).

For $\mu \in [0,(\sqrt{\overline{\mu}}-a)^2)$, we define the space $H = H_0^1(\Omega,|x|^{-2a}) \times H_0^1(\Omega,|x|^{-2a})$ with the norm

$$||(u,v)||^2 = ||u||^2 + ||v||^2, \quad \text{where} \quad ||u||^2 = \int_{\Omega}\left(|x|^{-2a}|\nabla u|^2 - \mu\frac{|u|^2}{|x|^{2(1+a)}}\right)dx,$$

which $||u||$ is equivalent to the usual norm of $H_0^1(\Omega,|x|^{-2a})$, resulting from (1.3). We can also define the best Hardy-Sobolev constant:

$$\widetilde{A}_{a,b,\mu}(\Omega) := \inf_{(u,v)\in H\setminus\{(0,0)\}} \frac{||(u,v)||^2}{\left(\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^{bp}}dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{1.5}$$

Modifying the proof of Theorem 5 in [1], we can easily deduce that

$$\widetilde{A}_{a,b,\mu}(\Omega) = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right] A_{a,b,\mu}(\Omega), \tag{1.6}$$

where

$$A_{a,b,\mu}(\Omega) := \inf_{u\in H_0^1(\Omega,|x|^{-2a})\setminus\{0\}} \frac{||u||^2}{\left(\int_\Omega \frac{|u|^p}{|x|^{bp}}dx\right)^{\frac{2}{p}}}.$$

Here are the main results of this paper.

THEOREM 1. *Suppose that*

$$N \geqslant 3(1+a),\ 0\leqslant a<\sqrt{\overline{\mu}},\ 0\leqslant \mu < (\sqrt{\overline{\mu}}-a)^2,\ a\leqslant b<a+1,$$

and F satisfies:
(H1) $F\in C^1(\overline{\Omega},\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ and $F(x,0,0)=\frac{\partial F}{\partial u}(x,0,v)=\frac{\partial F}{\partial v}(x,u,0)=0$;
(H2) $0<F(x,s,t)\leqslant e_1 s\frac{\partial F}{\partial u}(x,s,t)+e_2 t\frac{\partial F}{\partial v}(x,s,t),\ \forall (s,t)\in\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\},\ x\in\overline{\Omega}$, where $e_1,e_2\in(\frac{1}{p},\frac{1}{2})$;
(H3) there exist $1<p_i<q$ (where $q\in(2,2^*]$), $i=1,2$, R_1 and R_2 such that

$$s\frac{\partial F}{\partial u}(x,s,t)+t\frac{\partial F}{\partial v}(x,s,t)\leqslant R_1(s^{p_1}+t^{p_2}),\ \textit{if}\ s+t\geqslant R_2\in\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\},$$

for all $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$ and for a.e $x\in\overline{\Omega}$;
(H4) let $f_0=\inf_{|(s,t)|=1}F(x,s,t)>0,\ (s,t)\in\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\},\ x\in\overline{\Omega}$.
Assume that

$$\rho := \frac{1}{\max\{e_1,e_2\}} > \max\left\{2,\frac{N}{\gamma},\frac{N-2\beta}{\sqrt{\overline{\mu}}-a}\right\} \triangleq r_0, \tag{1.7}$$

where

$$\beta \triangleq \sqrt{(\sqrt{\overline{\mu}}-a)^2-\mu}\ \textit{and}\ \gamma\triangleq\sqrt{\overline{\mu}}-a+\beta.$$

Then there exists $\lambda^*>0$ such that the problem (1.1) possesses one positive solution for every $\lambda\in(0,\lambda^*)$.

THEOREM 2. *Suppose that*

$$N \geqslant 3(1+a),\ 0 \leqslant a < \sqrt{\overline{\mu}},\ 0 \leqslant \mu < (\sqrt{\overline{\mu}}-a)^2,\ a \leqslant b < a+1$$

and F *satisfies:*
(H1') $F \in C^1(\overline{\Omega}, \mathbb{R}^2, \mathbb{R})$ *and* $F(x,0,0) = \frac{\partial F}{\partial u}(x,0,v) = \frac{\partial F}{\partial v}(x,u,0) = 0$*;*
(H2') $0 < F(x,s,t) \leqslant e_1 s \frac{\partial F}{\partial u}(x,s,t) + e_2 t \frac{\partial F}{\partial v}(x,s,t),\ \forall (s,t) \in \mathbb{R}^2 \setminus \{(0,0)\},\ x \in \overline{\Omega}$, *where* $e_1, e_2 \in (\frac{1}{p}, \frac{1}{2})$*;*
(H3') there exist $1 < p_i < q$ *(where* $q \in (2, 2^*]$*),* $i = 1,2$*,* R_1 *and* R_2 *such that*

$$\left| s\frac{\partial F}{\partial u}(x,s,t) + t\frac{\partial F}{\partial v}(x,s,t) \right| \leqslant R_1(|s|^{p_1} + |t|^{p_2}), \qquad \text{if } |s|+|t| \geqslant R_2 \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\},$$

for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ *and for a.e* $x \in \overline{\Omega}$;
(H4') let $f_0 = \inf_{|(s,t)|=1} F(x,s,t) > 0,\ (s,t) \in \mathbb{R}^2 \setminus \{(0,0)\},\ x \in \overline{\Omega}$.

Assume that (1.7) holds. Then there exists $\lambda^* > 0$ *such that the problem (1.1) possesses one positive solution for every* $\lambda \in (0, \lambda^*)$.

For example, in the following, it holds that the conditions (H1)-(H4) and (H1')-(H4') of Theorems 1 and 2 holds:

$$F(x,u,v) = |u|^{\theta}\sin(u) + |v|^{\gamma}\sin(v), \quad (u,v) \in (0, \frac{\gamma\pi}{2}] \times (0, \frac{\gamma\pi}{2}], \quad \frac{1}{p} < \theta, \gamma < \frac{1}{2}.$$

Then

$$\frac{\partial F}{\partial u} = \theta|u|^{\theta-2}u\sin(u) + |u|^{\theta}\cos(u),\ (u,v) \in \left(0, \frac{\gamma\pi}{2}\right] \times \left(0, \frac{\gamma\pi}{2}\right],$$
$$\frac{\partial F}{\partial v} = \gamma|v|^{\gamma-2}v\sin(v) + |v|^{\gamma}\cos(v),\ (u,v) \in \left(0, \frac{\gamma\pi}{2}\right] \times \left(0, \frac{\gamma\pi}{2}\right],$$

now, by the Formulas of $F(x,u,v)$, $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ it is obvious that the (H1) and (H1') hold true. We know that $\sin(u) \leqslant \cos(u)$ for $u \in (0, \frac{\gamma\pi}{2}]$; therefore, (H2) and (H2') hold true. By the inequality $\sin(u) \leqslant u$, if we get $p_1 = \theta + 1$ and $p_2 = \gamma + 1$; then (H3) and (H3') hold true with p_i $(i = 1,2)$ in the certain interval. By the Mountain-Pass Theorem, we can show that a given functional F having a local extremum, so (H4) and (H4') hold true.

In recent years, much attention has been paid to the existence of nontrivial solutions for the singular elliptic problems concerning the operator $\triangle u - \mu\frac{u}{|x|^2}$ $(0 \leqslant \mu < \overline{\mu})$ with Sobolev critical exponents (the case that $a = b = 0$) (see [5, 6, 8] and their references). Some authors have also studied the singular problems with Hardy-Sobolev critical exponents (the case that $a \neq 0,\ b \neq 0$) (see [9, 10, 11, 12, 13, 16]). Since the embedding $H_0^1(\Omega, |x|^{-2a}) \hookrightarrow L^{2^*}(\Omega)$ is not compact, the corresponding energy functional does not satisfy the (PS) condition globally, which caused a serious difficulty when trying to find critical points by standard variational methods. However, we use argument of Brezis and Nirenberg [3] to verify that the associated functional satisfies the Palais-Smale condition on a given interval of the real line.

In this work, motivated by the above works we are interested to study the problem (1) by using the Mountain-Pass Theorem due to Rabinowitz [14].

This paper is divided into three sections, organized as follows. In Section 2, we establish preliminaries and some elementary results. Finally, in Section 3, we prove our main results (Theorems 1 and 2).

2. Preliminaries

Let $u^{\pm} = \max\{\pm u, 0\}$. The corresponding energy functional of problem (1.1) is defined by

$$J(u,v) = \frac{1}{2}\int_{\Omega}\Big(|x|^{-2a}|\nabla u|^2 + |x|^{-2a}|\nabla v|^2 - \mu\frac{|u|^2}{|x|^{2(1+a)}} - \mu\frac{|v|^2}{|x|^{2(1+a)}}\Big)dx$$
$$- \frac{2}{\alpha+\beta}\int_{\Omega}\frac{(u^+)^{\alpha}(v^+)^{\beta}}{|x|^{bp}}dx - \lambda\int_{\Omega}F(x,u^+,v^+)dx,$$

for each $(u,v) \in H$. Then $J \in C^1(H,\mathbb{R})$. Now, it is well known that there exists a one to one correspondence between the weak solutions of problem (1.1) and the critical points of J on H. More precisely, we say that $(u,v) \in H$ is a weak solution of problem (1.1), if for any $(\varphi_1,\varphi_2) \in H$, there holds

$$\langle J'(u,v),(\varphi_1,\varphi_2)\rangle = \frac{1}{2}\int_{\Omega}\Big(|x|^{-2a}\nabla u\nabla\varphi_1 + |x|^{-2a}\nabla v\nabla\varphi_2 - \mu\frac{u\varphi_1 + v\varphi_2}{|x|^{2(1+a)}}\Big)dx$$
$$- \frac{2\alpha}{\alpha+\beta}\int_{\Omega}\frac{(u^+)^{\alpha-1}(v^+)^{\beta}}{|x|^{bp}}\varphi_1 dx$$
$$- \frac{2\beta}{\alpha+\beta}\int_{\Omega}\frac{(u^+)^{\alpha}(v^+)^{\beta-1}}{|x|^{bp}}\varphi_2 dx$$
$$- \lambda\int_{\Omega}\big(\frac{\partial F}{\partial u}(x,u^+,v^+)\varphi_1 + \frac{\partial F}{\partial v}(x,u^+,v^+)\varphi_2\big)dx.$$

LEMMA 1. *Assume that*

$$N \geqslant 3(1+a),\ 0 \leqslant a < \sqrt{\overline{\mu}},\ 0 \leqslant \mu < (\sqrt{\overline{\mu}} - a)^2,\ a \leqslant b < a+1 \text{ and } \lambda > 0.$$

Suppose that (H1)-(H3) and (1.7) hold. Then the functional J satisfies the $(PS)_c$ condition for all

$$0 < c < c^* := \frac{p(a,b)-2}{2p}(\widetilde{A}_{a,b,\mu}(\Omega))^{\frac{p}{p-2}}.$$

Proof. Suppose $\{(u_n,v_n)\} \subset H$ satisfies $J(u_n,v_n) \to c$ and $J'(u_n,v_n) \to 0$ with $c < c^*$. Together (H2), we get as $n \to \infty$ the following:

$$
\begin{aligned}
c+||(u_n,v_n)||+o_n(1) &\geqslant J(u_n,v_n)-\langle J'(u_n,v_n),(e_1u_n,e_2v_n)\rangle \\
&= \big(\frac{1}{2}-e_1\big)||u_n||^2+\big(\frac{1}{2}-e_2\big)||v_n||^2 \\
&\quad \lambda\int_\Omega \Big(e_1u_n^+\frac{\partial F}{\partial u}(x,u_n^+,v_n^+)+e_2v_n^+\frac{\partial F}{\partial v}(x,u_n^+,v_n^+) \\
&\quad -F(x,u_n^+,v_n^+)\Big)dx+\frac{2(\alpha e_1+\beta e_2-1)}{\alpha+\beta}\int_\Omega \frac{(u^+)^\alpha(v^+)^\beta}{|x|^{bp}}dx \\
&\geqslant \big(\frac{1}{2}-e_1\big)||u_n||^2+\big(\frac{1}{2}-e_2\big)||v_n||^2 \\
&\geqslant \min\Big\{\frac{1}{2}-e_1,\frac{1}{2}-e_2\Big\}||(u_n,v_n)||^2.
\end{aligned}
$$

Hence, we conclude $\{(u_n,v_n)\}$ is a bounded sequence in H and there exists (u,v) such that $(u_n,v_n)\rightharpoonup(u,v)$ up to a subsequence. Moreover, we may assume

$$
\begin{cases}
u_n\rightharpoonup u,\ \ v_n\rightharpoonup v, & \text{weakly in } H_0^1(\Omega),\\
u_n\to u,\ \ u_n\to u, & \text{strongly in } L^r(\Omega),\ \ 1<r<2^*\\
u_n\to u,\ \ u_n\to u, & \text{a.e. on } \Omega.
\end{cases}
$$

By (H1) and (H3), there exists a positive constant $M>0$ such that

$$
F(x,u_n^+,v_n^+)\leqslant \frac{R_1}{2}((u_n^+)^{p_1}+(v_n^+)^{p_2})+M. \tag{2.1}
$$

Now, by absolutely continuity of integral, for any $\varepsilon>0$, there exists $\delta=\frac{\varepsilon}{2M}>0$, when $E\subset\Omega$, $\text{mes}(E)<\delta$, we have

$$
\int_E((u_n^+)^{p_1}+(v_n^+)^{p_2})dx<\frac{\varepsilon}{R_1}.
$$

Hence, by (2.1), we have

$$
\begin{aligned}
\int_E F(x,u_n^+,v_n^+)dx &\leqslant \frac{R_1}{2}\int_E((u_n^+)^{p_1}+(v_n^+)^{p_2})dx+\text{mes}(E)M \\
&\leqslant \frac{R_1}{2}\frac{\varepsilon}{R_1}+M\delta=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Thus, $\Big\{\int_\Omega F(x,u_n^+,v_n^+)dx,\,j\in\mathbb{N}\Big\}$ is equi-absolutely continuous. It follows easily from Vitali Convergence Theorem that

$$
\int_\Omega F(x,u_n^+,v_n^+)dx\to\int_\Omega F(x,u^+,v^+)dx,
$$

as $n\to\infty$. Using the same methods, we can prove that

$$\int_\Omega \frac{\partial F(x,u_n^+,v_n^+)}{\partial u} u_n^+ dx \to \frac{\partial F(x,u^+,v^+)}{\partial u} u^+ dx,$$
$$\int_\Omega \frac{\partial F(x,u_n^+,v_n^+)}{\partial v} v_n^+ dx \to \frac{\partial F(x,u^+,v^+)}{\partial v} v^+ dx, \tag{2.2}$$

as $n \to \infty$. Hence, we have $J'(u,v) = 0$ by the weak continuity of J. Let $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$. Then we have

$$\begin{aligned}\int_\Omega \Big(|x|^{-2a}|\nabla \widetilde{u}_n|^2 + |x|^{-2a}|\nabla \widetilde{v}_n|^2 - \mu \frac{|\widetilde{u}_n|^2 + |\widetilde{v}_n|^2}{|x|^{2(1+a)}} \Big) dx \\ = \int_\Omega \Big(|x|^{-2a}|\nabla u_n|^2 + |x|^{-2a}|\nabla v_n|^2 - \mu \frac{|u_n|^2 + |v_n|^2}{|x|^{2(1+a)}} \Big) dx \\ - \int_\Omega \Big(|x|^{-2a}|\nabla u|^2 + |x|^{-2a}|\nabla v|^2 - \mu \frac{|u|^2 + |v|^2}{|x|^{2(1+a)}} \Big) dx + o(1).\end{aligned}$$

By the Brèzis-Lieb lemma [2], we obtain

$$||(\widetilde{u}_n, \widetilde{v}_n)||^2 \to ||(u_n,v_n)||^2 - ||(u,v)||^2, \quad \text{as} \quad n \to \infty, \tag{2.3}$$

and

$$\int_\Omega \frac{|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta}{|x|^{bp}} dx = \int_\Omega \frac{|u_n|^\alpha |v_n|^\beta}{|x|^{bp}} dx - \int_\Omega \frac{|u|^\alpha |v|^\alpha}{|x|^{bp}} dx + o(1). \tag{2.4}$$

Since $J'(u_n,v_n) \to 0$, we obtain

$$\begin{aligned}||(u_n,v_n)||^2 - 2\int_\Omega \frac{(u_n^+)^\alpha (v_n^+)^\alpha}{|x|^{bp}} dx \\ - \lambda \int_\Omega \Big(\frac{\partial F(x,u_n^+,v_n^+)}{\partial u} u_n^+ + \frac{\partial F(x,u_n^+,v_n^+)}{\partial v} v_n^+ \Big) dx = o(1).\end{aligned}$$

Now, by (2.2), (2.3) and (2.4), we have

$$\begin{aligned}\|(\widetilde{u}_n, \widetilde{v}_n)\|^2 + ||(u,v)||^2 - 2\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}} dx - 2\int_\Omega \frac{(u^+)^\alpha (v^+)^\alpha}{|x|^{bp}} dx \\ - \lambda \int_\Omega \Big(\frac{\partial F(x,u^+,v^+)}{\partial u} u^+ + \frac{\partial F(x,u^+,v^+)}{\partial v} v^+ \Big) dx = o(1).\end{aligned} \tag{2.5}$$

And

$$\begin{aligned}\lim_{n\to\infty} \langle J'(u_n,v_n),(u,v)\rangle = ||(u,v)||^2 - 2\int_\Omega \frac{(u^+)^\alpha (v^+)^\alpha}{|x|^{bp}} dx \\ - \lambda \int_\Omega \Big(\frac{\partial F(x,u^+,v^+)}{\partial u} u^+ + \frac{\partial F(x,u^+,v^+)}{\partial v} v^+ \Big) dx = 0.\end{aligned} \tag{2.6}$$

It derives from (2.6) that

$$J(u,v) = \Big(1 - \frac{2}{\alpha+\beta}\Big)\int_\Omega \frac{(u^+)^\alpha (v^+)^\alpha}{|x|^{bp}}dx + \lambda \int_\Omega \Big[\frac{1}{2}\Big(\frac{\partial F(x,u^+,v^+)}{\partial u}u^+ + \frac{\partial F(x,u^+,v^+)}{\partial v}v^+\Big) - F(x,u^+,v^+)\Big]dx.$$

Together with (H2), we can conclude that

$$J(u,v) \geqslant 0. \tag{2.7}$$

Since $J(u_n,v_n) \to c$ as $n \to \infty$, we obtain

$$\begin{aligned} J(u_n,v_n) &= \frac{1}{2}||(\widetilde{u}_n,\widetilde{v}_n)||^2 + \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx \\ &\quad - \frac{2}{\alpha+\beta}\int_\Omega \frac{(u^+)^\alpha (v^+)^\alpha}{|x|^{bp}}dx - \lambda\int_\Omega F(x,u^+,v^+)dx + o(1) \\ &= J(u,v) + \frac{1}{2}||(\widetilde{u}_n,\widetilde{v}_n)||^2 - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx + o(1) \\ &= c + o(1). \end{aligned}$$

Therefore

$$J(u,v) + \frac{1}{2}||(\widetilde{u}_n,\widetilde{v}_n)||^2 - \frac{2}{\alpha+\beta}\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx = c + o(1). \tag{2.8}$$

From (2.5) and (2.6), we have

$$||(\widetilde{u}_n,\widetilde{v}_n)||^2 - 2\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx = o(1).$$

Let us prove that $||(\widetilde{u}_n,\widetilde{v}_n)|| \to 0$ as $n \to \infty$. Otherwise, there exists a subsequence (still denoted by $(\widetilde{u}_n,\widetilde{v}_n)$) such that

$$||(\widetilde{u}_n,\widetilde{v}_n)||^2 \to l, \quad 2\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx \to l. \tag{2.9}$$

From definition of $\widetilde{A}_{a,b,\mu}(\Omega)$, we obtain

$$||(\widetilde{u}_n,\widetilde{v}_n)||^2 \geqslant \widetilde{A}_{a,b,\mu}(\Omega)\Big(\int_\Omega \frac{(\widetilde{u}_n^+)^\alpha (\widetilde{v}_n^+)^\alpha}{|x|^{bp}}dx\Big)^{\frac{2}{\alpha+\beta}},$$

then $l \geqslant \widetilde{A}_{a,b,\mu}(\Omega) l^{\frac{2}{p}}$, i.e., $l \geqslant \left(\widetilde{A}_{a,b,\mu}(\Omega)\right)^{\frac{p}{p-2}}$, which, together with (2.8) and (2.9), shows that

$$J(u,v) = c - \frac{1}{2}l + \frac{1}{p}l \leqslant c - \frac{p-2}{2p}\widetilde{A}_{a,b,\mu}^{\frac{p}{p-2}} < 0,$$

which contradicts (2.7). Therefore, we get

$$||(\widetilde{u}_n, \widetilde{v}_n)|| \to 0 \quad \text{as} \quad n \to \infty.$$

This proves $(u_n, v_n) \to (u,v)$ in H as $n \to \infty$.
Thus, J satisfies $(PS)_c$ condition.

The author in [11] proved that, for:

$$0 \leqslant a < \sqrt{\overline{\mu}},\ 0 \leqslant \mu < (\sqrt{\overline{\mu}} - a)^2, \quad \text{and} \quad a \leqslant b < a+1,$$

$A_{a,b,\mu}$ is attained when $\Omega = \mathbb{R}^N$ by the functions

$$y_\varepsilon(x) = \frac{(2\varepsilon p \beta^2)^{\frac{1}{p}}}{|x|^{\gamma'}(\varepsilon + |x|^{(p-2)\beta})^{\frac{2}{p}}},$$

for all $\varepsilon > 0$, where $\gamma' \triangleq \sqrt{\overline{\mu}} - a - \beta$. Moreover, the functions $y_\varepsilon(x)$ solve the equation

$$-\mathrm{div}(|x|^{-2a}\nabla u) - \mu \frac{u}{|x|^{2(1+a)}} = \frac{|u|^{p-2}}{|x|^{bp}} u, \ \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Let

$$C_\varepsilon = (2\varepsilon p \beta^2)^{\frac{1}{p}} \quad \text{and} \quad U_\varepsilon(x) = \frac{y_\varepsilon(x)}{C_\varepsilon}.$$

Define a cut-off function $\varphi \in C_0^+(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leqslant r$, $\varphi(x) = 0$ for $|x| \geqslant 2r$, $0 \leqslant \varphi(x) \leqslant 1$, where $B_{2r}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x) U_\varepsilon(x)$,

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{(\int_\Omega |u_\varepsilon|^p |x|^{-bp} dx)^{1/p}},$$

so that $\int_\Omega |v_\varepsilon|^p |x|^{-bp} dx = 1$. The author in [9] proved that

$$A_{a,b,\mu}(\Omega) + C_2 \varepsilon^{\frac{2}{p-2}} \leqslant ||v_\varepsilon||^2 \leqslant A_{a,b,\mu}(\Omega) + C_3 \varepsilon^{\frac{2}{p-2}}, \tag{2.10}$$

and

$$\begin{cases} C_4 \varepsilon^{\frac{q}{p-2}} \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_5 \varepsilon^{\frac{q}{p-2}}, & 1 \leqslant q < \frac{N}{\gamma}, \\ C_4 \varepsilon^{\frac{q}{p-2}} |\ln \varepsilon| \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_5 \varepsilon^{\frac{q}{p-2}} |\ln \varepsilon|, & q = \frac{N}{\gamma}, \\ C_4 \varepsilon^{\frac{N - q(\sqrt{\overline{\mu}} - a)}{(p-2)\beta}} \leqslant \int_\Omega |v_\varepsilon|^q dx \leqslant C_5 \varepsilon^{\frac{N - q(\sqrt{\overline{\mu}} - a)}{(p-2)\beta}}, & \frac{N}{\gamma} < q < 2^*. \end{cases} \tag{2.11}$$

LEMMA 2. *Assume that*

$$0 \leqslant a < \sqrt{\overline{\mu}},\ 0 \leqslant \mu < (\sqrt{\overline{\mu}} - a)^2, \quad \text{and} \quad a \leqslant b < a+1.$$

Suppose that (H1)-(H4) hold. Then there exists $(u_0, v_0) \in H,\ u_0 \neq 0,\ v_0 \neq 0$ *and* $\lambda_1^* > 0$ *such that*

$$\sup_{t \geqslant 0} J(tu_0, tv_0) < \frac{p-2}{2p}\Big(\frac{\widetilde{A}_{a,b,\mu}(\Omega)}{2}\Big)^{\frac{p}{p-2}},$$

for every $\lambda \in (0, \lambda_1^*)$.

Proof. Let $u = \sqrt{\alpha} v_\varepsilon$ and $v = \sqrt{\beta} v_\varepsilon$, then we consider the functions

$$g(t) = J(t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon) = \frac{t^2}{2}(\alpha+\beta)||v_\varepsilon||^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}} - \lambda\int_\Omega F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)dx,$$

and

$$\widetilde{g}(t) = J(t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon) = \frac{t^2}{2}(\alpha+\beta)||v_\varepsilon||^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}.$$

Note that $\lim_{t\to+\infty} g(t) = -\infty,\ g(0) = 0,\ g(t) > 0$ for $t \to 0^+$, so $\sup_{t\geqslant 0} g(t)$ is attained for some $t_\varepsilon > 0$. Since (H2) and

$$0 = g'(t_\varepsilon) = t_\varepsilon(\alpha+\beta)||v_\varepsilon||^2 - 2t_\varepsilon^{\alpha+\beta-1}\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}} - \lambda\int_\Omega\Big(\frac{\partial F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon + \frac{\partial F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\Big)dx,$$

we have

$$\begin{aligned} ||v_\varepsilon||^2 &= \frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2} + \frac{\lambda}{t_\varepsilon(\alpha+\beta)}\int_\Omega\Big(\frac{\partial F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)}{\partial u}\sqrt{\alpha}v_\varepsilon \\ &\quad + \frac{\partial F(x, t\sqrt{\alpha}v_\varepsilon, t\sqrt{\beta}v_\varepsilon)}{\partial v}\sqrt{\beta}v_\varepsilon\Big)dx \\ &\geqslant \frac{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{\alpha+\beta}t_\varepsilon^{\alpha+\beta-2}. \end{aligned} \tag{2.12}$$

Therefore, by the last inequality, we can write

$$t_\varepsilon \leqslant \Big[\frac{(\alpha+\beta)||v_\varepsilon||^2}{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}\Big]^{\frac{1}{\alpha+\beta-2}} \triangleq t_\varepsilon^0. \tag{2.13}$$

By (2.10) and (2.11), we get

$$||v_\varepsilon||^2 \to A_{a,b,\mu}, \ \int_\Omega v_\varepsilon^{p_1} dx \to 0, \ \text{and} \ \int_\Omega v_\varepsilon^{p_1 2} dx \to 0, \tag{2.14}$$

as $n \to \infty$. Now, by (H1) and (H3), we deduce that

$$\begin{aligned} &\frac{\partial F(x, t\sqrt{\alpha} v_\varepsilon, t\sqrt{\beta} v_\varepsilon)}{\partial u} \sqrt{\alpha} v_\varepsilon + \frac{\partial F(x, t\sqrt{\alpha} v_\varepsilon, t\sqrt{\beta} v_\varepsilon)}{\partial v} \sqrt{\beta} v_\varepsilon \\ &\qquad \leqslant R_1 (t_\varepsilon^{p_1 - 1} \alpha^{\frac{p_1}{2}} v_\varepsilon^{p_1} + t_\varepsilon^{p_2 - 1} \beta^{\frac{p_2}{2}} v_\varepsilon^{p_2}) + C_5 t_\varepsilon, \end{aligned}$$

for some constant $C_5 > 0$. By (2.12)-(2.14) and Hölder inequality, we obtain

$$\begin{aligned} ||v_\varepsilon||^2 &\leqslant \frac{2\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}}}{\alpha + \beta} t_\varepsilon^{\alpha + \beta - 2} + \frac{\lambda}{(\alpha + \beta)} R_1 \Big((t_\varepsilon^0)^{p_1 - 2} \alpha^{\frac{p_1}{2}} \int_\Omega v_\varepsilon^{p_1} dx \\ &\qquad + (t_\varepsilon^0)^{p_2 - 2} \beta^{\frac{p_2}{2}} \int_\Omega v_\varepsilon^{p_2} dx \Big) + \frac{\lambda C_5 |\Omega|}{\alpha + \beta} \\ &= \frac{2\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}}}{\alpha + \beta} t_\varepsilon^{\alpha + \beta - 2} + \frac{\lambda}{(\alpha + \beta)} (R_1 + C_5 |\Omega|) + o(1), \end{aligned}$$

as $n \to \infty$. Thus, there exists $\lambda_1^* = \frac{\alpha + \beta}{2(R_1 + C_5 |\Omega|)} A_{a,b,\mu}(\Omega) > 0$ such that

$$t_\varepsilon \geqslant \left(\frac{\alpha + \beta}{2\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}}} \cdot \frac{A_{a,b,\mu}}{2} \right)^{\frac{1}{\alpha + \beta - 2}} \triangleq T_0, \tag{2.15}$$

for all $\lambda \in (0, \lambda_1^*)$.

From (H2), we get

$$\begin{aligned} F(x, u, v) &\leqslant e_1 u \frac{\partial F}{\partial u}(x, u, v) + e_2 v \frac{\partial F}{\partial v}(x, u, v) \\ &\leqslant \max\{e_1, e_2\} \cdot \langle \nabla F(x, u, v), (u, v) \rangle \\ &= \frac{1}{\rho} \langle \nabla F(x, u, v), (u, v) \rangle. \end{aligned} \tag{2.16}$$

Now, we consider the function $L : [1, +\infty) \to \mathbb{R}$ defined by

$$L(t) := F(x, \frac{u}{t}, \frac{v}{t}) t^\rho, \tag{2.17}$$

clearly, by (2.16), the function L is non-increasing. Thus, for $|(u, v)| \geqslant 1$, we have $L(1) \geqslant L(|(u, v)|)$. Together with (H4), it derives

$$\begin{aligned} F(x, u, v) &\geqslant F(x, \frac{(u, v)}{|(u, v)|}) |(u, v)|^\rho \\ &\geqslant \inf_{|(u, v)| = 1} F(x, u, v) |(u, v)|^\rho = \eta |(u, v)|^\rho. \end{aligned} \tag{2.18}$$

If $|(u,v)| \leqslant 1$, by the continuity of F, we can write,

$$F(x,u,v) \geqslant \eta|(u,v)|^{\rho} - C_7,$$

where $C_7 \geqslant \max\{0, \eta - \min_{|(u,v)|\leqslant 1} F(x,u,v)\}$. Together with (2.18), we deduce that

$$F(x,u,v) \geqslant \eta|(u,v)|^{\rho} - C_7, \quad \text{for all} \quad (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+. \tag{2.19}$$

On one hand, from (2.10), we get

$$||v_\varepsilon||^{\frac{2p}{p-2}} \leqslant \left(A_{a,b,\mu}(\Omega)\right)^{\frac{p}{p-2}} + C_6 \varepsilon^{\frac{2}{p-2}}. \tag{2.20}$$

On the other hand, the function $\widetilde{g}(t)$ attains its maximum at t_ε^0 and is increasing the interval $[0, t_\varepsilon^0]$, together with (2.11), (2.15), (2.19) and (2.20), we deduce that

$$\begin{aligned}
g(t_\varepsilon) &= \widetilde{g}(t_\varepsilon) - \lambda \int_\Omega F(x, t_\varepsilon\sqrt{\alpha}v_\varepsilon, t_\varepsilon\sqrt{\beta}v_\varepsilon)dx \\
&\leqslant \widetilde{g}(t_\varepsilon^0) - \lambda \int_\Omega F(x, t_\varepsilon\sqrt{\alpha}v_\varepsilon, t_\varepsilon\sqrt{\beta}v_\varepsilon)dx \\
&\leqslant \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\left[\frac{(\alpha+\beta)||v_\varepsilon||^2}{2\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}\right]^{\frac{2}{\alpha+\beta-2}}(\alpha+\beta)||v_\varepsilon||^2 \\
&\qquad\qquad -\lambda\eta(\alpha+\beta)^{\frac{\rho}{2}} t_\varepsilon^\rho \int_\Omega v_\varepsilon^\rho dx - \lambda C_7|\Omega| \\
&\leqslant 2\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\left[\frac{(\alpha+\beta)}{2\alpha^{\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}}\right]^{\frac{\alpha+\beta}{\alpha+\beta-2}}(\alpha+\beta)||v_\varepsilon||^{\frac{p}{p-2}} - \lambda C_7|\Omega| \\
&\qquad\qquad -\lambda\eta(\alpha+\beta)^{\frac{\rho}{2}} T_0^\rho C_3 \varepsilon^{\frac{N-\rho(\sqrt{\overline{\mu}}-a)}{(p-2)\beta}} \\
&\leqslant \frac{p-2}{2p}\left[\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right)\frac{A_{a,b,\mu}(\Omega)}{2}\right]^{\frac{p}{p-2}} + C_8\varepsilon^{\frac{2}{p-2}} \\
&\qquad\qquad -C_9\varepsilon^{\frac{N-\rho(\sqrt{\overline{\mu}}-a)}{(p-2)\beta}} - \lambda C_7|\Omega|,
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
C_8 &= \frac{p-2}{2p}\left[\frac{1}{2}\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right)\frac{A_{a,b,\mu}(\Omega)}{2}\right]^{\frac{p}{p-2}} C_6, \\
C_9 &= \lambda\eta(\alpha+\beta)^{\frac{\rho}{2}} T_0^\rho C_3.
\end{aligned}$$

By the definition of ρ in Theorem 1, we obtain that

$$\frac{2}{p-2} > \frac{N-\rho(\sqrt{\overline{\mu}}-a)}{(p-2)\beta}.$$

Choosing ε small enough, we have

$$\sup_{t\geqslant 0} J(tu,tv) = g(t_\varepsilon) < \frac{p-2}{2p}\Big(\frac{\widetilde{A}_{a,b,\mu}(\Omega)}{2}\Big)^{\frac{p}{p-2}}.$$

3. Proof of main results

Proof of Theorem 1. From the Caffarelli-Kohn-Nirenberg inequality (1.3), we can easily get:

$$\int_\Omega u^{p_1} dx \leqslant C_{10}||u||^{p_1}, \quad \int_\Omega u^{p_2} dx \leqslant C_{10}||u||^{p_2}, \tag{3.1}$$

for all $u \in H_0^1(\Omega, |x|^{-2a})$. For every $\varepsilon > 0$, fix $\lambda^{**} \in (0,\varepsilon)$. If $\lambda \in (0,\lambda^{**})$, by (1.7), (2.1), (3.1) and (H2), for any $(u,v) \in H$, we have

$$\begin{aligned}
J(u,v) &\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p - \lambda\int_\Omega F(x,u^+,v^+)dx \\
&\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p \\
&\qquad -\lambda\int_\Omega \Big(e_1\frac{\partial F(x,u^+,v^+)}{\partial u}u + e_2\frac{\partial F(x,u^+,v^+)}{\partial v}v\Big)dx \\
&\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p \\
&\qquad -\frac{\lambda R_1}{2}\int_\Omega \big((u^+)^{p_1} + (v^+)^{p_2}\big)dx - \lambda M|\Omega| \\
&\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p \\
&\qquad -\frac{\lambda R_1}{2}C_{10}\Big(||u^+||^{p_1} + ||v^+||^{p_2}\Big)dx - \lambda M|\Omega| \\
&\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p \\
&\qquad -\frac{\lambda^{**} R_1}{2}C_{10}\Big(||(u,v)||^{p_1} + ||(u,v)||^{p_2}\Big)dx - \lambda^{**} M|\Omega| \\
&\geqslant \frac{1}{2}||(u,v)||^2 - \frac{2}{\alpha+\beta}(\widetilde{A}_{a,b,\mu}(\Omega))^{-\frac{p}{2}}||(u,v)||^p \\
&\qquad -\frac{\varepsilon R_1}{2}C_{10}\Big(||(u,v)||^{p_1} + ||(u,v)||^{p_2}\Big)dx - \varepsilon M|\Omega|,
\end{aligned}$$

for ε small enough. Thus, there exists $\rho > 0$ such that $J(u,v) \geqslant \rho$ for all $(u,v) \in \partial B_r = \{(u,v) \in H,\ ||(u,v)|| = r\}$, where $r > 0$ small enough. Let $\lambda^* = \min\{\lambda_1^*, \lambda^{**}\}$. By Lemma 2, for $\lambda \in (0,\lambda^*)$, there exists $(u_0,v_0) \in H$, $u_0 \neq 0$, $v_0 \neq 0$, such that

$$\sup_{t\geqslant 0} J(tu_0,tv_0) < \frac{p-2}{2p}\Big(\frac{\widetilde{A}_{a,b,\mu}(\Omega)}{2}\Big)^{\frac{p}{p-2}}.$$

By the nonnegativity of $F(x,u,v)$, we get

$$\begin{aligned} J(tu_0,tv_0) &= \frac{1}{2}t^2||(u_0,v_0)||^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega \frac{(u_0^+)^\alpha (v_0^+)^\beta}{|x|^{bp}}dx - \lambda\int_\Omega F(x,tu_0,tv_0)dx \\ &\leqslant \frac{1}{2}t^2||(u_0,v_0)||^2 - \frac{2t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega \frac{(u_0^+)^\alpha (v_0^+)^\beta}{|x|^{bp}}dx, \end{aligned}$$

which implies that $\lim_{t\to+\infty} J(tu_0,tv_0) \to -\infty$. Hence, we can choose $t_0 > 0$ such that $||(t_0u_0,t_0v_0)|| > r$ and $J(tu_0,tv_0) \leqslant 0$. Applying the Mountain Pass Lemma in [14], there is a sequence $(u_n,v_n) \subset H$ satisfying

$$J(u_n,v_n) \to c \geqslant \rho, \quad \text{and} \quad J'(u_n,v_n) \to 0,$$

where

$$\begin{aligned} c &= \inf_{h\in\tau}\max_{t\in[0,1]} J(h(t)), \\ \tau &= \{h \in ([0,1],H^2) | h(0) = 0, h(1) = (t_0u_0,t_0v_0)\}. \end{aligned}$$

Note that

$$\begin{aligned} 0 < \rho \leqslant c = \inf_{h\in\tau}\max_{t\in[0,1]} J(h(t)) &\leqslant \max_{t\in[0,1]} J(tt_0u_0,tt_0v_0) \\ &\leqslant \sup_{t\geqslant 0} J(tu_0,tv_0) < \frac{p-2}{2p}\Big(\frac{\widetilde{A}_{a,b,\mu}(\Omega)}{2}\Big)^{\frac{p}{p-2}}. \end{aligned}$$

By Lemma 1 there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $(u_n,v_n) \to (u,v)$ strongly in H. Thus, we get a critical point (u,v) of J satisfying (1.1) and c is a critical value.
In $J(u,v)$, by replacing the terms of

$$\int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^{bp}}dx, \quad \int_\Omega F(x,u,v)dx$$

instead of

$$\int_\Omega \frac{|u^+|^\alpha |v^+|^\beta}{|x|^{bp}}dx, \quad \int_\Omega F(x,u^+,v^+)dx$$

respectively and repeating the above process, we get a nonnegative solution (u,v) to (1.1). Also, by the maximum principle we deduce that $u>0$, $v>0$ in Ω. $\square$

Proof of Theorem 2. By Theorem 1, there exist $\lambda^*>0$ such that the problem (1.1) has a positive solution (u_1,v_1) for each $\lambda\in(0,\lambda^*)$. Set $G(x,s,t)=-F(x,-s,-t)$ for $(s,t)\in\mathbb{R}^2$. It follows from Theorem 1 that there exists $\lambda'^*>0$ such that the system

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u)-\mu\frac{u}{|x|^{2(1+a)}}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^{bp}}+\lambda\frac{\partial}{\partial u}G(x,u,v), & x\in\Omega,\\ -\mathrm{div}(|x|^{-2a}\nabla v)-\mu\frac{v}{|x|^{2(1+a)}}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{bp}}+\lambda\frac{\partial}{\partial v}G(x,u,v), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega, \end{cases}$$

has at least a positive solution (u,v) for each $\lambda\in(0,\lambda'^*)$. Let $(u_2,v_2)=-(u,v)$, then (u_2,v_2) is a solution of system

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u)-\mu\frac{u}{|x|^{2(1+a)}}=\frac{2\alpha}{\alpha+\beta}\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^{bp}}+\lambda\frac{\partial}{\partial u}F(x,u,v), & x\in\Omega,\\ -\mathrm{div}(|x|^{-2a}\nabla v)-\mu\frac{v}{|x|^{2(1+a)}}=\frac{2\beta}{\alpha+\beta}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^{bp}}+\lambda\frac{\partial}{\partial v}F(x,u,v), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases}$$

Set $\overline{\lambda}=\min\{\lambda^*,\lambda'^*\}$. It is obvious that:

$$(u_1,v_1)\neq(0,0),\ (u_2,v_2)\neq(0,0)\ \text{and}\ (u_1,v_1)\neq(u_2,v_2).$$

So the system (1.1) has at least two distinct nontrivial solutions for every $\lambda\in(0,\overline{\lambda})$. Therefore, Theorem 2 holds. $\square$

Acknowledgements. The authors would like to thank the anonymous referees for his/her valuable suggestions and comments.

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(Received November 5, 2012)
(Revised June 24, 2013)

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